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POLYNOMIAL BOUNDS FOR A CLASS OF UNIVALENT FUNCTION INVOLVING SIGMOID FUNCTION

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Abstract. In this work, a new subclass of univalent function was defined using the Sălăgean differential operator involving the modified sigmoid function and the Chebyshev polynomials. The coefficient bounds and the Fekete-Szego functional of this class were obtained using subordination principle. The results obtained agree and extend some earlier results.

1. Introduction and preliminaries

Special functions play an important role in geometric function theory. Example of special function is activation function. The most popular activation function in the hardware implementation of Artificial Neural Network (ANN) is the sigmoid function.The sigmoid function increases the size of the hypothesis space that the network can represent. Neural networks can be used for complex learning tasks. It is therefore necessary to investigate the role of sigmoid function in geometric function theory.
The sigmoid function

$$h(s) = \frac{1}{1+e^{-s}} \qquad s \geq 0,$$

is a bounded differentiable function and has the following properties:

(1) It outputs real numbers between 0 and 1.
(2) It maps a very large output domain to a small range of inputs.
(3) It never looses information because it is a one-to-one function.

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(4) It increases monotonically.

Let **A** be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

that are analytic in the unit disk $\mathbf{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$

Salagean[10] introduced the differential operator $D^n f, n \in \mathbb{N}_0 = 0, 1, 2, \dots$ for functions $f(z)$ belonging to class **A** of analytic functions in the unit disk **D** as follows:

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k; \quad n \in \mathbb{N}_0.$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbf{D} = \{|z| < 1\}$. We say that $f(z)$ is subordinate to $g(z)$ if there exists a function $\omega(z)$ analytic (not necessarily univalent) in **D** satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$f(z) = g(\omega(z)) \quad (|z| < 1)$$

(see [7]).

Fadipe-Joseph et al studied the modified sigmoid function

$$G(z) = \frac{2}{1 + e^{-z}}$$

and obtained another series of the modified sigmoid function as

$$G(z) = 1 + \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\frac{(-1)^n}{n!} z^n \right)^m \right)$$

$$= 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 \dots$$

(see [6]).

The Chebyshev polynomials are a sequence of orthogonal polynomials which are related to De'Moivre's formula and which are defined recursively. The role of Chebyshev polynomials in numerical analysis is increased in both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. The majority of books and research papers dealing with specific orthogonal polynomials of Chebyshev family, contain mainly results of Chebyshev polynomials of first and second kinds $T_n(t)$ and $U_n(t)$ and their numerous uses in different applications, for example, see [1],[2].

The Chebyshev polynomials of the first and second kind are defined as:

$$T_n(t) = \cos n\alpha \qquad t \in (-1, 1),$$

$$U_n(t) = \frac{\sin(n+1)\alpha}{\sin \alpha} \qquad t \in (-1, 1),$$

where n denotes the degree of the polynomial and $t = \cos \alpha$.

The Chebyshev polynomials of the first kind $T_n(t), t \in [-1, 1]$ have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1-tz}{1-2tz+z^2} \quad (z \in \mathbf{D})$$

and that of second kind is:

$$H(z,t) = \frac{1}{1-2tz+z^2} = 1 + \sum_{n=1}^{\infty} \frac{sin(n+1)\alpha}{sin\alpha} z^n \quad (z \in \mathbf{D}) \; for \; |t| < 1.$$

Note that if $t = cos\alpha$, $\alpha \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$, then

$$H(z,t) = \frac{1}{1-2cos\alpha z + z^2} = 1 + \sum_{n=1}^{\infty} \frac{sin(n+1)\alpha}{sin\alpha} z^n.$$

Thus

$$H(z,t) = 1 + 2cos\alpha z + (3cos^2\alpha - sin^2\alpha)z^2 + \dots.$$

Following [11], we write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \dots \qquad (z \in \mathbf{D}, t \in (-1,1)),$$

where

$$U_{n-1} = \frac{sin(n\; arcost)}{\sqrt{1-t^2}} \qquad (n \in \mathbb{N})$$

are the Chebyshev polynomials of the second kind. It is also known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

so that

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad \dots.$$

Lemma 1.1. [3] *If* $\omega(z) = b_1 z + b_2 z^2 + \cdots, b_1 \neq 0$ *is analytic and satisfy* $|\omega(z)| < 1$ *in the unit disk* $\mathbf{D}$*, then for each* $0 < r < 1, |\omega'(z)| < 1$ *and* $\left|\omega(re^{i\theta})\right| < 1$ *unless* $\omega(z) = e^{i\sigma} z$ *for some real number* σ*.*

Lemma 1.2. [4] *Let* $\omega \in \Omega = \{\omega \in A : |w(z)| \leq |z| \; z \in \mathbf{D}\}$.

If $\omega \in \Omega, \omega(z) = \sum_{n=1}^{\infty} c_n z^n \quad (z \in \boldsymbol{D})$*, then*

$$|c_n| \leq 1 \quad n = 1, 2, \cdots, \quad |c_2| \leq 1 - |c_1|^2 \tag{1.1}$$

and

$$\left|c_2 - \mu c_1^2\right| \leq max\{1, |\mu|\} \, (\mu \in \mathbb{C}). \tag{1.2}$$

The result is sharp. The functions

$$\omega(z) = z, \quad \omega_a(z) = z\frac{z+a}{1+\overline{a}z} \quad (z \in \mathbf{D}, |a| < 1)$$

are extremal functions.

