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LAPLACIAN AND SIGNLESS LAPLACIAN SPECTRUM OF COMMUTING GRAPHS OF FINITE GROUPS

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Abstract. The commuting graph of a finite non-abelian group G with center $Z(G)$, denoted by Γ_G, is a simple undirected graph whose vertex set is $G\setminus Z(G)$, and two distinct vertices x and y are adjacent if and only if $xy = yx$. A finite non-abelian group G is called super integral if the spectrum, Laplacian spectrum and signless Laplacian spectrum of its commuting graph contain only integers. In this paper, we first compute Laplacian spectrum and signless Laplacian spectrum of several families of finite non-abelian groups and conclude that those groups are super integral. As an application of our results we obtain some positive rational numbers r such that G is super integral if commutativity degree of G is r. In the last section, we show that G is super integral if G is not isomorphic to S_4 and its commuting graph is planar. We conclude the paper showing that G is super integral if its commuting graph is toroidal.

1. Introduction

Let $A(\mathcal{G})$ and $D(\mathcal{G})$ denote the adjacency matrix and degree matrix of a graph $\mathcal{G}$ respectively. Then the Laplacian matrix and signless Laplacian matrix of $\mathcal{G}$ are given by $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$ and $Q(\mathcal{G}) = D(\mathcal{G}) + A(\mathcal{G})$ respectively. We write Spec($\mathcal{G}$), L-spec($\mathcal{G}$) and Q-spec($\mathcal{G}$) to denote the spectrum, Laplacian spectrum and Signless Laplacian spectrum of $\mathcal{G}$. Also, $\text{Spec}(\mathcal{G}) = \{\alpha_1^{a_1}, \alpha_2^{a_2}, \ldots, \alpha_l^{a_l}\}$, $\text{L-spec}(\mathcal{G}) = \{\beta_1^{b_1}, \beta_2^{b_2}, \ldots, \beta_m^{b_m}\}$ and $\text{Q-spec}(\mathcal{G}) = \{\gamma_1^{c_1}, \gamma_2^{c_2}, \ldots, \gamma_n^{c_n}\}$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the eigenvalues of $A(\mathcal{G})$ with multiplicities $a_1, a_2, \ldots, a_l$; $\beta_1, \beta_2, \ldots, \beta_m$ are the eigenvalues of $L(\mathcal{G})$ with multiplicities $b_1, b_2, \ldots, b_m$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the eigenvalues of $Q(\mathcal{G})$ with multiplicities $c_1, c_2, \ldots, c_n$ respectively. A graph $\mathcal{G}$ is called integral or L-integral or Q-integral according as Spec($\mathcal{G}$) or L-spec($\mathcal{G}$)

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or Q-spec($\mathcal{G}$) contains only integers. $\mathcal{G}$ is called super integral if it is integral, L-integral and Q-integral. The notion of integral graph was introduced by Harary and Schwenk [16] in the year 1974. A very impressive survey on integral graphs can be found in [6]. Ahmadi et al. noted that integral graphs have some interests for designing the network topology of perfect state transfer networks, see [4] and the references there in. L-integral graphs are also studied extensively over the years while Q-integral graphs are not studied much. One may conf. [2, 8, 18, 20, 28] and some of the references in [18] for several interesting results of these graphs.

Let G be a finite non-abelian group with center $Z(G)$. The commuting graph of G, denoted by Γ_G, is a simple undirected graph whose vertex set is $G \setminus Z(G)$, and two distinct vertices x and y are adjacent if and only if $xy = yx$. Various aspects of commuting graphs of finite groups can be found in [5, 11, 12, 17, 21, 25]. A finite non-abelian group G is called integral or L-integral or Q-integral according as Γ_G is integral or L-integral or Q-integral. One may ask the following questions.

Question 1. Which finite non-abelian groups are integral?

Question 2. Which finite non-abelian groups are L-integral?

Question 3. Which finite non-abelian groups are Q-integral?

A finite non-abelian group G is called super integral if Γ_G is integral, L-integral and Q-integral. Therefore, in the line of above questions, we can also ask the following question.

Question 4. Which finite non-abelian groups are super integral?

