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ON CERTAIN RESULTS INVOLVING A MULTIPLIER TRANSFORMATION IN A PARABOLIC REGION

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Abstract. We, here, obtain certain results in subordination form involving a multiplier transformation in a parabolic region. In particular, using different dominants in our main result, we derive certain results on parabolic starlikeness, starlikeness, convexity, uniform convexity, strongly starlikeness, close-to-convexity, and uniform close-to-convexity of p-valent analytic functions as well as univalent analytic functions.

1. Introduction and preliminaries

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N}; z \in \mathbb{E})$$

which are analytic and p-valent in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. Obviously, $\mathcal{A}_1 = \mathcal{A}$, the class of all analytic functions f, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g in $\mathbb{E}$ (written as $f \prec g$), if there exists a Schwartz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1, \phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)), \qquad |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, let p be an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$, and let h be univalent in $\mathbb{E}$. Then

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the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \qquad \Phi(p(0), 0; 0) = h(0). \tag{1.1}$$

A univalent function q is called a dominant of the differential subordination (1.1) if $p(0) = q(0)$ and $p(z) \prec q(z)$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants q of (1.1), is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of $\mathbb{E}$.

A function $f \in \mathcal{A}_p$ is said to be p-valent starlike of order $\alpha(0 \leq \alpha < p)$ in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad z \in \mathbb{E}. \tag{1.2}$$

Let $\mathcal{S}_p^*(\alpha)$ denote the class of p-valent starlike functions of order α. Write $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$, the class of p-valent starlike functions.

A function $f \in \mathcal{A}_p$ is said to be p-valent convex of order $\alpha(0 \leq \alpha < p)$ in $\mathbb{E}$ if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad z \in \mathbb{E}.$$

Let the class of such functions be denoted by $\mathcal{K}_p(\alpha)$. Let $\mathcal{K}_p(0) = \mathcal{K}_p$, the class of p-valent convex functions.

A function $f \in \mathcal{A}_p$ is said to be p-valent close-to-convex in $\mathbb{E}$, if it satisfies

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \qquad z \in \mathbb{E}, \tag{1.3}$$

for some $g \in \mathcal{S}_p^*$. Let the class of such functions be denoted by $\mathcal{C}_p$. Note that $\mathcal{C}_1 = \mathcal{C}$. Select $g(z) \equiv z^p \in \mathcal{A}_p$. Note that it satisfies condition (1.2). Therefore, condition (1.3), becomes

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right) > 0, \qquad z \in \mathbb{E}. \tag{1.4}$$

Hence, a function $f \in \mathcal{A}_p$ is said to be p-valent close-to-convex in $\mathbb{E}$, if it satisfies condition (1.4).

A function $f \in \mathcal{A}_p$ is said to be p-valent strongly starlike of order $\alpha, 0 < \alpha \leq 1$, if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}, \qquad z \in \mathbb{E}, \tag{1.5}$$

or, equivalently,

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^\alpha, \qquad z \in \mathbb{E}.$$

Let $\tilde{\mathcal{S}}_p(\alpha)$ denote the class of p-valent strongly starlike functions of order α. Note that $\tilde{\mathcal{S}}_p(1) \equiv \mathcal{S}_p^*$.

A function $f \in \mathcal{A}_p$ is said to be p-valent strongly convex of order $\alpha, 0 < \alpha \leq 1$, if

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2}, \qquad z \in \mathbb{E}, \tag{1.6}$$

or, equivalently,

$$1+\frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \qquad z \in \mathbb{E}.$$

Let $\tilde{\mathcal{K}}_p(\alpha)$ denote the class of p-valent strongly convex functions of order α. Note that $\tilde{\mathcal{K}}_p(1) \equiv \mathcal{K}_p$.
A function $f \in \mathcal{A}_p$ is said to be p-valent parabolic starlike in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - p\right|, \qquad z \in \mathbb{E}. \tag{1.7}$$

Let $\mathcal{S}_{\mathcal{P}}^{p}$ denote the class of p-valent parabolic starlike functions. Write $\mathcal{S}_{\mathcal{P}}^{1} = \mathcal{S}_{\mathcal{P}}$, the class of parabolic starlike functions.
A function $f \in \mathcal{A}_p$ is said to be uniformly p-valent convex in $\mathbb{E}$ if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)} - (p-1)\right|, \qquad z \in \mathbb{E}, \tag{1.8}$$

and is denoted by UCV_p, the class of uniformly p-valent convex functions, and let $UCV_1 = UCV$, the class of uniformly convex functions.

A function $f \in \mathcal{A}_p$ is said to be p-valent uniformly close-to-convex in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \left|\frac{zf'(z)}{g(z)} - p\right|, \qquad z \in \mathbb{E}, \tag{1.9}$$

for some $g \in \mathcal{S}_{\mathcal{P}}^{p}$. Let UCC_p denote the class of all such functions. Let $UCC_1 = UCC$. Note that the function $g(z) \equiv z^p \in \mathcal{S}_{\mathcal{P}}^{p}$. Therefore, for $g(z) \equiv z^p$, condition (1.9) becomes

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right) > \left|\frac{f'(z)}{z^{p-1}} - p\right|, \qquad z \in \mathbb{E}. \tag{1.10}$$

Ronning [18] and Ma and Minda [15] studied the domain Ω, and the function $q(z)$ is defined below:

$$\Omega = \{u+iv : u > \sqrt{(u-1)^2+v^2}\}.$$

Clearly the function

$$q(z) = 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \tag{1.11}$$

maps the unit disk $\mathbb{E}$ onto the domain Ω. Hence the conditions (1.7), (1.8), and (1.10) are equivalent to

$$\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right) \prec q(z), \qquad z \in \mathbb{E},$$

$$\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec q(z), \qquad z \in \mathbb{E},$$

and

$$\frac{f'(z)}{pz^{p-1}} \prec q(z),$$

respectively, where $q(z)$ is given by (1.11).