Sălăgean Differential Operator Involving Modified Sigmoid Function
Consider the function

$$f_\gamma(z) = z + \sum_{k=2}^{\infty} \gamma(s) a_k z^k, \tag{1.3}$$

where

$$\gamma(s) = \frac{2}{1+e^{-s}} \quad s \geq 0.$$

Functions of the form (1.3) belong to the class $\mathbf{A}_\gamma$, where $\mathbf{A_1} \equiv \mathbf{A}$.
Let $D^n f_\gamma(z); n \in \mathbb{N}_0$ denote the Sălăgean differential operator involving modified sigmoid function, then

$$D^0 f_\gamma(z) = f_\gamma(z)$$
$$D^1 f_\gamma(z) = \gamma(s) z f'_\gamma(z)$$
$$\vdots$$
$$D^n f_\gamma(z) = D(D^{n-1} f\gamma(z)) = \gamma(s) z (D^{n-1} f_\gamma(z))' \tag{1.4}$$

when $\gamma(s) = 1$, we have the Sălăgean differential operator in [10]. For the details, see [5].
Ramachandran and Dhanalakshmi [8] obtained the coefficient estimates for a class of spirallike functions in the space of sigmoid function. Furthermore, the Fekete-Szegö functional for a subclass of analytic functions related to sigmoid function was obtained in [9].

2. Main results

Motivated by the work of Altinkaya and Yalcin [1], Bulut and Magesh [2] we have the following results:

Definition 2.1. A function $f_\gamma(z) \in \mathbf{A}_\gamma$ is said to be in the class $H_\gamma(n, \mu, \lambda)$, $0 \leq \lambda \leq 1, \quad \mu \geq 0, \quad \gamma(s) = \frac{2}{1+e^{-s}} \quad s \geq 0, \quad n \in \mathbb{N}_0$ if the following subordination holds

$$(1-\lambda)\left(\frac{D^n f_\gamma(z)}{z}\right)^{\mu} + \lambda f'_\gamma(z)\left(\frac{D^n f_\gamma(z)}{z}\right)^{\mu-1} \prec H(z,t), \tag{2.1}$$

where, $D^n f_\gamma(z)$ is the Salagean differential operator involving modified sigmoid function defined as follows:

$$D^0 f_\gamma(z) = f_\gamma(z) = z + \sum_{k=2}^{\infty} \gamma(s) a_k z^k,$$

$$D^1 f_\gamma(z) = D f_\gamma(z) = \gamma(s) z + \sum_{k=2}^{\infty} k \gamma^2(s) a_k z^k,$$

$$\vdots$$

$$D^n f_\gamma(z) = D(D^{n-1} f_\gamma(z)) = \gamma^n(s) z + \sum_{k=2}^{\infty} k^n \gamma^{n+1}(s) a_k z^k. \tag{2.2}$$

Theorem 2.2. *If $f(z)$ belongs to the class $H_\gamma(n,\mu,\lambda) : n \in \mathbb{N}_0$, $\mu \geq 1, \quad 0 \leq \lambda \leq 1$, then*

$$|a_2| \leq \frac{2t}{(A+B)},$$

$$|a_3| \leq \frac{2t+(4t^2-1)}{(A'+B')} + \frac{4t^2(D+E)}{(A+B)^2(A'+B')},$$

$$|a_4| \leq \begin{array}{c} \frac{8t^3+8t^2-2t-2}{(D'+E')} + \frac{(F+G)(4t^2+2t(4t^2-1))}{(A+B)(A'+B')(D'+E')} \\ +\frac{8t^3(D+E)(F+G)}{(A+B)^3(A'+B')(D'+E')} + \frac{(F'+G')8t}{(A+B)^3(D'+E')} \end{array},$$

$$\begin{array}{c} |a_5| \leq \frac{16t^4+24t^3-10t-4}{(H+I)} + \frac{(H'+I')(16t^4+16t^3-4t^2-4t)}{(A+B)(D'+E')(H+I)} \\ +\frac{(F+G)(H'+I')(16t^4+8t^3-4t^2)}{(A+B)^2(A'+B')(D'+E')(H+I)} + \frac{16t^4(D+E)(F+G)(H'+I')}{(A+B)^4(A'+B')(D'+E')(H+I)} \\ +\frac{16t^4(F'+G')(H'+I')}{(A+B)^4(D'+E')(H+I)} + \frac{(16t^4-4t^2+1)(J+K)}{(A'+B')^2(H+I)} \\ +\frac{(D+E)(J+K)(32t^4+16t^3-8t^2)}{(A+B)^2(A'+B')^2(H+I)} + \frac{(D+E)^2(J+K)2t^4}{(A+B)^4(A'+B')^2(H+I)} + \frac{(J'+K')(2t^3-4t)}{(A+B)^2(A'+B')(H+I)} \\ +\frac{16t^4(D+E)(J'+K')}{(A+B)^4(A'+B')(H+I)} + \frac{(L+M)16t^4}{(A+B)^4(H+I)} \end{array},$$

where

$$A = (1-\lambda)2^n\mu\gamma^{n\mu+1}(s),$$
$$B = \lambda\gamma^{n(\mu-1)+1}(s)(2^n(\mu-1)+2),$$
$$A' = (1-\lambda)3^n\mu\gamma^{n\mu+1}(s),$$
$$B' = \lambda\gamma^{n(\mu-1)+1}(s)(3^n(\mu-1)+3),$$
$$D = -(1-\lambda)2^{2n}\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s),$$
$$E = -\lambda\gamma^{n(\mu-1)+2}(s)\left(\frac{(\mu-1)(\mu-2)}{2!}2^{2n} + 2^{n+1}(\mu-1)\right),$$
$$D' = -(1-\lambda)4^n\mu\gamma^{n\mu+1}(s),$$
$$E' = -\lambda\gamma^{n(\mu-1)+1}(s)(4^n(\mu-1)+4),$$
$$F = -(1-\lambda)2^{n+1}\cdot 3^n\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s),$$
$$G = -\lambda\gamma^{n(\mu-1)+2}(s)\left(\frac{(\mu-1)(\mu-2)}{2!}2^{n+1}\cdot 3^n + (\mu-1)(2\cdot 3^n + 3\cdot 2^n)\right),$$
$$F' = -(1-\lambda)2^{3n}\frac{\mu(\mu-1)(\mu-2)}{3!}\gamma^{n\mu+3}(s),$$
$$G' = -\lambda\gamma^{n(\mu-1)+3}(s)\left(\frac{(\mu-1)(\mu-2)(\mu-3)}{3!}2^{3n} + \frac{(\mu-1)(\mu-2)}{2!}2^{2n+1}\right),$$
$$H = (1-\lambda)5^n\mu\gamma^{n\mu+1}(s),$$