In [11, 12], we have computed the spectrum of Γ_G for several families of finite non-abelian groups and determined several finite non-abelian integral groups. In [24], we have computed Laplacian spectrum and signless Laplacian spectrum of n-centralizer finite groups for $n = 4, 5$ and showed that those groups are super integral. In this paper, we compute Laplacian spectrum and signless Laplacian spectrum of commuting graphs of finite groups whose central quotient is isomorphic to $Sz(2)$ (the Suzuki group of order 20) or $\mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) or D_{2m} (the dihedral group of order $2m$). We also consider the quasidihedral groups, generalized quaternion groups, general linear groups, some projective special linear groups, the groups constructed by Hanaki in [15] etc. Our computations in Section 2 reveal that all the above mentioned groups are super integral. Further, we shall show that all finite AC-groups are super integral. The rest part of this paper is devoted to some applications of the results obtained in Section 2.

The commutativity degree of G denoted by $\Pr(G)$ is the probability that a randomly chosen pair of elements of G commute. Clearly, $\Pr(G) = 1$ if and only if G is abelian. For a non-abelian group, it was shown in [14] that $\Pr(G) \leq 5/8$. Since then many mathematicians have studied this notion. The readers may conf. [7, 22] for various results on $\Pr(G)$. A survey of recent works on $\Pr(G)$ can be found in [9]. Using our results, in Section 3, we shall show that G is super integral if $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{5}{8}\}$.

Recall that genus of a graph is the smallest non-negative integer n such that the graph can be embedded on the surface obtained by attaching n handles to a sphere. A graph is said to be planar or toroidal if the genus of the graph is zero or one respectively. It is worth mentioning that Afkhami et al. [3] and Das et al. [10] have classified all finite non-abelian groups whose commuting graphs are planar or toroidal recently. In the last section, we shall show that a finite non-abelian group G is super integral if it is not isomorphic to S_4, the symmetric group of degree 4, and the commuting graph of G is planar. We also show that a finite non-abelian group is super integral if its commuting graph is toroidal.

2. Laplacian and signless Laplacian spectrum

It is well-known that L-spec$(K_n) = \{0^1, n^{n-1}\}$ and Q-spec$(K_n) = \{(2n-2)^1, (n-2)^{n-1}\}$ where K_n denotes the complete graph on n vertices. Further, we have the following theorem.

Theorem 2.1. *If $\mathcal{G} = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup \cdots \sqcup l_kK_{m_k}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of K_{m_i} for $1 \leq i \leq k$, then*

$$\text{L-spec}(\mathcal{G}) = \left\{0^{\sum_{i=1}^{k} l_i}, m_1^{l_1(m_1-1)}, m_2^{l_2(m_2-1)}, \dots, m_k^{l_k(m_k-1)}\right\}$$

and

$$\text{Q-spec}(\mathcal{G}) = \{(2m_1-2)^{l_1}, (m_1-2)^{l_1(m_1-1)}, (2m_2-2)^{l_2}, (m_2-2)^{l_2(m_2-1)}, \dots, (2m_k-2)^{l_k}, (m_k-2)^{l_k(m_k-1)}\}.$$

Now we compute the Laplacian spectrum and signless Laplacian spectrum of the commuting graphs of some families of finite non-abelian groups. We begin with some families of groups whose central factors are some well-known groups.

Theorem 2.2. *Let G be a finite group and $\frac{G}{Z(G)} \cong Sz(2)$, where $Sz(2)$ is the Suzuki group presented by $\langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2\rangle$. Then*

$$\text{L-spec}(\Gamma_G) = \{0^6, (4|Z(G)|)^{4|Z(G)|-1}, (3|Z(G)|)^{15|Z(G)|-5}\} \quad \textit{and}$$

$$\text{Q-spec}(\Gamma_G) = \{(8|Z(G)|-2)^1, (4|Z(G)|-2)^{4|Z(G)|-1}, (6|Z(G)|-2)^5, (3|Z(G)|-2)^{15|Z(G)|-5}\}.$$

Proof. It was shown in [11, Theorem 2.2] that $\Gamma_G = K_{4|Z(G)|} \sqcup 5K_{3|Z(G)|}$. Therefore, by Theorem 2.1, the result follows. □

