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RICCI SOLITONS ON KENMOTSU MANIFOLDS UNDER D-HOMOTHETIC DEFORMATION

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Abstract. The aim of the present paper is to study Ricci solitons in Kenmotsu manifolds under D-homothetic deformation. We analyzed behaviour of Ricci solitons when potential vector field is orthogonal to Reeb vector field and pointwise collinear with Reeb vector field. Further we prove Ricci solitons in D-homothetically transformed Kenmotsu manifolds are shrinking.

1. Introduction

One of the important topics in the study of almost contact metric manifolds is the study of Ricci flow and Ricci solitons. Ricci solitons introduced by Hamilton [12] are natural generalization of an Einstein metric. A complete Riemannian metric g on a smooth manifold M is a Ricci soliton if there is a vector field V and a constant λ such that

$$(L_{\mathrm{V}}g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{1.1}$$

where S is a Ricci tensor and L_{V} denote the Lie derivative along V. A Ricci soliton is said to be shrinking or steady or expanding according as λ is negative, zero or positive. A Ricci soliton is said to be a gradient Ricci soliton if the vector field V is gradient of some smooth function $-f$ on M.
Sharma [11] initiated the study of Ricci solitons in K- contact geometry. Also in subsequent papers, [2] [3] [11] Ghosh and Sharma, Sharma established results by considering K-contact, Kenmotsu, Sasakian and (κ, μ)- contact metrics as Ricci

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solitons. De and others [4][8] [7] [16] [17] studied Ricci solitons in f-Kenmotsu manifolds, Kenmotsu manifolds and QR-hypersurfaces of a quaternionic space form Q^n. In [1] authors analyze the behaviour of trans-Sasakian manifolds under D-homothetic deformations. Several authors Nagaraja and Premalatha [6], De and Ghosh [15] studied the behaviour of K-contact, normal almost contact metric manifolds under D-homothetic deformations. We make use of the invariance of certain contact structures under D-homothetic deformations to study Ricci solitons. The present paper is organized as follows:
After a brief review of Kenmotsu manifolds in Section 2, we study the behaviour of Ricci solitons in Kenmotsu manifolds under D-homothetic deformation when potential vector field is orthogonal to Reeb vector field and pointwise collinear with Reeb vector field in which cases the Ricci soliton is shrinking and expanding respectively. Also we investigate Ricci solitons in D-homothetically transformed Kenmotsu manifolds under certain curvature conditions and proved that Ricci solitons in these cases are shrinking in Section 3.

2. PRELIMINARIES

A $(2n+1)$-dimensional smooth manifold M is said to be an almost contact metric manifold if it admits an almost contact metric structure (ϕ, ξ, η, g) consisting of a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g compatible with (ϕ, ξ, η) satisfying

$$\phi^2 X = -X + \eta(X)\xi,\ \phi\xi = 0,\ g(X,\xi) = \eta(X),\ \eta(\xi) = 1, \eta \circ \phi = 0, \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

An almost contact metric manifold is said to be a Kenmotsu manifold [5] if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.3}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y), \tag{2.5}$$

where ∇ denotes the Riemannian connection of g.
In a Kenmotsu manifold the following relations hold [14]:

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.7}$$

$$S(\phi X, \phi Y) = S(X,Y) + 2n\eta(X)\eta(Y), \tag{2.8}$$

for any vector fields X, Y, Z on M, where R denote the curvature tensor of type $(1,3)$ on M.

3. Ricci solitons in Kenmotsu manifolds under D-homothetic deformation

Let (M, ϕ, ξ, η, g) be a Kenmotsu manifold, where g is a Ricci soliton. A D-homothetic deformation [13] on M is given by

$$\phi^* = \phi,\ \xi^* = \frac{1}{a}\xi,\ \eta^* = a\eta,\ g^* = ag + a(a-1)\eta \otimes \eta, \tag{3.1}$$

for a positive constant a. Then the D-homothetically changed structure $(M, \phi^*, \xi^*, \eta^*, g^*)$ is also a Kenmotsu structure [1]. Throughout this paper the quantity with $*$ denote the quantities in $(M, \phi^*, \xi^*, \eta^*, g^*)$ and quantities without $*$ are from (M, ϕ, ξ, η, g).
The relation between the connections ∇ and ∇^* is given by [10]

$$\nabla^*_X Y = \nabla_X Y + \frac{(a-1)}{a} g(\phi X, \phi Y)\xi, \tag{3.2}$$

for any vector fields X, Y on M.
Using (3.2), we now calculate the Riemann curvature tensor R^* of $(M, \phi^*, \xi^*, \eta^*, g^*)$ as follows:

$$R^*(X,Y)Z = R(X,Y)Z + \frac{a-1}{a}\{g(\phi Y, \phi Z)X - g(\phi X, \phi Z)Y\}, \tag{3.3}$$

for any vector fields X, Y, Z on M.
On contracting (3.3), we obtain the Ricci tensor S^* of D-homothetically deformed Kenmotsu manifold as

$$S^*(Y,Z) = S(Y,Z) + \frac{2n(a-1)}{a} g(\phi Y, \phi Z). \tag{3.4}$$

Contracting (3.4) with respect to Y and Z, we get

$$r^* = r + 2n(2n+1), \tag{3.5}$$

where r^* and r are the scalar curvatures of (M, g) and (M, g^*) respectively.
Using (3.1) and (3.4), we can write

$$\begin{aligned}
&(L^*_V g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y) \\
&= g^*(\nabla^*_X V, Y) + g^*(X, \nabla^*_Y V) + 2S(X,Y) + \frac{4n(a-1)}{a} g(\phi X, \phi Y) \\
&+ 2\lambda\{ag(X,Y) + a(a-1)\eta(X)\eta(Y)\}.
\end{aligned} \tag{3.6}$$

Making use of (3.2) in (3.6), we obtain

$$\begin{aligned}
&(L^*_V g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y) \\
&= a\{g(\nabla_X V, Y) + g(X, \nabla_Y V)\} \\
&+ (a-1)\{g(\phi X, \phi V)\eta(Y) + g(\phi Y, \phi V)\eta(X)\} \\
&+ a(a-1)\{\eta(\nabla_X V)\eta(Y) + \eta(\nabla_Y V)\eta(X)\} + 2S(X,Y) \\
&+ \frac{4n(a-1)}{a} g(\phi X, \phi Y) + (a-1)^2\{g(\phi X, \phi V)\eta(Y) + g(\phi Y, \phi V)\eta(X)\} \\
&+ 2\lambda\{ag(X,Y) + a(a-1)\eta(X)\eta(Y)\}.
\end{aligned} \tag{3.7}$$

If $V \perp \xi$, it provides $\eta(\nabla_X V) = -g(\phi V, \phi X)$. Hence (3.7) can be written as

$$\begin{aligned}&(L^*_V g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y)\\ &= a\{g(\nabla_X V, Y) + g(X, \nabla_Y V) + 2S(X,Y) + 2\lambda g(X,Y)\}\\ &+ \frac{4n(a-1)}{a} g(\phi X, \phi Y) + 2(1-a)S(X,Y) + 2\lambda a(a-1)\eta(X)\eta(Y).\end{aligned} \tag{3.8}$$

Since g is a Ricci soliton, the above equation takes the form

$$\begin{aligned}&(L^*_V g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y)\\ &= 2(1-a)S(X,Y) + \frac{4n(a-1)}{a} g(\phi X, \phi Y) + 2\lambda a(a-1)\eta(X)\eta(Y).\end{aligned} \tag{3.9}$$

Thus g^* is a Ricci soliton if and only if

$$S(X,Y) = \frac{2n}{a} g(X,Y) + \{\lambda a - \frac{2n}{a}\}\eta(X)\eta(Y). \tag{3.10}$$

By taking $X = Y = \xi$ in (3.10), we obtain

$$\lambda = -\frac{2n}{a}. \tag{3.11}$$

Thus we have the following:

Theorem 3.1. *Let $(M, \phi^*, \xi^*, \eta^*, g^*)$ be obtained by D-homothetic deformation of a Kenmotsu manifold (M, ϕ, ξ, η, g) with the potential vector field V orthogonal to Reeb vector field. If (M, g) is η-Einstein then (M, g^*) is a Ricci soliton and the Ricci soliton is shrinking.*

Now, let V be pointwise collinear with the Reeb vector field ξ^* i.e. $V = \beta\xi^*$, where β is a function on M. Then using (3.6), we write

$$g(\nabla^*_X(\beta\xi^*), Y) + g(\nabla^*_Y(\beta\xi^*), X) + 2S^*(X,Y) + 2\lambda g^*(X,Y) = 0. \tag{3.12}$$

By using (3.1), (3.2), (3.4) in (3.12), we obtain

$$\begin{aligned}&2\beta g(\phi X, \phi Y) + (X\beta)\eta(Y) + (Y\beta)\eta(X)\} + 2S(X,Y)\\ &+ \frac{4n(a-1)}{a} g(\phi X, \phi Y) + 2\lambda\{a g(X,Y) + a(a-1)\eta(X)\eta(Y)\} = 0.\end{aligned} \tag{3.13}$$