$$I=\lambda\gamma^{n(\mu-1)+1}(s)(5^n(\mu-1)+5),$$

$$H'=-(1-\lambda)2^{3n+1}\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s),$$

$$I'=-\lambda\gamma^{n(\mu-1)+2}(s)\left(\frac{(\mu-1)(\mu-2)}{2!}2^{3n+1}+(\mu-1)(4^n\cdot 2+4\cdot 2^n)\right),$$

$$J=-(1-\lambda)3^{2n}\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s),$$

$$K=-\lambda\gamma^{n(\mu-1)+2}(s)\left(\frac{(\mu-1)(\mu-2)}{2!}3^{2n}+3^{n+1}(\mu-1)\right),$$

$$J'=-(1-\lambda)2^{2n}\cdot 3^{n+1}\frac{\mu(\mu-1)(\mu-2)}{3!}\gamma^{n\mu+3}(s),$$

$$K'=-\lambda\gamma^{n(\mu-1)+3}(s)\left(\begin{array}{c}\frac{(\mu-1)(\mu-2)(\mu-3)}{3!}(2\cdot 12^n+2^{2n}\cdot 3^n)\\ +\frac{(\mu-1)(\mu-2)}{2!}(4\cdot 6^n+3\cdot 2^{4n})\end{array}\right),$$

$$L=-(1-\lambda)2^{4n}\frac{\mu(\mu-1)(\mu-2)(\mu-3)}{4!}\gamma^{n\mu+4}(s),$$

$$M=-\lambda\gamma^{n\mu+4}(s)\left(\begin{array}{c}\frac{(\mu-1)(\mu-2)(\mu-3)(\mu-4)}{4!}2^{4n}\\ +\frac{(\mu-1)(\mu-2)(\mu-3)}{3!}2^{3n+1}\end{array}\right).$$

Proof. If $f(z)\in H_\gamma(n,\mu,\lambda)$, then from (2.1)

$$(1-\lambda)\left(\frac{D^nf_\gamma(z)}{z}\right)^{\mu}+\lambda f'_\gamma(z)\left(\frac{D^nf_\gamma(z)}{z}\right)^{\mu-1}\prec H(z,t)$$

where $\gamma(s)$ as given in (1.3). From (2.2), we have

$$\frac{D^nf_\gamma(z)}{z}=\gamma^n(s)+\sum_{k=2}^{\infty}k^n\gamma^{n+1}(s)a_kz^{k-1} \tag{2.3}$$

$=\gamma^n(s)+2^n\gamma^{n+1}(s)a_2z+3^n\gamma^{n+2}(s)a_3z^2+4^n\gamma^{n+3}(s)a_4z^3+5^n\gamma^{n+4}(s)a_5z^4+\cdots.$

Hence

$$\left(\frac{D^nf_\gamma(z)}{z}\right)^{\mu}=\begin{array}{c}\gamma^{n\mu}(s)+2^n\mu\gamma^{n\mu+1}(s)a_2z\\ +\left(3^n\mu\gamma^{n\mu+1}(s)a_3+2^{2n}\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s)a_2^2\right)z^2\\ +\left(\begin{array}{c}4^n\mu\gamma^{n\mu+1}(s)a_4+2^n\cdot 3^n2^{2n}\frac{2\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s)a_2a_3\\ +2^{3n}\frac{\mu(\mu-1)(\mu-2)}{3!}\gamma^{n\mu+3}(s)a_2^3\end{array}\right)z^3\\ +\left(\begin{array}{c}5^n\mu\gamma^{n\mu+1}(s)a_5+\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s)(2^{2n+1}a_2a_4+3^{2n}a_3^2)\\ +2^{2n}\cdot 3^{n+1}\frac{\mu(\mu-1)(\mu-2)}{3!}\gamma^{n\mu+3}(s)a_2^2a_3\\ +2^{4n}\frac{\mu(\mu-1)(\mu-2)(\mu-3)}{4!}\gamma^{n\mu+4}(s)a_2^4\end{array}\right)z^4\\ +\cdots.\end{array} \tag{2.4}$$

Also,

$$\left(\frac{D^n f_\gamma(z)}{z}\right)^{\mu-1} = \gamma^{n(\mu-1)}(s) + 2^n(\mu-1)\gamma^{n(\mu-1)+1}(s)a_2 z$$
$$+\left(3^n(\mu-1)\gamma^{n(\mu-1)+1}(s)a_3 + 2^{2n}\frac{(\mu-1)(\mu-2)}{2!}\gamma^{n(\mu-1)+2}(s)a_2^2\right)z^2$$
$$+\begin{pmatrix} 4^n(\mu-1)\gamma^{n(\mu-1)+1}(s)a_4 \\ +2^n\cdot 3^n 2^{2n}\frac{2(\mu-1)(\mu-2)}{2!}\gamma^{n(\mu-1)+2}(s)a_2a_3 \\ +2^{3n}\frac{(\mu-1)(\mu-2)(\mu-3)}{3!}\gamma^{n(\mu-1)+3}(s)a_2^3 \end{pmatrix} z^3$$
$$+\begin{pmatrix} 5^n(\mu-1)\gamma^{n(\mu-1)+1}(s)a_5 \\ +\frac{(\mu-1)(\mu-2)}{2!}\gamma^{n(\mu-1)+2}(s)(2^{2n+1}a_2a_4 + 3^{2n}a_3^2) \\ +2^{2n}\cdot 3^{n+1}\frac{(\mu-1)(\mu-2)(\mu-3)}{3!}\gamma^{n(\mu-1)+3}(s)a_2^2a_3 \\ +2^{4n}\frac{(\mu-1)(\mu-2)(\mu-3)(\mu-4)}{4!}\gamma^{n(\mu-1)+4}(s)a_2^4 \end{pmatrix} z^4$$
$$+\cdots. \tag{2.5}$$