Theorem 2.3. *Let G be a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime integer. Then*

$$\text{L-spec}(\Gamma_G) = \{0^{p+1}, ((p-1)|Z(G)|)^{(p^2-1)|Z(G)|-p-1}\} \textit{ and}$$

$$\text{Q-spec}(\Gamma_G) = \{(2(p-1)|Z(G)|-2)^{p+1}, ((p-1)|Z(G)|-2)^{(p^2-1)|Z(G)|-p-1}\}.$$

Proof. It was shown in [12, Theorem 2.1] that $\Gamma_G = (p+1)K_{(p-1)|Z(G)|}$. Hence the result follows from Theorem 2.1. □

As a corollary we have the following result.

Corollary 2.4. *Let G be a non-abelian group of order p^3, for any prime p, then*

$$\text{L-spec}(\Gamma_G) = \{0^{p+1}, (p^2-p)^{p^3-2p-1}\} \textit{ and}$$

$$\text{Q-spec}(\Gamma_G) = \{(2p^2-2p-2)^{p+1}, (p^2-p-2)^{p^3-2p-1}\}.$$

Proof. Note that $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 2.3. □

Theorem 2.5. *Let G be a finite group such that $\frac{G}{Z(G)} \cong D_{2m}$, for $m \geq 2$. Then*

$$\text{L-spec}(\Gamma_G) = \{0^{m+1}, ((m-1)|Z(G)|)^{(m-1)|Z(G)|-1}, (|Z(G)|)^{m(|Z(G)|-1)}\} \text{ and}$$
$$\text{Q-spec}(\Gamma_G) = \{(2(m-1)|Z(G)|-2)^1, ((m-1)|Z(G)|-2)^{(m-1)|Z(G)|-1}, (2|Z(G)|-2)^m, (|Z(G)|-2)^{m(|Z(G)|-1)}\}.$$

Proof. It was shown in [12, Theorem 2.5] that $\Gamma_G = K_{(m-1)|Z(G)|} \sqcup mK_{|Z(G)|}$. Hence the result follows from Theorem 2.1. □

Using Theorem 2.5, we now compute the Laplacian and signless Laplacian spectrum of the commuting graphs of the groups M_{2mn}, D_{2m} and Q_{4n} respectively.

Corollary 2.6. *Let $M_{2mn} = \langle a, b : a^m = b^{2n} = 1, bab^{-1} = a^{-1}\rangle$ be a metacyclic group, where $m > 2$. Then*

$$\text{L-spec}(\Gamma_{M_{2mn}}) = \begin{cases} \{0^{m+1}, (mn-n)^{mn-n-1}, n^{mn-m}\} & \text{if } m \text{ is odd} \\ \{0^{\frac{m}{2}+1}, (mn-2n)^{mn-2n-1}, (2n)^{mn-\frac{m}{2}}\} & \text{if } m \text{ is even} \end{cases}$$

and

$$\text{Q-spec}(\Gamma_{M_{2mn}}) = \begin{cases} \{(2mn-2n-2)^1, (mn-n-2)^{mn-n-1}, (2n-2)^m, (n-2)^{mn-m}\} & \text{if } m \text{ is odd} \\ \{(2mn-4n-2)^1, (mn-2n-2)^{mn-2n-1}, (4n-2)^{\frac{m}{2}}, (2n-2)^{mn-\frac{m}{2}}\} & \text{if } m \text{ is even.} \end{cases}$$

Proof. Observe that $Z(M_{2mn}) = \langle b^2\rangle$ or $\langle b^2\rangle \cup a^{\frac{m}{2}}\langle b^2\rangle$ according as m is odd or even. Also, it is easy to see that $\frac{M_{2mn}}{Z(M_{2mn})} \cong D_{2m}$ or D_m according as m is odd or even. Hence, the result follows from Theorem 2.5. □

As a corollary to the above result we have the following results.