For $f \in \mathcal{A}_p$, we define the multiplier transformation $I_p(n,\lambda)$ as

$$I_p(n,\lambda)[f](z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k, \qquad \text{where } \lambda \geq 0, n \in \mathbb{Z}.$$

The operator $I_1(n,0)$ is the well-known Sălăgean [19] derivative operator $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}$. In 1992, Uralegaddi and Somanatha [25] investigated the operator $I_1(n,1)$. In 1993, Jung et al. [13] studied the transformation

$$I_1(-1,\lambda)[f](z) = z + \sum_{k=2}^{\infty} \left(\frac{1+\lambda}{k+\lambda}\right) a_k z^k, \qquad \text{where } \lambda > -1, f \in \mathcal{A}.$$

In 2003, Cho and Srivastava [9] and Cho and Kim [8] investigated the operator $I_1(n,\lambda)$. In 2005, Aghalary et al. [1] studied the operator $I_p(n,\lambda)$, whereas Uralegaddi and Somanatha [25] studied the operator $I_1(n,1)$. Recently, Billing [2, 3, 4, 5, 6], Singh et al. [20, 21], Brar and Billing [7] investigated the operator $I_p(n,\lambda)$ and obtained certain sufficient conditions for starlike and convex functions.

Let $\mathcal{S}_n(\alpha)$ denote the class of functions $f \in \mathcal{A}$ for which

$$\Re\left(\frac{D^{n+1}[f](z)}{D^n[f](z)}\right) > \alpha, \qquad z \in \mathbb{E}, 0 \leq \alpha < 1.$$

In 1989, Owa, Shen and Obradovič [17] studied this class and proved the following result.

Theorem 1.1. *For $n \in \mathbb{N}_0$, if $f \in \mathcal{A}$ satisfies*

$$\left|\frac{D^{n+1}[f](z)}{D^n[f](z)} - 1\right|^{1-\beta} \left|\frac{D^{n+2}[f](z)}{D^{n+1}[f](z)} - 1\right|^{\beta} < (1-\alpha)^{1-2\beta}\left(1 - \frac{3}{2}\alpha + \alpha^2\right)^{\beta}, \qquad z \in \mathbb{E},$$

for some $\alpha(0 \leq \alpha \leq 1/2)$ and $\beta(0 \leq \beta \leq 1)$, then $\Re\left(\frac{D^{n+1}[f](z)}{D^n[f](z)}\right) > \alpha$; that is, $f \in \mathcal{S}_n(\alpha)$.

Later on, Li and Owa [14] extended this result by proving the following result.

Theorem 1.2. *For $n \in \mathbb{N}_0$, if $f \in \mathcal{A}$ satisfies*

$$\left|\frac{D^{n+1}[f](z)}{D^n[f](z)} - 1\right|^{\gamma} \left|\frac{D^{n+2}[f](z)}{D^{n+1}[f](z)} - 1\right|^{\beta} < \begin{cases} (1-\alpha)^{\gamma}(\frac{3}{2}-\alpha)^{\beta}, & 0 \leq \alpha \leq \frac{1}{2}, \\ 2^{\beta}(1-\alpha)^{\beta+\gamma}, & \frac{1}{2} \leq \alpha < 1. \end{cases}$$

for some $\alpha(0 \leq \alpha < 1), \beta \geq 0$ and $\gamma \geq 0$ with $\beta + \gamma > 0$, then $f \in S_n(\alpha)$, $n \in \mathbb{N}_0$.

Let $\mathcal{S}_n(p,\lambda,\alpha)$ denote the class of functions $f \in \mathcal{A}_p$ for which

$$\Re\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right) > \frac{\alpha}{p}, \qquad z \in \mathbb{E}, 0 \leq \alpha < p.$$

In 2008, Singh et al. [20] investigated the above class and proved the following sufficient condition for a multivalent function to be a member of this class.

Theorem 1.3. *Let $f \in \mathcal{A}_p$ satisfy*

$$\left|\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}-1\right|^{\gamma}\left|\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)}-1\right|^{\beta}<M(p,\lambda,\alpha,\beta,\gamma), \qquad z\in\mathbb{E},$$

for some real numbers α, β, and γ such that $0 \le \alpha < p, \beta \ge 0, \gamma \ge 0, \beta+\gamma > 0$; then $f \in S_n(p,\lambda,\alpha)$, where $n \in \mathbb{N}_0$ and

$$M(p,\lambda,\alpha,\beta,\gamma)=\begin{cases}\left(1-\frac{\alpha}{p}\right)^{\gamma}\left(1-\frac{\alpha}{p}+\frac{1}{2(p+\lambda)}\right)^{\beta}, & 0\le\alpha\le\frac{p}{2},\\ \left(1-\frac{\alpha}{p}\right)^{\gamma+\beta}\left(1+\frac{1}{(p+\lambda)}\right)^{\beta}, & \frac{p}{2}\le\alpha<p.\end{cases}$$

Recently, Billing [3, 5] also proved the following results and obtained sufficient conditions for starlikeness and convexity of univalent functions.