Differentiating (1.3) with respect to z, we have

$$f'_\gamma(z) = 1 + \gamma(s)(2a_2z + 3a_3z^2 + 4a_4z^3 + 5a_5z^4 + \cdots). \tag{2.6}$$

Multiplying (2.5) and (2.6), we have

$$f'_\gamma(z)\left(\frac{D^n f_\gamma(z)}{z}\right)^{\mu-1} = \gamma^{n(\mu-1)}(s) + \gamma^{n(\mu-1)+1}(s)(2^n(\mu-1)+2)a_2z$$
$$+\begin{pmatrix} \gamma^{n(\mu-1)+1}(s)(3^n(\mu-1)+3)a_3 \\ +\gamma^{n(\mu-1)+2}\left(2^{2n}\frac{(\mu-1)(\mu-2)}{2!} + 2^{n+1}(\mu-1)\right)a_2^2 \end{pmatrix} z^2$$
$$+\begin{pmatrix} \gamma^{n(\mu-1)+1}(s)(4^n(\mu-1)+4)a_4 \\ +\gamma^{n(\mu-1)+2}(s)\left(2^{n+1}\cdot 3^n\frac{(\mu-1)(\mu-2)}{2!} + (\mu-1)(2^n\cdot 3 + 2\cdot 3^n)\right)a_2a_3 \\ +\gamma^{n(\mu-1)+3}(s)\left(2^{3n}\frac{(\mu-1)(\mu-2)(\mu-3)}{3!} + 2^{2n+1}\frac{(\mu-1)(\mu-2)}{2!}\right)a_2^3 \end{pmatrix} z^3$$
$$+\begin{pmatrix} \gamma^{n(\mu-1)+1}(s)(5^n(\mu-1)+5)a_5 \\ +\gamma^{n(\mu-1)+2}(s)\begin{pmatrix} \left(\frac{(\mu-1)(\mu-2)}{2!}2^{3n+1} + (\mu-1)(4^n\cdot 2 + 4\cdot 2^n)\right)a_2a_4 \\ +\left(\frac{(\mu-1)(\mu-2)}{2!}3^{2n} + 3^{n+1}(\mu-1)\right)a_3^2 \end{pmatrix} \\ +\gamma^{n(\mu-1)+3}(s)\begin{pmatrix} \frac{(\mu-1)(\mu-2)(\mu-3)}{3!}(2^{2n+1}\cdot 3^n + 2^{2n}\cdot 3^n) \\ +\frac{(\mu-1)(\mu-2)}{2!}(2^{n+2}\cdot 3^n + 3\cdot 2^{2n}) \end{pmatrix} a_2^2a_3 \\ +\gamma^{n(\mu-1)+4}(s)\left(\frac{(\mu-1)(\mu-2)(\mu-3)(\mu-4)}{4!}2^{4n} + \frac{(\mu-1)(\mu-2)(\mu-3)}{3!}2^{3n+1}\right)a_2^4 \end{pmatrix} z^4$$
$$+\cdots$$

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + U_3(t)z^3 + U_4(t)z^4 + \cdots \tag{2.7}$$

$$H(\omega(z,t)) = 1 + U_1(t)\omega(z) + U_2(t)\omega(z)^2 + U_3(t)\omega(z)^3 + U_4(t)\omega(z)^4 + \cdots \tag{2.8}$$

$$\omega(z) = c_1z + c_2z^2 + c_3z^3 + c_4z^4 + c_5z^5 + \cdots, \tag{2.9}$$

$$\omega^2(z) = c_1^2z^2 + 2c_1c_2z^3 + (2c_1c_3 + c_2^2)z^4 + (c_1c_4 + 2c_2c_3)z^5 + \cdots, \tag{2.10}$$

$$\omega^3(z) = c_1^3z^3 + 3c_1^2c_2z^4 + (3c_1^2c_3 + 3c_1c_2^2)z^5 + \cdots, \tag{2.11}$$

$$\omega^4(z) = c_1^4z^4 + 4c_1^3c_2z^5 + \cdots, \omega^5(z) = c_1^5z^5 + \cdots. \tag{2.12}$$

Substituting (2.9),(2.10), (2.11) and (2.12) into (2.8), we have

$$H(\omega(z,t)) = 1 + U_1(t)c_1z + (c_2U_1(t) + c_1^2U_2(t))z^2 + (c_3U_1(t) + 2c_1c_2U_2(t) \\ c_1^3U_3(t))z^3 + (c_4U_1(t) + (2c_1c_3 + c_2^2)U_2(t) + (3c_1^2c_2)U_3(t) + c_1^4U_4(t))z^4 + \cdots . \tag{2.13}$$

Thus,

$$\left(\frac{D^nf_\gamma(z)}{z}\right)^\mu + \lambda\left(f_\gamma'(z)\left(\frac{D^nf_\gamma(z)}{z}\right)^{\mu-1} - \left(\frac{D^nf_\gamma(z)}{z}\right)^\mu\right) = (1+\lambda)\gamma^{n\mu}(s) + \lambda\gamma^{n(\mu-1)}(s)$$