Corollary 2.7. *Let $D_{2m} = \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1}\rangle$ be the dihedral group of order $2m$, where $m > 2$. Then*

$$\text{L-spec}(\Gamma_{D_{2m}}) = \begin{cases} \{0^{m+1}, (m-1)^{m-2}\} & \text{if } m \text{ is odd} \\ \{0^{\frac{m}{2}+1}, (m-2)^{m-3}, 2^{\frac{m}{2}}\} & \text{if } m \text{ is even} \end{cases}$$

and

$$\text{Q-spec}(\Gamma_{D_{2m}}) = \begin{cases} \{(2m-4)^1, (m-3)^{m-2}, (2n-2)^m, 0^m\} & \text{if } m \text{ is odd} \\ \{(2m-6)^1, (m-4)^{m-3}, 2^{\frac{m}{2}}, 0^{\frac{m}{2}}\} & \text{if } m \text{ is even.} \end{cases}$$

Corollary 2.8. *Let $Q_{4n} = \langle x, y : y^{2n} = 1, x^2 = y^n, xyx^{-1} = y^{-1}\rangle$, where $n \geq 2$, be the generalized quaternion group of order $4n$. Then*

$$\text{L-spec}(\Gamma_{Q_{4n}}) = \{0^{n+1}, (2n-2)^{2n-3}, 2^n\} \text{ and}$$
$$\text{Q-spec}(\Gamma_{Q_{4n}}) = \{(4n-6)^1, (2n-4)^{2n-3}, 2^n, 0^n\}.$$

Proof. The result follows from Theorem 2.5 noting that $Z(Q_{4n}) = \{1, a^n\}$ and $\frac{Q_{4n}}{Z(Q_{4n})} \cong D_{2n}$. □

Now we compute the Laplacian spectrum and signless Laplacian spectrum of the commuting graphs of some well-known families of finite non-abelian groups.

Proposition 2.9. *Let G be a non-abelian group of order pq, where p and q are primes with $p \mid (q-1)$. Then*

$$\text{L-spec}(\Gamma_G) = \{0^{q+1}, (q-1)^{q-2}, (p-1)^{pq-2q}\} \quad \textit{and}$$
$$\text{Q-spec}(\Gamma_G) = \{(2q-4)^1, (q-3)^{q-2}, (2p-4)^q, (p-3)^{pq-2q}\}.$$

Proof. It was shown in [11, Lemma 3.2] that $\Gamma_G = K_{q-1} \sqcup qK_{p-1}$. Hence, the result follows from Theorem 2.1. □

Proposition 2.10. *Let QD_{2^n} denote the quasidihedral group $\langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1}\rangle$, where $n \geq 4$. Then*

$$\text{L-spec}(\Gamma_{QD_{2^n}}) = \{0^{2^{n-2}+1}, (2^{n-1}-2)^{2^{n-1}-3}, 2^{2^{n-2}}\} \quad \textit{and}$$
$$\text{Q-spec}(\Gamma_{QD_{2^n}}) = \{(2^n-6)^1, (2^{n-1}-4)^{2^{n-1}-3}, 2^{2^{n-2}}, 0^{2^{n-2}}\}.$$

Proof. It was shown in [11, Proposition 2.1] that $\Gamma_{QD_{2^n}} = K_{2^{n-1}-2} \sqcup 2^{n-2}K_2$. Hence, the result follows from Theorem 2.1. □

Proposition 2.11. *The Laplacian spectrum and signless Laplacian spectrum of the commuting graph of the projective special linear group $PSL(2, 2^k)$, where $k \geq 2$, are given by*

$$\{0^{2^{2k}+2^k+1}, (2^k-1)^{2^{2k}-2^k-2}, (2^k-2)^{2^{k-1}(2^{2k}-2^{k+1}-3)}, (2^k)^{2^{k-1}(2^{2k}-2^{k+1}+1)}\} \ \textit{and}$$

$$\{(2^{k+1}-4)^{2^k+1}, (2^k-3)^{2^{2k}-2^k-2}, (2^{k+1}-6)^{2^{k-1}(2^k+1)},$$
$$(2^k-4)^{2^{k-1}(2^{2k}-2^{k+1}-3)}, (2^{k+1}-2)^{2^{k-1}(2^k-1)}, (2^k-2)^{2^{k-1}(2^{2k}-2^{k+1}+1)}\}$$

respectively.