Theorem 1.4. *Let α, β be real numbers such that $\alpha > \frac{2}{1-\beta}, 0 \le \beta < 1$, and let*

$$0<M\equiv M(\alpha,\beta,\gamma,p)=\frac{(\alpha+p+\lambda)[\alpha(1-\beta)-2]}{\alpha[1+(1-\beta)(p+\lambda)]}.$$

If $f \in \mathcal{A}_p$ satisfies the differential inequality

$$\left|(1-\alpha)\frac{I_p(n,\lambda)f(z)}{z^p}+\alpha\frac{I_p(n+1,\lambda)f(z)}{z^p}-1\right|<M(\alpha,\beta,\gamma,p),$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right)>\beta, \qquad z\in\mathbb{E}.$$

Theorem 1.5. *Let α be a nonzero complex number such that $\Re(\alpha) > 0$, and let $h(z)$ be analytic and convex in $\mathbb{E}$ with $h(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\left\{1+\alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)}-\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)\right\}$$
$$\prec\frac{p+\lambda}{\alpha}\int_0^1 t^{\frac{p+\lambda}{\alpha}-1}h(zt)dt,$$

then

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\prec h(z), \qquad z\in\mathbb{E}.$$

For more references see [10, 11, 12, 22, 23, 24]. In the present paper, we study a differential subordination involving multiplier transformation $I_p(n,\lambda)$ defined above. In particular cases to our main result, we obtain sufficient conditions for starlikeness, convexity, and close-to-convexity of multivalent and univalent analytic functions in a parabolic region.

To prove our main results, we shall use the following lemma of Miller and Mocanu ([16, p.132]).

Lemma 1.6. *Let q be univalent in $\mathbb{E}$, and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$ with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\phi[q(z)], h(z) = \theta[q(z)] + Q(z)$, and suppose that either*
(i) h is convex, or
(ii) Q is starlike.
In addition, assume that
(iii) $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$ for all z in $\mathbb{E}$.
If p is analytic in $\mathbb{E}$ with $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$, and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)], \qquad z \in \mathbb{E},$$

then $p(z) \prec q(z)$ and q is the best dominant.

2. Main results

Theorem 2.1. *Let $q, q(z) \neq 0$ be a univalent function in $\mathbb{E}$ such that*
(i) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)}\right] > 0$,
(ii) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + \dfrac{p+\lambda}{\alpha}\right] > 0$.
If $f \in \mathcal{A}_p$ satisfies

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\left[1 + \alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)} - \frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)\right] \prec q(z) + \frac{\alpha}{p+\lambda}zq'(z), \tag{2.1}$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec q(z), \qquad z \in \mathbb{E},$$

and $q(z)$ is the best dominant, where $\lambda \geq 0, n \in \mathbb{N}_0$, and α is a nonzero complex number.

Proof. On writing $\dfrac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} = u(z)$, in (2.1), we obtain

$$u(z) + \frac{\alpha}{p+\lambda}zu'(z) \prec q(z) + \frac{\alpha}{p+\lambda}zq'(z).$$

Let us define the functions θ and ϕ as follows:

$$\theta(w) = w$$

and

$$\phi(w) = \frac{\alpha}{p+\lambda}.$$

Clearly, both the functions θ and ϕ are analytic in $\mathbb{D} = \mathbb{C}$ and $\phi(w) \neq 0$ in $\mathbb{D}$. Therefore,

$$Q(z) = \phi(q(z))zq'(z) = \frac{\alpha}{p+\lambda}zq'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = q(z) + \frac{\alpha}{p+\lambda} zq'(z).$$

On differentiating, we obtain $\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)}$ and

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \frac{p+\lambda}{\alpha}.$$

In view of the given conditions, we see that Q is starlike and $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$. Therefore, the proof, now follows from Lemma 1.6. □

Theorem 2.2. *Let $q, q(z) \neq 0$ be a univalent function in $\mathbb{E}$ such that*
(i) $\Re\left[1 + \frac{zq''(z)}{q'(z)}\right] > 0$,
(ii) $\Re\left[1 + \frac{zq''(z)}{q'(z)} + \frac{p+\lambda}{\alpha}\right] > 0$.
If $f \in \mathcal{A}_p$ satisfies

$$(1-\alpha)\frac{I_p(n,\lambda)[f](z)}{z^p} + \alpha\frac{I_p(n+1,\lambda)[f](z)}{z^p} \prec q(z) + \frac{\alpha}{p+\lambda} zq'(z), \tag{2.2}$$

then

$$\frac{I_p(n,\lambda)[f](z)}{z^p} \prec q(z), \qquad z \in \mathbb{E},$$

and $q(z)$ is the best dominant, where $\lambda \geq 0, n \in \mathbb{N}_0$, and α is a nonzero complex number .