$$+((1-\lambda)2^n\mu\gamma^{n\mu+1}(s) + \lambda\gamma^{n(\mu-1)+1}(s)(2^n(\mu-1)+2))a_2z$$
$$+\left(\begin{array}{c} ((1-\lambda)3^n\mu\gamma^{n\mu+1}(s) + \lambda\gamma^{n(\mu-1)+1}(s)(3^n(\mu-1)+3))a_3 \\ \left(\begin{array}{c} (1-\lambda)2^{2n}\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s) \\ +\lambda\gamma^{n(\mu-1)+2}(s)\left(\frac{(\mu-1)(\mu-2)}{2!}2^{2n} + 2^{n+1}(\mu-1)\right)\end{array}\right)a_2^2 \end{array}\right)z^2$$
$$+\left(\begin{array}{c} ((1-\lambda)4^n\mu\gamma^{n\mu+1}(s) + \lambda\gamma^{n(\mu-1)+1}(s)(4^n(\mu-1)+4))a_4 \\ +\left(\begin{array}{c} (1-\lambda)2^{n+1}\cdot 3^n\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s) \\ +\lambda\gamma^{n(\mu-1)+2}(s)\left(\begin{array}{c}\frac{(\mu-1)(\mu-2)}{2!}2^{n+1}\cdot 3^n \\ +(\mu-1)(2\cdot 3^n + 3\cdot 2^n)\end{array}\right)\end{array}\right)a_2a_3 \\ +\left(\begin{array}{c} (1-\lambda)2^{3n}\frac{\mu(\mu-1)(\mu-2)}{3!}\gamma^{n\mu+3}(s) \\ +\lambda\gamma^{n(\mu-1)+3}(s)\left(\frac{(\mu-1)(\mu-2)(\mu-3)}{3!}2^{3n} + \frac{(\mu-1)(\mu-2)}{2!}2^{2n+1}\right)\end{array}\right)a_2^3 \end{array}\right)z^3$$
$$+\left(\begin{array}{c} ((1-\lambda)5^n\mu\gamma^{n\mu+1}(s) + \lambda\gamma^{n(\mu-1)+1}(s)(5^n(\mu-1)+5))a_5 \\ +\left(\begin{array}{c} (1-\lambda)2^{3n+1}\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s) + \lambda\gamma^{n(\mu-1)+2}(s) \\ \left(\frac{(\mu-1)(\mu-1)}{2!}2^{3n+1} + (\mu-1)(4^n\cdot 2 + 4\cdot 2^n)\right)\end{array}\right)a_2a_4 \\ +\left(\begin{array}{c} (1-\lambda)3^{2n}\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s) \\ +\lambda\gamma^{n(\mu-1)+2}(s)\left(\frac{(\mu-1)(\mu-1)}{2!}3^{2n} + 3^{n+1}(\mu-1)\right)\end{array}\right)a_3^2 \\ +\left(\begin{array}{c} (1-\lambda)2^{2n}\cdot 3^{n+1}\frac{\mu(\mu-1)(\mu-2)}{3!}\gamma^{n\mu+3}(s) + \gamma^{n(\mu-1)+3}(s) \\ \left(\begin{array}{c}\frac{(\mu-1)(\mu-2)(\mu-3)}{3!}(2^{2n+1}\cdot 3^n + 2^{2n}\cdot 3^n) \\ +\frac{(\mu-1)(\mu-2)}{2!}(2^{n+2}\cdot 3^n + 3\cdot 2^{2n})\end{array}\right)\end{array}\right)a_2^2a_3 \\ +\left(\begin{array}{c} (1-\lambda)2^{4n}\frac{\mu(\mu-1)(\mu-2)(\mu-3)}{4!}\gamma^{n\mu+4}(s) \\ +\gamma^{n(\mu-1)+4}(s)\left(\begin{array}{c}\frac{(\mu-1)(\mu-2)(\mu-3)(\mu-4)}{4!}2^{4n} \\ +\frac{(\mu-1)(\mu-2)(\mu-3)}{3!}2^{3n+1}\end{array}\right)\end{array}\right)a_2^4 \end{array}\right)z^4 \tag{2.14}$$
$$+\cdots$$

Equating (2.13) and (2.14) and comparing the coefficients, we have

$$((1-\lambda)2^n\mu\gamma^{n\mu+1}(s) + \lambda\gamma^{n(\mu-1)+1}(s)(2^n(\mu-1)+2))a_2z = U_1(t)c_1,$$

$$((1-\lambda)3^n\mu\gamma^{n\mu+1}(s) + \lambda\gamma^{n(\mu-1)+1}(s)(3^n(\mu-1)+3))a_3 \\ \left((1-\lambda)2^{2n}\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s) + \lambda\gamma^{n(\mu-1)+2}(s)\left(\frac{(\mu-1)(\mu-2)}{2!}2^{2n} + 2^{n+1}(\mu-1)\right)\right)a_2^2 \\ = c_2U_1(t) + c_1^2U_2(t),$$

$$\begin{aligned}
&((1-\lambda)4^n\mu\gamma^{n\mu+1}(s)+\lambda\gamma^{n(\mu-1)+1}(s)(4^n(\mu-1)+4))a_4\\
&+\left(\begin{array}{c}(1-\lambda)2^{n+1}\cdot 3^n\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s)\\ +\lambda\gamma^{n(\mu-1)+2}(s)\left(\frac{(\mu-1)(\mu-2)}{2!}2^{n+1}\cdot 3^n+(\mu-1)(2\cdot 3^n+3\cdot 2^n)\right)\end{array}\right)a_2a_3\\
&+\left(\begin{array}{c}(1-\lambda)2^{3n}\frac{\mu(\mu-1)(\mu-2)}{3!}\gamma^{n\mu+3}(s)\\ +\lambda\gamma^{n(\mu-1)+3}(s)\left(\frac{(\mu-1)(\mu-2)(\mu-3)}{3!}2^{3n}+\frac{(\mu-1)(\mu-2)}{2!}2^{2n+1}\right)\end{array}\right)a_2^3\\
&=c_3U_1(t)+2c_1c_2U_2(t)+c_1^3U_3(t),
\end{aligned}$$