Proof. It was shown in [11, Proposition 2.2] that

$$\Gamma_{PSL(2,2^k)} = (2^k+1)K_{2^k-1} \sqcup 2^{k-1}(2^k+1)K_{2^k-2} \sqcup 2^{k-1}(2^k-1)K_{2^k}.$$

Hence, the result follows from Theorem 2.1. □

Proposition 2.12. *The Laplacian spectrum and signless Laplacian spectrum of the commuting graph of the general linear group $GL(2, q)$, where $q = p^n > 2$ and p is a prime integer, are given by*

$$\{0^{q^2+q+1}, (q^2-3q+2)^{\frac{q(q+1)(q^2-3q+1)}{2}}, (q^2-q)^{\frac{q(q-1)(q^2-q-1)}{2}}, (q^2-2q+1)^{q(q+1)(q-2)}\} \ \textit{and}$$

$$\{(2q^2-6q-2)^{\frac{q(q+1)}{2}}, (q^2-3q)^{\frac{q(q+1)(q^2-3q+1)}{2}}, (2q^2-2q-2)^{\frac{q(q-1)}{2}},$$
$$(q^2-q-2)^{\frac{q(q-1)(q^2-q-1)}{2}}, (2q^2-4q)^{q+1}, (q^2+2q-1)^{q(q+1)(q-2)}\}$$

respectively.

Proof. It was shown in [11, Proposition 2.3] that

$$\Gamma_{GL(2,q)} = \frac{q(q+1)}{2} K_{q^2-3q+2} \sqcup \frac{q(q-1)}{2} K_{q^2-q} \sqcup (q+1) K_{q^2-2q+1}.$$

Hence, the result follows from Theorem 2.1. □

Proposition 2.13. *Let $F = GF(2^n), n \geq 2$ and ϑ be the Frobenius automorphism of F, i. e., $\vartheta(x) = x^2$ for all $x \in F$. Let $A(n, \vartheta)$ denote the group*

$$\left\{ U(a,b) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \vartheta(a) & 1 \end{bmatrix} : a, b \in F \right\}$$

under matrix multiplication given by $U(a,b)U(a',b') = U(a+a', b+b'+a'\vartheta(a))$. Then

$$\text{L-spec}(\Gamma_{A(n,\vartheta)}) = \{0^{2^n-1}, (2^n)^{2^{2n}-2^{n+1}+1}\} \text{ and}$$
$$\text{Q-spec}(\Gamma_{A(n,\vartheta)}) = \{(2^{n+1}-2)^{2^n-1}, (2^n-2)^{2^{2n}-2^{n+1}+1}\}.$$

Proof. It was shown in [11, Proposition 2.4] that $\Gamma_{A(n,\vartheta)} = (2^n - 1)K_{2^n}$. Hence the result follows from Theorem 2.1. □

Proposition 2.14. *Let $F = GF(p^n)$ where p is a prime. Let $A(n,p)$ denote the group*

$$\left\{ V(a,b,c) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} : a, b, c \in F \right\}$$

under matrix multiplication $V(a,b,c)V(a',b',c') = V(a+a', b+b'+ca', c+c')$. Then

$$\text{L-spec}(\Gamma_{A(n,p)}) = \{0^{p^n+1}, (p^{2n}-p^n)^{p^{3n}-2p^n-1}\} \text{ and}$$
$$\text{Q-spec}(\Gamma_{A(n,p)}) = \{(2p^{2n}-2p^n-2)^{p^n+1}, (p^{2n}-p^n-2)^{p^{3n}-2p^n-1}\}.$$

Proof. It was shown in [11, Proposition 2.5] that $\Gamma_{A(n,p)} = (p^n+1)K_{p^{2n}-p^n}$. Hence the result follows from Theorem 2.1. □

Our computations reveal that all the groups considered above are both L-integral and Q-integral. Also, it was shown in [11, 12] that theses groups are integral. Hence, all these groups are super integral.