Proof. On writing $\frac{I_p(n,\lambda)[f](z)}{z^p} = u(z)$. Equation 2.2, reduces to

$$u(z) + \frac{\alpha}{p+\lambda} zu'(z) \prec q(z) + \frac{\alpha}{p+\lambda} zq'(z).$$

Further, the proof follows on the same lines as in Theorem 2.1. □

Setting $\lambda = 0$ and $p = 1$ in Theorem 2.1 and Theorem 2.2, we, respectively, obtain the next two results for Salagean operator.

Theorem 2.3. *Let $q, q(z) \neq 0$ be a univalent function in $\mathbb{E}$ such that*
(i) $\Re\left[1 + \frac{zq''(z)}{q'(z)}\right] > 0$,
(ii) $\Re\left[1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\alpha}\right] > 0$.
If $f \in \mathcal{A}$ satisfies

$$\frac{D^{n+1}[f](z)}{D^n[f](z)}\left[1 + \alpha\left(\frac{D^{n+2}[f](z)}{D^{n+1}[f](z)} - \frac{D^{n+1}[f](z)}{D^n[f](z)}\right)\right] \prec q(z) + \alpha zq'(z),$$

then

$$\frac{D^{n+1}[f](z)}{D^n[f](z)} \prec q(z), \qquad z \in \mathbb{E},$$

and $q(z)$ is the best dominant, where $n \in \mathbb{N}_0$ and α is a nonzero complex number.

Theorem 2.4. *Let $q, q(z) \neq 0$ be a univalent function in $\mathbb{E}$ such that*
(i) $\Re\left[1 + \frac{zq''(z)}{q'(z)}\right] > 0,$
(ii) $\Re\left[1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\alpha}\right] > 0.$
If $f \in \mathcal{A}$ satisfies

$$(1-\alpha)\frac{D^n[f](z)}{z} + \alpha\frac{D^{n+1}[f](z)}{z} \prec q(z) + \alpha z q'(z),$$

then

$$\frac{D^n[f](z)}{z} \prec q(z), z \in \mathbb{E},$$

and $q(z)$ is the best dominant, where $n \in \mathbb{N}_0$ and α is a nonzero complex number.

3. Applications

Remark 3.1. When we select the dominant $q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$ in Theorems 2.1 and 2.2, a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}$$

$$1 + \frac{zq''(z)}{q'(z)} + \frac{p+\lambda}{\alpha} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} + \frac{p+\lambda}{\alpha}.$$

For a positive real number α, we note that q(z) satisfies the conditions (i) and (ii) of Theorems 2.1 and 2.2. Therefore, we immediately conclude the following results, respectively.

Theorem 3.2. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\left[1 + \alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)} - \frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)\right]$$
$$\prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$
$$+\frac{\alpha}{p+\lambda}\left(\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right),$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \qquad z \in \mathbb{E},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Theorem 3.3. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{I_p(n,\lambda)[f](z)}{z^p}+\alpha\frac{I_p(n+1,\lambda)[f](z)}{z^p} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\alpha}{p+\lambda}\left(\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right),$$

then

$$\frac{I_p(n,\lambda)[f](z)}{z^p} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \qquad z \in \mathbb{E},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.2, we have the following result.

Corollary 3.4. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf'(z)}{pf(z)}\left[1+\frac{\alpha}{p}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right] \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 +\frac{4\alpha\sqrt{z}}{p\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right), \quad z \in \mathbb{E},$$

then $f \in \mathcal{S}^p_{\mathcal{P}}$.

Setting $p = 1$ in above corollary, we get the following.

Corollary 3.5. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)}\left[1+\alpha\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right] \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right), \qquad z \in \mathbb{E},$$

then $f \in \mathcal{S}_{\mathcal{P}}$.

Setting $\lambda = 0, n = 1$ in Theorem 3.2, we obtain the following corollary.

Corollary 3.6. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\left[1+\frac{\alpha}{p}\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right)\right] \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{4\alpha\sqrt{z}}{p\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right), \qquad z \in \mathbb{E},$$

then $f \in UCV_p$.

Setting $p = 1$ in above corollary, we have the following result.

Corollary 3.7. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\left(1+\frac{zf''(z)}{f'(z)}\right)\left[1+\alpha\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right)\right] \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right), \qquad z \in \mathbb{E},$$

then $f \in UCV$.

Setting $\lambda = 0, n = 1$ in Theorem 3.3, we obtain the following.

Corollary 3.8. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{f'(z)}{pz^{p-1}}+\frac{\alpha}{p^2}\left[\frac{f'(z)}{z^{p-1}}+\frac{f''(z)}{z^{p-2}}\right] \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{4\alpha\sqrt{z}}{p\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right), \quad z \in \mathbb{E},$$

then $f \in UCC_p$.

Taking $p = 1$ in above corollary, we get the following result.

Corollary 3.9. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$f'(z) + \alpha z f''(z) \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right), \qquad z \in \mathbb{E},$$

then $f \in UCC$.

Remark 3.10. When we select the dominant $q(z) = \dfrac{1+(1-2\beta)z}{1-z}, 0 \le \beta < 1$ in Theorems 2.1 and 2.2, a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{1-z},$$

$$1 + \frac{zq''(z)}{q'(z)} + \frac{p+\lambda}{\alpha} = \frac{1+z}{1-z} + \frac{p+\lambda}{\alpha}.$$

For a positive real number α, we see that q(z) satisfies the conditions (i) and (ii) of Theorems 2.1 and 2.2. Therefore, we get the following results.