$$\begin{aligned}
&((1-\lambda)5^n\mu\gamma^{n\mu+1}(s)+\lambda\gamma^{n(\mu-1)+1}(s)(5^n(\mu-1)+5))a_5\\
&+\left(\begin{array}{c}(1-\lambda)2^{3n+1}\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s)\\ +\lambda\gamma^{n(\mu-1)+2}(s)\left(\frac{(\mu-1)(\mu-1)}{2!}2^{3n+1}+(\mu-1)(4^n\cdot 2+4\cdot 2^n)\right)\end{array}\right)a_2a_4\\
&+\left(\begin{array}{c}(1-\lambda)3^{2n}\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s)\\ +\lambda\gamma^{n(\mu-1)+2}(s)\left(\frac{(\mu-1)(\mu-1)}{2!}3^{2n}+3^{n+1}(\mu-1)\right)\end{array}\right)a_3^2\\
&+\left(\begin{array}{c}(1-\lambda)2^{2n}\cdot 3^{n+1}\frac{\mu(\mu-1)(\mu-2)}{3!}\gamma^{n\mu+3}(s)\\ +\gamma^{n(\mu-1)+3}(s)\left(\begin{array}{c}\frac{(\mu-1)(\mu-2)(\mu-3)}{3!}(2^{2n+1}\cdot 3^n+2^{2n}\cdot 3^n)\\ +\frac{(\mu-1)(\mu-2)}{2!}(2^{n+2}\cdot 3^n+3\cdot 2^{2n})\end{array}\right)\end{array}\right)a_2^2a_3\\
&+\left(\begin{array}{c}(1-\lambda)2^{4n}\frac{\mu(\mu-1)(\mu-2)(\mu-3)}{4!}\gamma^{n\mu+4}(s)\\ +\gamma^{n(\mu-1)+4}(s)\left(\begin{array}{c}\frac{(\mu-1)(\mu-2)(\mu-3)(\mu-4)}{4!}2^{4n}\\ +\frac{(\mu-1)(\mu-2)(\mu-3)}{3!}2^{3n+1}\end{array}\right)\end{array}\right)a_2^4\\
&=c_4U_1(t)+(2c_1c_3+c_2^2)U_2(t)+3c_1^2c_2U_3(t)+c_1^4U_4(t)
\end{aligned}$$

Simplifying the above equations, we have

$$a_2=\frac{2tc_1}{(A+B)},$$

$$a_3=\frac{c_2(2t)+c_1^2(4t^2-1)}{(A'+B')}+\frac{c_1^2 4t^2(D+E)}{(A+B)^2(A'+B')},$$

$$a_4=\begin{array}{c}\frac{c_3 2(t)+2c_1c_2(4t^2-1)+c_1^3(8t^3-4t)}{(D'+E')}+\frac{(F+G)(c_1c_2 4t^2+c_1^3(2t)(4t^2-1))}{(A+B)(A'+B')(D'+E')}\\ +\frac{c_1^3 8t^3(D+E)(F+G)}{(A+B)^3(A'+B')(D'+E')}+\frac{(F'+G')8tc_1^3}{(D'+E')(A+B)^3}\end{array},$$

$$a_5=\begin{array}{c}\frac{c_4 2(t)+(2c_1c_3+c_2^2)(4t^2-1)+3c_1^2c_2(8t^3-4t)+c_1^4(16t^4-12t^2-1)}{H+I}\\ +\frac{c_3c_1 4t^2+c_1^2c_2 16t^3-4t+c_1 16t^4-8t^2(H'+I')}{(A+B)(D'+E')(H+I)}\\ +\frac{c_1^2c_2 8t^3+c_1^4 16t^4-4t^2(F+G)(H'+I')}{(A+B)^2(A'+B')(D'+E')(H+I)}-\frac{c_1 16t^4(D+E)(F+G)(H'+I')}{(A+B)^4(A'+B')(D'+E')(H+I)}\\ +\frac{c_1^4 16t^4(F'+G')(H'+I')}{(A+B)^4(D'+E')(H+I)}+\frac{c_2^2 4t^2+c_1^4(4t^2-1)^2(J+K)}{(A'+B')^2(H+I)}\\ +\frac{(16t^3c_1^2c_2+32t^4c_1^4-8t^2c_1^4)(D+E)(J+K)}{(A+B)^2(A'+B')^2(H+I)}+\frac{2t^4c_1^4(D+E)^2(J+K)}{(A+B)^4(A'+B')^2(H+I)}\\ +\frac{c_1^2c_2 8t^3+c_1^3 16t^3-4t)(J'+K')}{(A+B)^2(A'+B')(H+I)}+\frac{c_1^4 16t^4(D+E)(J'+K')}{(A+B)^4(A'+B')(H+I)}\\ +\frac{16t^4c_1^4(L+M)}{(A+B)^4(H+I)}\end{array}, \tag{2.15}$$

where

$$U_1(t) = 2t,$$
$$U_2(t) = 4t^2 - 1,$$
$$U_3(t) = 8t^3 - 4t,$$
$$U_4(t) = 16t^4 - 12t^2 - 1.$$

Then, from Lemma 1.1, we have

$$|a_2| = \left|\frac{2tc_1}{(A+B)}\right| \leq \frac{2t}{(A+B)},$$

$$\begin{aligned} |a_3| &= \left|\frac{c_2(2t)+c_1^2(4t^2-1)}{(A'+B')} + \frac{(c_1^2)4t^2(D+E))}{(A+B)^2(A'+B')}\right| \\ &\leq \frac{2t+(4t^2-1)}{(A'+B')} + \frac{4t^2(D+E)}{(A+B)^2(A'+B')}, \end{aligned}$$

$$\begin{aligned} |a_4| &= \left| \begin{array}{c} \frac{c_3 2(t)+2c_1c_2(4t^2-1)+c_1^3(8t^3-4t)}{(D'+E')} + \frac{(F+G)(c_1c_2 4t^2+c_1^2 2t(4t^2-1))}{(A+B)(A'+B')(D'+E')} + \\ \frac{(c_1^3)8t^3(D+E)(F+G)}{(A+B)^3(A'+B')(D'+E')} + \frac{(F'+G')8t(c_1^3)}{(D'+E')(A+B)^3} \end{array} \right|, \\ &\leq \frac{8t^3+8t^2-2t-2}{(D'+E')} + \frac{(F+G)(4t^2+2t(4t^2-1))}{(A+B)(A'+B')(D'+E')} \\ &\quad + \frac{8t^3(D+E)(F+G)}{(A+B)^3(A'+B')(D'+E')} + \frac{(F'+G')8t}{(D'+E')(A+B)^3} \end{aligned}$$