A group G is called an AC-group if $C_G(x)$ is abelian for all $x \in G \setminus Z(G)$, where $C_G(x)$ is the centralizer of x given by the set $\{y \in G : xy = yx\}$. Various aspects of AC-groups can be found in [1, 10, 26]. In [11], the authors have shown that finite AC-groups are integral. In the following two results we shall show that if G is an AC-group or G is isomorphic to $H \times A$, where H is a finite non-abelian AC-group and A is any finite abelian group then G is L-integral as well as Q-integral. Hence, G is super integral.

Theorem 2.15. *Let G be a finite non-abelian AC-group. Then*

$$\text{L-spec}(\Gamma_G) = \{0^n, (|X_1| - |Z(G)|)^{|X_1|-|Z(G)|-1}, \dots, (|X_n| - |Z(G)|)^{|X_n|-|Z(G)|-1}\}$$

and

$$\text{Q-spec}(\Gamma_G) = \{(2(|X_1| - |Z(G)|) - 2)^1, (|X_1| - |Z(G)| - 2)^{|X_1|-|Z(G)|-1}, \dots, (2(|X_n| - |Z(G)|) - 2)^1, (|X_n| - |Z(G)| - 2)^{|X_n|-|Z(G)|-1}\},$$

where $X_1, \dots, X_n$ *are the distinct centralizers of non-central elements of* G*.*

Proof. By [11, Lemma 2.1], we have $\Gamma_G = \bigsqcup_{i=1}^{n} K_{|X_i|-|Z(G)|}$. Therefore, the result follows from Theorem 2.1. □

Corollary 2.16. *Let* $G \cong H \times A$ *where* H *is a finite non-abelian AC-group and* A *is any finite abelian group. Then*

$$\text{L-spec}(\Gamma_G) = \{0^n, (|A|(|X_1| - |Z(H)|))^{|A|(|X_1|-|Z(H)|)-1}, \dots, (|A|(|X_n| - |Z(H)|))^{|A|(|X_n|-|Z(H)|)-1}\}$$

and Q-spec(Γ_G) *is given by*

$$\{(2|A|(|X_1| - |Z(H)|) - 2)^1, (|A|(|X_1| - |Z(H)|) - 2)^{|A|(|X_1|-|Z(H)|)-1}, \dots, (2|A|(|X_n| - |Z(H)|) - 2)^1, (|A|(|X_n| - |Z(H)|) - 2)^{|A|(|X_n|-|Z(H)|)-1}\},$$

where $X_1, \dots, X_n$ *are the distinct centralizers of non-central elements of* H*.*

Proof. It is easy to see that $Z(H \times A) = Z(H) \times A$ and $X_1 \times A, X_2 \times A, \dots, X_n \times A$ are the distinct centralizers of non-central elements of $H \times A$. Therefore, if H is an AC-group then $H \times A$ is also an AC-group. Hence, the result follows from Theorem 2.15. □

3. Commutativity degree and super integral group

Let G be a finite group. The commutativity degree of G is given by the ratio

$$\Pr(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G|^2}.$$

The origin of the commutativity degree of a finite group lies in a paper of Erdös and Turán (see [13]). Readers may conf. [7, 9, 22] for various results on $\Pr(G)$. In this section, we ask the following question.

Question 5. Can we determine all the positive rational numbers r such that any group G with $\Pr(G) = r$ is super integral?

The following theorems give some rational numbers r such that G is super integral if $\Pr(G) = r$.

Theorem 3.1. *If* $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{5}{8}\}$ *then* G *is super integral.*

Proof. If $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{5}{8}\}$ then as shown in [27, pp. 246] and [23, pp. 451], we have $\frac{G}{Z(G)}$ is isomorphic to one of the groups in $\{D_{14}, D_{10}, D_8, D_6, \mathbb{Z}_2 \times \mathbb{Z}_2\}$. If $\frac{G}{Z(G)}$ is isomorphic to D_{14}, D_{10}, D_8 or D_6 then by [12, Theorem 2.5], we have G is integral and by Theorem 2.5, we have G is both L-integral and Q-integral. Hence, in this case G is super integral. If $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ then by [11, Theorem 3.1], it follows that G is integral. Also, by Theorem 2.3 we have G

is both L-integral and Q-integral. Hence, G is super integral. This completes the proof. □

Theorem 3.2. *Let G be a finite group and p the smallest prime divisor of $|G|$. If* $\Pr(G) = \frac{p^2+p-1}{p^3}$ *then G is super integral.*

Proof. If $\Pr(G) = \frac{p^2+p-1}{p^3}$ then by [19, Theorem 3] we have $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Now, by [12, Theorem 2.1], we have that G is integral. Again, by Theorem 2.3, it follows that G is L-integral as well as Q-integral. Hence, G is super integral. □

We conclude this section by the following result.