Theorem 3.11. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\left[1+\alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)} - \frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)\right] \prec \frac{1+(1-2\beta)z}{1-z} + \frac{2\alpha(1-\beta)z}{(p+\lambda)(1-z)^2},$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \frac{1+(1-2\beta)z}{1-z}, \qquad 0 \le \beta < 1, z \in \mathbb{E},$$

where $\lambda \ge 0, n \in \mathbb{N}_0$.

Theorem 3.12. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{I_p(n,\lambda)[f](z)}{z^p} + \alpha\frac{I_p(n+1,\lambda)[f](z)}{z^p} \prec \frac{1+(1-2\beta)z}{1-z} + \frac{2\alpha(1-\beta)z}{(p+\lambda)(1-z)^2},$$

then

$$\frac{I_p(n,\lambda)[f](z)}{z^p} \prec \frac{1+(1-2\beta)z}{1-z}, \qquad 0 \le \beta < 1, z \in \mathbb{E},$$

where $\lambda \ge 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.11, we obtain the following criterion for starlikeness.

Corollary 3.13. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf'(z)}{pf(z)}\left[1+\frac{\alpha}{p}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right] \prec \frac{1+(1-2\beta)z}{1-z}+\frac{2\alpha(1-\beta)z}{p(1-z)^2}, \qquad 0 \le \beta < 1, z \in \mathbb{E},$$

then $\frac{zf'(z)}{pf(z)} \prec \frac{1+(1-2\beta)z}{1-z}$*; that is,* $f \in \mathcal{S}_p^*(\gamma)$*, where* $\gamma = p\beta < p$.

Setting $p = 1$ in above corollary we have the following.

Corollary 3.14. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)}\left[1+\alpha\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right] \prec \frac{1+(1-2\beta)z}{1-z}+\frac{2\alpha(1-\beta)z}{(1-z)^2}, \qquad 0 \le \beta < 1, z \in \mathbb{E},$$

then $\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}$*; that is,* $f \in \mathcal{S}^*(\beta)$.

Setting $\lambda = 0, n = 1$ in Theorem 3.11, we obtain the following result of convexity.

Corollary 3.15. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\left[1+\frac{\alpha}{p}\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right)\right] \prec \frac{1+(1-2\beta)z}{1-z}+\frac{2\alpha(1-\beta)z}{p(1-z)^2}, \qquad 0 \le \beta < 1, z \in \mathbb{E},$$

then $\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+(1-2\beta)z}{1-z}$*; that is,* $f \in \mathcal{K}_p(\gamma)$*, where* $\gamma = p\beta < p$.

Selecting $p = 1$ in above corollary, we obtain:

Corollary 3.16. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\left(1+\frac{zf''(z)}{f'(z)}\right)\left[1+\alpha\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right)\right] \prec \frac{1+(1-2\beta)z}{1-z}+\frac{2\alpha(1-\beta)z}{(1-z)^2}, \qquad 0 \le \beta < 1, z \in \mathbb{E},$$

then $1+\frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\beta)z}{1-z}$*; that is,* $f \in \mathcal{K}(\beta)$.

Setting $\lambda = 0, n = 1$, and $\beta = 0$ in Theorem 3.12, we obtain the following result of close-to-convexity.

Corollary 3.17. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{f'(z)}{pz^{p-1}} + \frac{\alpha}{p^2}\left[\frac{f'(z)}{z^{p-1}} + \frac{f''(z)}{z^{p-2}}\right] \prec \frac{1+z}{1-z} + \frac{2\alpha z}{p(1-z)^2}, \qquad z \in \mathbb{E},$$

then $f \in \mathcal{C}_p$.

Setting $p = 1$ in above result, we have the following.

Corollary 3.18. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$f'(z) + \alpha z f''(z) \prec \frac{1+z}{1-z} + \frac{2\alpha z}{(1-z)^2}, \qquad z \in \mathbb{E},$$

then $f \in \mathcal{C}$.

Remark 3.19. When we select the dominant $q(z) = e^z$ in Theorems 2.1 and 2.2 , a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} = 1 + z,$$

$$1 + \frac{zq''(z)}{q'(z)} + \frac{p+\lambda}{\alpha} = 1 + z + \frac{p+\lambda}{\alpha}.$$

For a positive real number α, we note that q(z) satisfies the conditions (i) and (ii) of Theorems 2.1 and 2.2 . Therefore, we get the following results.

Theorem 3.20. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\left[1 + \alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)} - \frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)\right]$$
$$\prec e^z + \frac{\alpha}{p+\lambda} z e^z,$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec e^z, \qquad z \in \mathbb{E},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Theorem 3.21. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{I_p(n,\lambda)[f](z)}{z^p} + \alpha\frac{I_p(n+1,\lambda)[f](z)}{z^p} \prec e^z + \frac{\alpha}{p+\lambda} z e^z,$$

then

$$\frac{I_p(n,\lambda)[f](z)}{z^p} \prec e^z, \qquad z \in \mathbb{E},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.20, we obtain the following criterion for starlikeness.

Corollary 3.22. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf'(z)}{pf(z)}\left[1+\frac{\alpha}{p}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right] \prec e^z+\frac{\alpha z e^z}{p}, \qquad z \in \mathbb{E},$$

then $\frac{zf'(z)}{pf(z)} \prec e^z$*; that is,* $f \in \mathcal{S}_p^*$.