$$|a_5| = \left| \begin{array}{c} \frac{c_4 2(t)+(2c_1c_3+c_2^2)(4t^2-1)+3c_1^2c_2(8t^3-4t)+c_1^4(16t^4-12t^2-1)}{(H+I)} \\ + \frac{c_3c_1 4t^2+c_1^2c_2 16t^3-4t+c_1 16t^4-8t^2(H'+I')}{(A+B)(D'+E')(H+I)} \\ + \frac{c_1^2c_2 8t^3+c_1^4 16t^4-4t^2(F+G)(H'+I')}{(A+B)^2(A'+B')(D'+E')(H+I)} \\ - \frac{c_1 16t^4(D+E)(F+G)(H'+I')}{(A+B)^4(A'+B')(D'+E')(H+I)} \\ + \frac{c_1^4 16t^4(F'+G')(H'+I')}{(A+B)^4(D'+E')(H+I)} + \frac{c_2^2 4t^2+c_1^4(4t^2-1)^2(J+K)}{(A'+B')^2(H+I)} \\ + \frac{(16t^3c_1^2c_2+32t^4c_1^4-8t^2c_1^4)(D+E)(J+K)}{(A+B)^2(A'+B')^2(H+I)} + \frac{2t^4c_1^4(D+E)^2(J+K)}{(A+B)^4(A'+B')^2(H+I)} \\ + \frac{c_1^2c_2 8t^3+c_1^3 16t^3-4t)(J'+K')}{(A+B)^2(A'+B')(H+I)} + \frac{c_1^4 16t^4(D+E)(J'+K')}{(A+B)^4(A'+B')(H+I)} + \frac{16t^4c_1^4(L+M)}{(A+B)^4(H+I)} \end{array} \right|. \tag{2.16}$$

$$\begin{aligned} &\leq \frac{16t^4+24t^3-10t-4}{(H+I)} + \frac{(H'+I')(16t^4+16t^3-4t^2-4t)}{(A+B)(D'+E')(H+I)} \\ &+ \frac{(F+G)(H'+I')(16t^4+8t^3-4t^2)}{(A+B)^2(A'+B')(D'+E')(H+I)} + \frac{16t^4(D+E)(F+G)(H'+I')}{(A+B)^4(A'+B')(D'+E')(H+I)} \\ &+ \frac{16t^4(F'+G')(H'+I')}{(A+B)^4(D'+E')(H+I)} + \frac{(16t^4-4t^2+1)(J+K)}{(A'+B')^2(H+I)} \\ &+ \frac{(D+E)(J+K)(32t^4+16t^3-8t^2)}{(A+B)^2(A'+B')^2(H+I)} + \frac{(D+E)^2(J+K)2t^4}{(A+B)^4(A'+B')^2(H+I)} \\ &+ \frac{(J'+K')(2t^3-4t)}{(A+B)^2(A'+B')(H+I)} + \frac{16t^4(D+E)(J'+K')}{(A+B)^4(A'+B')(H+I)} + \frac{(L+M)16t^4}{(A+B)^4(H+I)} \end{aligned}$$

□

Corollary 2.3. *If $f(z)$ belongs to the class $H_\gamma(0,1,0)$, then*

$$|a_2| \leq \frac{2t}{\gamma(s)},$$

$$|a_3| \leq \frac{2t + (4t^2 - 1)}{\gamma(s)},$$

$$|a_4| \leq \frac{8t^3 + 8t^2 - 2t - 2}{\gamma(s)},$$

$$|a_5| \leq \frac{16t^4 + 24t^3 - 10t - 4}{\gamma(s)}.$$

Proof. Setting $n = 0$, $\mu = 1$ and $\lambda = 0$ in (2.16), the result follows. □

Corollary 2.4. *If $f(z)$ belongs to the class $H_1(0, 1, 0)$, then*

$$|a_2| \leq 2t,$$

$$|a_3| \leq 2t + (4t^2 - 1),$$

$$|a_4| \leq 8t^3 + 8t^2 - 2t - 2,$$

$$|a_5| \leq 16t^4 + 24t^3 - 10t - 4.$$

Proof. Setting $\gamma = 1$, n $= 0$, $\mu = 1$ and $\lambda = 0$ in (2.16), the result follows. □

Corollary 2.5. *If $f(z)$ belongs to the class $H_\gamma(0, 1, 1)$, then*

$$|a_2| \leq \frac{2t}{2\gamma(s)},$$

$$|a_3| \leq \frac{2t + (4t^2 - 1)}{3\gamma(s)},$$

$$|a_4| \leq \frac{8t^3 + 8t^2 - 2t - 2}{4\gamma(s)},$$

$$|a_5| \leq \frac{16t^4 + 24t^3 - 10t - 4}{5\gamma(s)}.$$

Proof. Setting $n = 0$, $\mu = 1$ and $\lambda = 1$ in (2.16), the result follows. □

Corollary 2.6. *If $f(z)$ belongs to the class $H_\gamma(1, 1, 0)$, then*

$$|a_2| \leq \frac{2t}{2\gamma^2(s)},$$

$$|a_3| \leq \frac{2t + (4t^2 - 1)}{3\gamma^2(s)},$$

$$|a_4| \leq \frac{8t^3 + 8t^2 - 2t - 2}{4\gamma^2(s)},$$

$$|a_5| \leq \frac{16t^4 + 24t^3 - 10t - 4}{5\gamma^2(s)}.$$

Proof. Setting $n = 1$, $\mu = 1$ and $\lambda = 0$ in (2.16), the result follows. □

Corollary 2.7. *If $f(z)$ belongs to the class $H_\gamma(1,1,1)$, then*

$$|a_2| \le \frac{2t}{2\gamma(s)},$$

$$|a_3| \le \frac{2t + (4t^2 - 1)}{3\gamma(s)},$$

$$|a_4| \le \frac{8t^3 + 8t^2 - 2t - 2}{4\gamma(s)},$$

$$|a_5| \le \frac{16t^4 + 24t^3 - 10t - 4}{5\gamma(s)}.$$

Proof. Setting $n = 1$, $\mu = 1$ and $\lambda = 1$ in (2.16), the result follows. □