Theorem 3.3. *If G is a finite non-solvable group with* $\Pr(G) = \frac{1}{12}$ *then G is super integral.*

Proof. By [7, Proposition 3.3.7] we have that G is isomorphic to $A_5 \times B$ for some abelian group B. Therefore G is an AC-group and hence super integral. □

4. More Applications

In this section, we show that a finite non-abelian group G not isomorphic to S_4 is super integral if its commuting graph is planar. We also show that G is super integral if its commuting graph is toroidal. We begin with the following useful lemma.

Lemma 4.1. *Let G be a group isomorphic to any of the following groups*

(1) $\mathbb{Z}_2 \times D_8$
(2) $\mathbb{Z}_2 \times Q_8$
(3) $M_{16} = \langle a, b : a^8 = b^2 = 1, bab = a^5 \rangle$
(4) $\mathbb{Z}_4 \rtimes \mathbb{Z}_4 = \langle a, b : a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle$
(5) $D_8 * \mathbb{Z}_4 = \langle a, b, c : a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = a^2cb \rangle$
(6) $SG(16, 3) = \langle a, b : a^4 = b^4 = 1, ab = b^{-1}a^{-1}, ab^{-1} = ba^{-1} \rangle$.

Then $\text{L-spec}(\Gamma_G) = \{0^3, 4^9\}$ *and* $\text{Q-spec}(\Gamma_G) = \{6^3, 2^9\}$.

Proof. If G is isomorphic to any of the above listed groups, then $|G| = 16$ and $|Z(G)| = 4$. Therefore, $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus the result follows from Theorem 2.3. □

Now we state and proof the main results of this section.

Theorem 4.2. *Let Γ_G be the commuting graph of a finite non-abelian group G. If G is not isomorphic to S_4 and Γ_G is planar then G is super integral.*

Proof. It was shown in [11, Theorem 3.2] that G is an integral group if it is not isomorphic to S_4 and Γ_G is planar.

By [3, Theorem 2.2], we have that Γ_G is planar if and only if G is isomorphic to either D_6, D_8, D_{10}, $D_{12}, Q_8, Q_{12}, \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, M_{16}, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, D_8 * \mathbb{Z}_4, SG(16, 3), A_4$, $A_5, S_4, SL(2, 3)$ or $Sz(2)$.

If $G \cong D_6, D_8, D_{10}$ or D_{12} then by Corollary 2.7, one may conclude that Γ_G is both L-integral and Q-integral. Hence, G is both L-integral and Q-integral. If

$G \cong Q_8$ or Q_{12} then G is both L-integral and Q-integral, by Corollary 2.8. If $G \cong \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, M_{16}, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, D_8 * \mathbb{Z}_4$ or $SG(16,3)$ then, by Lemma 4.1, G is L-integral as well as Q-integral.

If $G \cong A_4 = \langle a, b : a^2 = b^3 = (ab)^3 = 1\rangle$ then the distinct centralizers of non-central elements of G are $C_G(a) = \{1, a, bab^2, b^2ab\}, C_G(b) = \{1, b, b^2\}$, $C_G(ab) = \{1, ab, b^2a\}, C_G(ba) = \{1, ba, ab^2\}$ and $C_G(aba) = \{1, aba, bab\}$. Note that these centralizers are abelian subgroups of G. Therefore, $\Gamma_G = K_3 \sqcup 4K_2$. Using Theorem 2.1, we have L-spec$(\Gamma_G) = \{0^5, 3^2, 2^4\}$ and Q-spec$(\Gamma_G) = \{4^1, 1^2, 2^4, 0^4\}$. Hence, G is both L-integral and Q-integral.