Setting $p = 1$ in above corollary, we have the following.

Corollary 3.23. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)}\left[1+\alpha\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right] \prec e^z+\alpha z e^z, \qquad z \in \mathbb{E},$$

then $\frac{zf'(z)}{f(z)} \prec e^z$*; that is,* $f \in \mathcal{S}^*$.

Setting $\lambda = 0, n = 1$ in Theorem 3.20, we obtain the following result of convexity.

Corollary 3.24. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\left[1+\frac{\alpha}{p}\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right)\right] \prec e^z+\frac{\alpha z e^z}{p}, \qquad z \in \mathbb{E},$$

then $\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec e^z$*; that is,* $f \in \mathcal{K}_p$.

Selecting $p = 1$ in above corollary, we obtain the following.

Corollary 3.25. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\left(1+\frac{zf''(z)}{f'(z)}\right)\left[1+\alpha\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right)\right] \prec e^z+\alpha z e^z, \qquad z \in \mathbb{E},$$

then $1+\frac{zf''(z)}{f'(z)} \prec e^z$*; that is,* $f \in \mathcal{K}$.

Setting $\lambda = 0, n = 1$ in Theorem 3.21, we get the following result of close-to-convexity.

Corollary 3.26. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{f'(z)}{pz^{p-1}}+\frac{\alpha}{p^2}\left[\frac{f'(z)}{z^{p-1}}+\frac{f''(z)}{z^{p-2}}\right] \prec e^z+\frac{\alpha}{p}ze^z, \qquad z \in \mathbb{E},$$

then $f \in \mathcal{C}_p$.

Setting $p = 1$ in above corollary, we get the following.

Corollary 3.27. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$f'(z)+\alpha z f''(z) \prec e^z+\alpha z e^z, \qquad z \in \mathbb{E},$$

then $f \in \mathcal{C}$.

Remark 3.28. When we select the dominant $q(z) = \dfrac{\alpha'(1-z)}{\alpha'-z}, 1 \le \alpha' < \dfrac{3}{2}$ in Theorems 2.1 and 2.2 , a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} = \frac{\alpha'+z}{\alpha'-z},$$

$$1 + \frac{zq''(z)}{q'(z)} + \frac{p+\lambda}{\alpha} = \frac{\alpha'+z}{\alpha'-z} + \frac{p+\lambda}{\alpha}.$$

For a positive real number α, we see that q(z) satisfies the conditions (i) and (ii) of Theorems 2.1 and 2.2. Therefore , we get the following results.

Theorem 3.29. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \left[1 + \alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)} - \frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)\right] \prec \frac{\alpha'(1-z)}{\alpha'-z} + \frac{\alpha}{(p+\lambda)}\frac{(\alpha'-\alpha'^2)z}{(\alpha'-z)^2},$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \frac{\alpha'(1-z)}{\alpha'-z}, \qquad 1 \le \alpha' < \frac{3}{2}, z \in \mathbb{E},$$

where $\lambda \ge 0, n \in \mathbb{N}_0$.

Theorem 3.30. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{I_p(n,\lambda)[f](z)}{z^p} + \alpha\frac{I_p(n+1,\lambda)[f](z)}{z^p} \prec \frac{\alpha'(1-z)}{\alpha'-z} + \frac{\alpha}{(p+\lambda)}\frac{(\alpha'-\alpha'^2)z}{(\alpha'-z)^2},$$

then

$$\frac{I_p(n,\lambda)[f](z)}{z^p} \prec \frac{\alpha'(1-z)}{\alpha'-z}, \qquad 1 \le \alpha' < \frac{3}{2}, z \in \mathbb{E},$$

where $\lambda \ge 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.29, we obtain the following criterion for starlikeness.

Corollary 3.31. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf'(z)}{pf(z)}\left[1 + \frac{\alpha}{p}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right] \prec \frac{\alpha'(1-z)}{\alpha'-z} + \frac{\alpha(\alpha'-\alpha'^2)z}{p(\alpha'-z)^2}, 1 \le \alpha' < \frac{3}{2}, \qquad z \in \mathbb{E},$$

then $\dfrac{zf'(z)}{pf(z)} \prec \dfrac{\alpha'(1-z)}{\alpha'-z}$; that is, $f \in \mathcal{S}_p^*$.

Setting $p = 1$ in above corollary, we have the following.

Corollary 3.32. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)}\left[1+\alpha\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right] \prec \frac{\alpha'(1-z)}{\alpha'-z} +\alpha\frac{(\alpha'-\alpha'^2)z}{(\alpha'-z)^2}, 1 \leq \alpha' < \frac{3}{2}, \qquad z \in \mathbb{E},$$

then $\frac{zf'(z)}{f(z)} \prec \frac{\alpha'(1-z)}{\alpha'-z}$*; that is,* $f \in \mathcal{S}^*$.

Setting $\lambda = 0, n = 1$ in Theorem 3.29, we obtain the following result of convexity.

Corollary 3.33. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\left[1+\frac{\alpha}{p}\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right)\right] \prec \frac{\alpha'(1-z)}{\alpha'-z}+\frac{\alpha(\alpha'-\alpha'^2)z}{p(\alpha'-z)^2}, \qquad 1 \leq \alpha' < \frac{3}{2}, z \in \mathbb{E},$$

then $\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{\alpha'(1-z)}{\alpha'-z}$*; that is,* $f \in \mathcal{K}_p$.