By setting $Y = \xi$ in (3.13) and using (2.7), we obtain

$$(X\beta) = \{4n - \xi\beta - 2\lambda a^2\}\eta(X). \tag{3.14}$$

Again replacing X by ξ in (3.14), we get

$$(\xi\beta) = 2n - \lambda a^2. \tag{3.15}$$

Substituting this in (3.14), we have

$$(X\beta) = \{2n - \lambda a^2\}\eta(X). \tag{3.16}$$

By applying d on (3.16), we get

$$(2n - \lambda a^2)d\eta = 0. \tag{3.17}$$

Since $d\eta \neq 0$, from (3.17) we have

$$2n - \lambda a^2 = 0. \tag{3.18}$$

Substituting (3.18) in (3.16), we conclude that β is a constant. Hence it is verified from (3.13) that

$$S(X,Y) = -\{\frac{2n(a-1)}{a} + \beta + \lambda a\}g(X,Y) + \{\beta + \lambda a - \frac{2n}{a}\}\eta(X)\eta(Y). \quad (3.19)$$

Further from (3.18), we have $\lambda = \frac{2n}{a^2}$. This leads to the following:

Theorem 3.2. *Under D-homothetic deformation, Kenmotsu manifold admitting a Ricci soliton (g^*, V, λ) with potential vector field V is pointwise collinear with ξ^*, the manifold is an η-Einstein manifold and the Ricci soliton is expanding.*

An interesting invariant of a concircular transformation is the concircular curvature tensor. The concircular curvature tensor [9] C^* under D-homothetic deformation ∇^* is defined by

$$C^*(X,Y)Z = R^*(X,Y)Z - \frac{r^*}{2n(2n+1)}\{g^*(Y,Z)X - g^*(X,Z)Y\}, \quad (3.20)$$

for all vector fields X, Y, Z on M.
Suppose M is concircularly flat. Then from (3.20), we have

$$R^*(X,Y)Z = \frac{r^*}{2n(2n+1)}\{g^*(Y,Z)X - g^*(X,Z)Y\}. \quad (3.21)$$

Using (3.1), (3.3) and (3.5) in (3.21), we obtain

$$\begin{aligned} R(X,Y)Z = & \frac{ar + 4n^2(a-1)}{2n(2n+1)}[\{g(Y,Z)X - g(X,Z)Y\} \\ & + (a-1)\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}] \\ , \quad & -\frac{a-1}{a}\{g(\phi Y,\phi Z)X - g(\phi X,\phi Z)Y\}. \end{aligned} \quad (3.22)$$

Let $\{e_i\}_{i=1}^{2n+1}$ be an orthonormal basis of the tangent space. Contracting (3.22) with respect to U, taking $X = U = e_i$ and summing over $i = 1, 2, ...(2n+1)$, we get

$$\begin{aligned} S(Y,Z) = & \{\frac{ar + 4n^2(a-1)}{2n+1} - \frac{2n(a-1)}{a}\}g(Y,Z) \\ , \quad & +(a-1)\{\frac{ar + 4n^2(a-1)}{2n+1} + \frac{2n}{a}\}\eta(Y)\eta(Z). \end{aligned} \quad (3.23)$$

Suppose g^* is a Ricci soliton. Then by comparing (3.23) and (3.10), we have the following:

Theorem 3.3. *Under D-homothetic deformation of concircularly flat Kenmotsu manifold (M, g, V, λ), where potential vector field V is orthogonal to Reeb vector field, g^* in (M, g^*) is a Ricci soliton if and only if $r = \frac{4n^2}{a}\{2 - a\} + 2n$ and $\lambda = \frac{2n(a+1)}{a^2}$.*

Taking $Z = \xi$ in (3.20) and using (3.3) and (2.6), we get

$$C^*(X,Y)\xi = \{1 + \frac{a^2 r^*}{2n(2n+1)}\}\{\eta(X)Y - \eta(Y)X\}. \quad (3.24)$$

Similarly using (2.6), (3.1), (3.3) in (3.20), we get

$$\eta(C^*(X,Y)Z) = \{\frac{1}{a} + \frac{ar^*}{2n(2n+1)}\}\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}. \tag{3.25}$$

We assume that the condition $R^*(\xi^*,X).C^* = 0$ holds. Then we have

$$\begin{aligned} &R^*(\xi^*,X)C^*(Y,Z)W - C^*(R^*(\xi^*,X)Y,Z)W - C^*(Y,R^*(\xi^*,X)Z)W \\ &- C^*(Y,Z)R^*(\xi^*,X)W = 0. \end{aligned} \tag{3.26}$$