If $G \cong Sz(2)$ then by Theorem 2.2, we have L-spec$(\Gamma_G) = \{0^6, 4^3, 3^{10}\}$ and Q-spec$(\Gamma_G) = \{6^1, 2^3, 4^5, 1^{10}\}$. Hence, G is both L-integral and Q-integral.

If G is isomorphic to $SL(2,3)$ then it was shown in the proof of [11, Theorem 3.2] that $\Gamma_G = 3K_2 \sqcup 4K_4$. Therefore, by Theorem 2.1, we have L-spec$(\Gamma_G) = \{0^7, 2^3, 4^{12}\}$ and Q-spec$(\Gamma_G) = \{0^3, 2^{15}, 6^4\}$. Hence, G is both L-integral and Q-integral.

If $G \cong A_5$ then by Proposition 2.11, we have

$$\text{L-spec}(\Gamma_G) = \{0^{21}, 3^{10}, 2^{10}, 4^{18}\} \text{ and } \text{Q-spec}(\Gamma_G) = \{4^5, 1^{10}, 2^{10}, 0^{10}, 6^6, 2^{18}\},$$

noting that $PSL(2,4) \cong A_5$. Hence, G is both L-integral and Q-integral.

Finally, if $G \cong S_4$ then it can be seen that the characteristic polynomial of $L(\Gamma_G)$ is $x^5(x-1)^3(x-2)^4(x-3)^6(x-5)(x^2-8x+3)^2$ and so

$$\text{L-spec}(\Gamma_G) = \left\{0^5, 1^3, 2^4, 3^6, 5^1(4+\sqrt{13})^2, (4-\sqrt{13})^2\right\}.$$

Also, the characteristic polynomial of $Q(\Gamma_G)$ is $x^4(x-1)^6(x-2)^4(x-3)^3(x^2-11x+20)(x^2-8x+11)^2$ and so Q-spec(Γ_G) is given by

$$\left\{0^4, 1^6, 2^4, 3^3, 5^1(4+\sqrt{5})^2, (4-\sqrt{5})^2, \left(\frac{11+\sqrt{41}}{2}\right)^1, \left(\frac{11-\sqrt{41}}{2}\right)^1\right\}.$$

This shows that if $G \cong S_4$ then it is neither L-integral nor Q-integral. This completes the proof. □

Theorem 4.3. *Let Γ_G be the commuting graph of a finite non-abelian group G. Then G is super integral if Γ_G is toroidal.*

Proof. By [10, Theorem 6.6], we have Γ_G is toroidal if and only if G is isomorphic to either $D_{14}, D_{16}, Q_{16}, QD_{16}, D_6 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$.

If $G \cong D_{14}$ or D_{16} then, by Corollary 2.7, one may conclude that G is both L-integral and Q-integral. If $G \cong Q_{16}$ then, by Corollary 2.8, G becomes both L-integral and Q-integral. If $G \cong QD_{16}$ then, by Proposition 2.10, G becomes both L-integral and Q-integral. If $G \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ then G is both L-integral and Q-integral, by Proposition 2.9. If G is isomorphic to $D_6 \times \mathbb{Z}_3$ or $A_4 \times \mathbb{Z}_2$ then G becomes both L-integral and Q-integral by Corollary 2.16, since D_6 and A_4 are AC-groups. Further, it was shown in [11, Theorem 3.4] that G is integral if Γ_G is toroidal. Hence, G is super integral if Γ_G is toroidal. □

We conclude the paper with the following result.

Proposition 4.4. *Let Γ_G be the commuting graph of a finite non-abelian group G. Then Γ_G is super integral if the complement of Γ_G is planar.*

Proof. If the complement of Γ_G is planar then, by [1, Proposition 2.3], we have G is isomorphic to either D_6, D_8 or Q_8. If $G \cong D_6$ or D_8 then, by Corollary 2.7 and [11, Proposition 3.1], we have that G is super integral. If $G \cong Q_8$ then, by Corollary 2.8 and [11, Proposition 3.2], it follows that G is super integral. This completes the proof. □

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