Selecting $p = 1$ in above corollary, we obtain the following.

Corollary 3.34. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\left(1+\frac{zf''(z)}{f'(z)}\right)\left[1+\alpha\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right)\right] \prec \frac{\alpha'(1-z)}{\alpha'-z}+\alpha\frac{(\alpha'-\alpha'^2)z}{(\alpha'-z)^2}, \qquad 1 \leq \alpha' < \frac{3}{2}, z \in \mathbb{E},$$

then $1+\frac{zf''(z)}{f'(z)} \prec \frac{\alpha'(1-z)}{\alpha'-z}$*; that is,* $f \in \mathcal{K}$.

Setting $\lambda = 0, n = 1$ in Theorem 3.30, we have the following result of close-to-convexity.

Corollary 3.35. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{f'(z)}{pz^{p-1}}+\frac{\alpha}{p^2}\left[\frac{f'(z)}{z^{p-1}}+\frac{f''(z)}{z^{p-2}}\right] \prec \frac{\alpha'(1-z)}{\alpha'-z}+\alpha\frac{(\alpha'-\alpha'^2)z}{p(\alpha'-z)^2}, \qquad 1 \leq \alpha' < \frac{3}{2}, z \in \mathbb{E},$$

then $f \in \mathcal{C}_p$.

Taking $p = 1$ in above corollary, we have:

Corollary 3.36. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$f'(z)+\alpha zf''(z) \prec \frac{\alpha'(1-z)}{\alpha'-z}+\alpha\frac{(\alpha'-\alpha'^2)z}{(\alpha'-z)^2}, \qquad 1 \leq \alpha' < \frac{3}{2}, z \in \mathbb{E},$$

then $f \in \mathcal{C}$.

Remark 3.37. When we select the dominant $q(z) = 1+az, 0 \leq a < 1$, in Theorems 2.1 and 2.2, a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} = 1,$$

$$1 + \frac{zq''(z)}{q'(z)} + \frac{p+\lambda}{\alpha} = 1 + \frac{p+\lambda}{\alpha}.$$

For a positive real number α, we note that q(z) satisfies the conditions (i) and (ii) of Theorems 2.1 and 2.2 . Therefore, we get the following results.

Theorem 3.38. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \left[1 + \alpha \left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)} - \frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)\right] \prec 1 + az + \frac{\alpha}{p+\lambda}az, \qquad 0 \leq a < 1,$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec 1 + az, \qquad z \in \mathbb{E},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Theorem 3.39. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{I_p(n,\lambda)[f](z)}{z^p} + \alpha\frac{I_p(n+1,\lambda)[f](z)}{z^p} \prec 1 + az + \frac{\alpha}{p+\lambda}az, \qquad 0 \leq a < 1,$$

then

$$\frac{I_p(n,\lambda)[f](z)}{z^p} \prec 1 + az, \qquad z \in \mathbb{E},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.38, we obtain the following criterion for starlikeness.

Corollary 3.40. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf'(z)}{pf(z)} \left[1 + \frac{\alpha}{p}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right] \prec 1 + \left(1 + \frac{\alpha}{p}\right)az, \qquad 0 \leq a < 1, z \in \mathbb{E},$$

then $\frac{zf'(z)}{pf(z)} \prec 1 + az$; that is, $f \in \mathcal{S}^*_p$.

Setting $p = 1$ in above corollary, we have the following.

Corollary 3.41. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)} \left[1 + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right] \prec 1 + (1+\alpha)az, \qquad 0 \leq a < 1, z \in \mathbb{E},$$

then $\frac{zf'(z)}{f(z)} \prec 1 + az$; that is, $f \in \mathcal{S}^*$.

Setting $\lambda = 0, n = 1$ in Theorem 3.38, we obtain the following result of convexity.

Corollary 3.42. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\left[1+\frac{\alpha}{p}\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right)\right]$$
$$\prec 1+\left(1+\frac{\alpha}{p}\right)az, \qquad 0 \le a < 1, z \in \mathbb{E},$$

then $\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+az$*; that is,* $f \in \mathcal{K}_p$.

Selecting $p = 1$ in above corollary, we obtain the following.

Corollary 3.43. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\left(1+\frac{zf''(z)}{f'(z)}\right)\left[1+\alpha\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right)\right]$$
$$\prec 1+(1+\alpha)az, \qquad 0 \le a < 1,$$

then $1+\frac{zf''(z)}{f'(z)} \prec 1+az, z \in \mathbb{E}$.

For $\lambda = 0, n = 1$, Theorem 3.39 gives the following sufficient condition for close-to-convexity.

Corollary 3.44. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{f'(z)}{pz^{p-1}}+\frac{\alpha}{p^2}\left[\frac{f'(z)}{z^{p-1}}+\frac{f''(z)}{z^{p-2}}\right] \prec 1+\left(1+\frac{\alpha}{p}\right)az, \qquad 0 \le a < 1, z \in \mathbb{E},$$

then $f \in \mathcal{C}_p$.

Setting $p = 1$ in above corollary, we have the following sufficient condition for close-to-convexity.