Corollary 2.8. *If $f(z)$ belongs to the class $H_1(0,\mu,\lambda)$, then*

$$|a_2| \le \frac{2t}{(A+B)},$$

$$|a_3| \le \frac{2t + (4t^2 - 1)}{(A'+B')} + \frac{2t^2(A'+B')(1-\mu)}{(A+B)^2(A'+B')},$$

$$|a_4| \le \frac{8t^3+8t^2-2t-2}{-(\mu+3\lambda)} - \frac{(1-\mu)(4t^2+2t(4t^2-1))}{(\mu+\lambda)(\mu+2\lambda)} + \frac{4t^3(1-\mu)^2}{(\mu+\lambda)^3} - \frac{(-\mu^2+3\mu-2)4t}{3(\mu+\lambda)^3},$$

$$\begin{aligned} |a_5| \le{} & \frac{16t^4+24t^3-10t-4}{4\lambda+\mu} + \frac{(1-\mu)(16t^4+16t^3-4t^2-4t)}{(\lambda+\mu)-(\mu+3\lambda)} \\ & - \frac{(1-\mu)(16t^4+8t^3-4t^2)}{(\mu+\lambda)^2(2\lambda+\mu)} - \frac{8t^4(1-\mu)^2}{(\mu+\lambda)^4} + \frac{8t^4(-\mu^2+3\mu-2)(1-\mu)}{3(\mu+\lambda)^4} + \frac{(16t64-4t^2+1)(1-\mu)}{2(2\lambda+\mu)^2} \\ & + \frac{(1-\mu)^2(8t^4+4t^3-t^2)}{(\mu+\lambda)^2(2\lambda+\mu)} + \frac{((2\lambda+\mu)(1-\mu))^2(1-\mu)t^4}{2(\mu+\lambda)^4(2\lambda+\mu)^2} + \frac{\lambda\mu^3-18\mu^2\lambda+47\mu\lambda-30\lambda-3\mu^3+9\mu^2-6\mu(2t^3-4t)}{6(\mu+\lambda)^2(\mu+2\lambda)(\mu+4\lambda)} \\ & + \frac{4t^4(1-\mu)(\lambda\mu^3-18\mu^2\lambda+47\mu\lambda-30\lambda-3\mu^3+9\mu^2-6\mu)}{3(\mu+\lambda)^4(\mu+4\lambda)} + \frac{(2\lambda-\mu)(-\mu^3+6\mu^2-11\mu+6)2t^4}{3(\mu+\lambda)^4(\mu+4\lambda)}. \end{aligned}$$

Proof. Setting $n = 0$, and $\gamma = 1$ in (2.16), the result agrees with $|a_2|$ and $|a_3|$ in Bulut and Magesh [2]. □

Theorem 2.9. *If $f(z)$ belongs to the class $H_\gamma(n,\mu,\lambda)$. then*

$$|a_3 - \rho a_2^2| \le \left\{ \begin{array}{ll} \frac{2t}{(A'+B')}; & \rho \in [\rho_1, \rho_2] \\ \frac{2t}{(A'+B')} \left| \frac{4t^2-1}{2t} - (\mu - 1 + 2\rho) \frac{(A'+B')}{(A+B)^2} t \right|; & \rho \notin [\rho_1, \rho_2] \end{array} \right\},$$

where

$$\rho_1 = \frac{1-\mu}{2} + \frac{(A+B)^2}{4(A'+B')} \frac{4t^2 - 2t - 1}{t^2},$$

$$\rho_2 = \frac{1-\mu}{2} + \frac{(A+B)^2}{4(A'+B')} \frac{4t^2 + 2t - 1}{t^2}$$

and

$$A = (1-\lambda)2^n \mu \gamma^{n\mu+1}(s),$$
$$B = \lambda \gamma^{n(\mu-1)+1}(s)(2^n(\mu-1)+2),$$
$$A' = (1-\lambda)3^n \mu \gamma^{n\mu+1}(s),$$
$$B' = \lambda \gamma^{n(\mu-1)+1}(s)(3^n(\mu-1)+3),$$
$$D = -(1-\lambda)2^{2n} \frac{\mu(\mu-1)}{2!} \gamma^{n\mu+2}(s),$$

$$E = -\lambda\gamma^{n(\mu-1)+2}(s)\left(\frac{(\mu-1)(\mu-2)}{2!}2^{2n} + 2^{n+1}(\mu-1)\right).$$

Proof. From (2.15),

$$a_2 = \frac{2tc_1}{(A+B)},$$

$$a_3 = \frac{c_2(2t) + c_1^2(4t^2-1)}{(A'+B')} + \frac{c_1^2 4t^2(D+E)}{(A+B)^2(A'+B')},$$

then it follows that

$$|a_3 - \rho a_2^2| \leq \frac{2t}{(A'+B')}\left|c_2 - c_1^2\left(\rho\frac{2t(A'+B')}{(A+B)^2} - \frac{4t^2-1}{2t} - \frac{2t(D+E)}{(A+B)^2}\right)\right|.$$

From Lemma 1.2, we have

$$|a_3 - \rho a_2^2| \leq \frac{2t}{(A'+B')}max\left\{1, \left|\rho\frac{2t(A'+B')}{(A+B)^2} - \frac{-4t^2-1}{2t} - \frac{2t(D+E)}{(A+B)^2}\right|\right\}.$$

Since $t > 0$, we have

$$\left|\rho\frac{2t(A'+B')}{(A+B)^2} - \frac{-4t^2-1}{2t} - \frac{2t(D+E)}{(A+B)^2}\right| \leq 1$$

$$\Leftrightarrow -\frac{4t^2-1}{2t} + \frac{2t(D+E)}{(A+B)^2} - 1 \leq \rho\frac{2t(A'+B')}{(A+B)^2} \leq 1 + \frac{4t^2-1}{2t} + \frac{2t(D+E)}{(A+B)^2}.$$

$$\Leftrightarrow \rho_1 \leq \rho \leq \rho_2.$$

□

2.1. **Conclusion.** The results agree and extend the work in Bulut and Magesh [2].

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