By using (3.3) and (3.20) in (3.26), we have

$$\begin{aligned} &\eta(C^*(Y,Z)W)X - \frac{1}{a}g(X,C^*(Y,Z)W)\xi - \frac{a-1}{a}\eta(X)\eta(C^*(Y,Z)W)\xi \\ &- \eta(Y)C^*(X,Z)W + \frac{1}{a}g(X,Y)C^*(\xi,Z)W + \frac{a-1}{a}\eta(X)\eta(Y)C^*(\xi,Z)W \\ &- \eta(Z)C^*(Y,X)W + \frac{1}{a}g(X,Z)C^*(Y,\xi)W + \frac{a-1}{a}\eta(X)\eta(Z)C^*(Y,\xi)W \\ &- \eta(W)C^*(Y,Z)X + \frac{1}{a}g(X,W)C^*(Y,Z)\xi + \frac{a-1}{a}\eta(X)\eta(W)C^*(Y,Z)\xi \\ &= 0. \end{aligned} \tag{3.27}$$

Cotracting the above equation with ξ, we get

$$\begin{aligned} &\eta(C^*(Y,Z)W)\eta(X) - \frac{1}{a}g(X,C^*(Y,Z)W) - \frac{a-1}{a}\eta(X)\eta(C^*(Y,Z)W) \\ &- \eta(Y)\eta(C^*(X,Z)W) + \frac{1}{a}g(X,Y)\eta(C^*(\xi,Z)W) \\ &+ \frac{a-1}{a}\eta(X)\eta(Y)\eta(C^*(\xi,Z)W) - \eta(Z)\eta(C^*(Y,X)W) \\ &+ \frac{1}{a}g(X,Z)\eta(C^*(Y,\xi)W) + \frac{a-1}{a}\eta(X)\eta(Z)\eta(C^*(Y,\xi)W) \\ &- \eta(W)\eta(C^*(Y,Z)X) + \frac{1}{a}g(X,W)\eta(C^*(Y,Z)\xi) \\ &+ \frac{a-1}{a}\eta(X)\eta(W)\eta(C^*(Y,Z)\xi) = 0. \end{aligned} \tag{3.28}$$

Now using (2.6), (3.1), (3.3), (3.20), (3.24) and (3.25) in (3.28), we obtain

$$R^{'}(Y,Z,W,X) = g(X,Z)g(Y,W) - g(X,Y)g(Z,W). \tag{3.29}$$

Taking $X = Y = e_i$ and summing over $i = 1, 2, ...(2n+1)$, we obtain

$$S(Z,W) = -2ng(Y,W). \tag{3.30}$$

Suppose $C^*.S^* = 0$. Then we have

$$S^*(C^*(X,Y)Z,U) + S^*(Z,C^*(X,Y)U) = 0. \tag{3.31}$$

Taking $U = \xi$ in (3.31) and using (3.4), we obtain

$$\begin{aligned} &-2n\eta(C^*(X,Y)Z) + \{1 + \frac{a^2r^*}{2n(2n+1)}\} \\ &\{\eta(X)S^*(Z,Y) - \eta(Y)S^*(Z,X)\} = 0. \end{aligned} \tag{3.32}$$

Again setting $X = \xi$ in (3.32) and making use of (3.4) and (3.25), we obtain

$$S(Y, Z) = -2ng(Y, Z). \tag{3.33}$$

We next assume that M is a locally symmetric Kenmotsu manifold under D-homothetic deformation. Then we have

$$(\nabla^*_X R^*)(Y, Z)W = 0. \tag{3.34}$$

On a suitable contraction of this equation, we have

$$(\nabla^*_X S^*)(Z, W) = \nabla^*_X S^*(Z, W) - S^*(\nabla^*_X Z, W) - S^*(Z, \nabla^*_X W) = 0. \tag{3.35}$$

Taking $W = \xi$ in the above equation yields

$$\nabla^*_X S^*(Z, \xi) - S^*(\nabla^*_X Z, \xi) - S^*(Z, \nabla^*_X \xi) = 0. \tag{3.36}$$

Using (3.2) and (3.4) in (3.36), we obtain

$$S(Y, Z) = -2ng(Y, Z). \tag{3.37}$$

Suppose g^* is a Ricci soliton. Then from the above discussion and using (3.30), (3.33) and (3.37) in (3.10) and taking $X = Y = \xi$, we obtain $\lambda = -\frac{2n}{a}$.
Thus we state the following:

Theorem 3.4. *Let (M, g, λ) be a Ricci soliton and (M, g^*, λ) be obtained by D-homothetic deformation of g. Then the Ricci soliton is shrinking under the following conditions:*
i) $R^*(\xi^*, X).C^* = 0$
ii) $C^*.S^* = 0$
iii) $(\nabla^*_X R^*)(Y, Z)W = 0.$

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