Corollary 3.45. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$f'(z)+\alpha zf''(z) \prec 1+(1+\alpha)az, \qquad 0 \le a < 1, z \in \mathbb{E},$$

then $f \in \mathcal{C}$.

Remark 3.46. When we select the dominant $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma, 0 < \gamma \le 1$ in Theorems 2.1 and 2.2, a little calculation yields that

$$1+\frac{zq''(z)}{q'(z)} = \frac{1+z^2+2\gamma z}{1-z^2},$$

$$1+\frac{zq''(z)}{q'(z)}+\frac{p+\lambda}{\alpha} = \frac{1+z^2+2\gamma z}{1-z^2}+\frac{p+\lambda}{\alpha}.$$

For a positive real number α, we note that q(z) satisfies the conditions (i) and (ii) of Theorems 2.1 and 2.2 . Therefore, we get the following results.

Theorem 3.47. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\left[1+\alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)}-\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)\right]$$

$$\prec\left(\frac{1+z}{1-z}\right)^{\gamma}\left[1+\frac{2\alpha\gamma z}{(p+\lambda)(1-z^2)}\right],\qquad 0<\gamma\leq 1,$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\prec\left(\frac{1+z}{1-z}\right)^{\gamma},\qquad z\in\mathbb{E},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Theorem 3.48. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{I_p(n,\lambda)[f](z)}{z^p}+\alpha\frac{I_p(n+1,\lambda)[f](z)}{z^p}$$

$$\prec\left(\frac{1+z}{1-z}\right)^{\gamma}\left[1+\frac{2\alpha\gamma z}{(p+\lambda)(1-z^2)}\right],\qquad 0<\gamma\leq 1,$$

then

$$\frac{I_p(n,\lambda)[f](z)}{z^p}\prec\left(\frac{1+z}{1-z}\right)^{\gamma},\qquad z\in\mathbb{E},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.47, we obtain the following criterion for strongly starlikeness.

Corollary 3.49. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf'(z)}{pf(z)}\left[1+\frac{\alpha}{p}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right]$$

$$\prec\left(\frac{1+z}{1-z}\right)^{\gamma}\left[1+\frac{2\alpha\gamma z}{p(1-z^2)}\right],\qquad 0<\gamma\leq 1, z\in\mathbb{E},$$

then $\dfrac{zf'(z)}{pf(z)}\prec\left(\dfrac{1+z}{1-z}\right)^{\gamma}$*; that is, $f\in\tilde{\mathcal{S}}_p(\gamma)$.*

Setting $p = 1$ in above corollary, we obtain the following.

Corollary 3.50. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)}\left[1+\alpha\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right]$$

$$\prec\left(\frac{1+z}{1-z}\right)^{\gamma}\left[1+\frac{2\alpha\gamma z}{1-z^2}\right],\qquad 0<\gamma\leq 1, z\in\mathbb{E},$$

then $\dfrac{zf'(z)}{f(z)}\prec\left(\dfrac{1+z}{1-z}\right)^{\gamma}$*; that is, $f\in\tilde{\mathcal{S}}(\gamma)$.*

Setting $\lambda = 0, n = 1$ in Theorem 3.47, we obtain the following result of convexity.

Corollary 3.51. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\left[1+\frac{\alpha}{p}\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right)\right]$$

$$\prec\left(\frac{1+z}{1-z}\right)^{\gamma}\left[1+\frac{2\alpha\gamma z}{p(1-z^2)}\right],\qquad 0<\gamma\leq 1, z\in\mathbb{E},$$

then $\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\left(\frac{1+z}{1-z}\right)^{\gamma}$*; that is,* $f\in\tilde{\mathcal{K}}_p(\gamma)$.

Selecting $p = 1$ in above corollary, we obtain the following.

Corollary 3.52. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\left(1+\frac{zf''(z)}{f'(z)}\right)\left[1+\alpha\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right)\right]$$

$$\prec\left(\frac{1+z}{1-z}\right)^{\gamma}\left[1+\frac{2\alpha\gamma z}{1-z^2}\right],\qquad 0<\gamma\leq 1, z\in\mathbb{E},$$

then $1+\frac{zf''(z)}{f'(z)}\prec\left(\frac{1+z}{1-z}\right)^{\gamma}$*; that is,* $f\in\tilde{\mathcal{K}}(\gamma)$.

Setting $\lambda = 0, n = 1$ in Theorem 3.48, we get the following result of close-to-convex functions.

Corollary 3.53. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{f'(z)}{pz^{p-1}}+\frac{\alpha}{p^2}\left[\frac{f'(z)}{z^{p-1}}+\frac{f''(z)}{z^{p-2}}\right]$$

$$\prec\left(\frac{1+z}{1-z}\right)^{\gamma}\left[1+\frac{2\alpha\gamma z}{p(1-z^2)}\right],\qquad 0<\gamma\leq 1, z\in\mathbb{E},$$

then $f\in\mathcal{C}_p$.

Setting $p = 1$ in above result, we have the following.

Corollary 3.54. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$f'(z)+\alpha zf''(z)\prec\left(\frac{1+z}{1-z}\right)^{\gamma}\left[1+\frac{2\alpha\gamma z}{1-z^2}\right],\qquad 0<\gamma\leq 1, z\in\mathbb{E},$$

then $f\in\mathcal{C}$.

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