ON CERTAIN RESULTS INVOLVING A MULTIPLIER TRANSFORMATION IN A PARABOLIC REGION

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Communicated by H.M. Srivastava

ABSTRACT. We, here, obtain certain results in subordination form involving a multiplier transformation in a parabolic region. In particular, using different dominants in our main result, we derive certain results on parabolic starlikeness, starlikeness, convexity, uniform convexity, strongly starlikeness, close-to-convexity, and uniform close-to-convexity of p-valent analytic functions as well as univalent analytic functions.

1. INTRODUCTION AND PRELIMINARIES

Let \( A_p \) denote the class of functions of the form

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}; z \in \mathbb{E})
\]

which are analytic and p-valent in the open unit disk \( \mathbb{E} = \{z : |z| < 1\} \). Obviously, \( A_1 = A \), the class of all analytic functions \( f \), normalized by the conditions \( f(0) = f'(0) - 1 = 0 \). Let the functions \( f \) and \( g \) be analytic in \( \mathbb{E} \). We say that \( f \) is subordinate to \( g \) in \( \mathbb{E} \) (written as \( f \prec g \)), if there exists a Schwartz function \( \phi \) in \( \mathbb{E} \) (i.e. \( \phi \) is regular in \( |z| < 1 \), \( \phi(0) = 0 \) and \( |\phi(z)| \leq |z| < 1 \)) such that

\[
f(z) = g(\phi(z)), \quad |z| < 1.
\]

Let \( \Phi : \mathbb{C}^2 \times \mathbb{E} \rightarrow \mathbb{C} \) be an analytic function, let \( p \) be an analytic function in \( \mathbb{E} \) with \( (p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E} \) for all \( z \in \mathbb{E} \), and let \( h \) be univalent in \( \mathbb{E} \). Then

Date: Received: 29 November 2017; Revised: 14 March 2018; Accepted: 19 March 2018.
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2010 Mathematics Subject Classification. Primary 30C80; Secondary 30C45.

Key words and phrases. Analytic function, parabolic starlike function, uniformly convex function, differential subordination, multiplier transformation.
the function \( p \) is said to satisfy first order differential subordination if
\[
\Phi(p(z), zp'(z); z) \prec h(z), \quad \Phi(p(0), 0; 0) = h(0). \tag{1.1}
\]
A univalent function \( q \) is called a dominant of the differential subordination (1.1) if \( p(0) = q(0) \) and \( p(z) \prec q(z) \) for all \( p \) satisfying (1.1). A dominant \( \tilde{q} \) that satisfies \( \tilde{q}(z) \prec q(z) \) for all dominants \( q \) of (1.1), is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of \( \mathbb{E} \).

A univalent function \( q \) is called a dominant of the differential subordination (1.1) if \( p(0) = q(0) \) and \( p(z) \prec q(z) \) for all \( p \) satisfying (1.1). A dominant \( \tilde{q} \) that satisfies \( \tilde{q}(z) \prec q(z) \) for all dominants \( q \) of (1.1), is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of \( \mathbb{E} \).

A function \( f \in \mathcal{A}_p \) is said to be \( p \)-valent starlike of order \( \alpha \) \((0 \leq \alpha < p)\) in \( \mathbb{E} \) if
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in \mathbb{E}. \tag{1.2}
\]

Let \( \mathcal{S}_p^*(\alpha) \) denote the class of \( p \)-valent starlike functions of order \( \alpha \). Write \( \mathcal{S}_p^*(0) = \mathcal{S}_p^* \), the class of \( p \)-valent starlike functions.

A function \( f \in \mathcal{A}_p \) is said to be \( p \)-valent convex of order \( \alpha \) \((0 \leq \alpha < p)\) in \( \mathbb{E} \) if it satisfies the condition
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in \mathbb{E}. \tag{1.3}
\]

Let the class of such functions be denoted by \( \mathcal{K}_p(\alpha) \). Let \( \mathcal{K}_p(0) = \mathcal{K}_p \), the class of \( p \)-valent convex functions.

A function \( f \in \mathcal{A}_p \) is said to be \( p \)-valent close-to-convex in \( \mathbb{E} \), if it satisfies
\[
\Re \left( \frac{zf'(z)}{g(z)} \right) > 0, \quad z \in \mathbb{E}, \tag{1.4}
\]

for some \( g \in \mathcal{S}_p^* \). Let the class of such functions be denoted by \( \mathcal{C}_p \). Note that \( \mathcal{C}_1 = \mathcal{C} \). Select \( g(z) \equiv z^p \in \mathcal{A}_p \). Note that it satisfies condition (1.2). Therefore, condition (1.3), becomes
\[
\Re \left( \frac{f'(z)}{z^p-1} \right) > 0, \quad z \in \mathbb{E}. \tag{1.4}
\]

Hence, a function \( f \in \mathcal{A}_p \) is said to be \( p \)-valent close-to-convex in \( \mathbb{E} \), if it satisfies condition (1.4).

A function \( f \in \mathcal{A}_p \) is said to be \( p \)-valent strongly starlike of order \( \alpha \), \( 0 < \alpha \leq 1 \), if
\[
\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\alpha \pi}{2}, \quad z \in \mathbb{E}, \tag{1.5}
\]
or, equivalently,
\[
\frac{zf'(z)}{f(z)} \prec \left( \frac{1 + z}{1 - z} \right)^\alpha, \quad z \in \mathbb{E}.
\]

Let \( \mathcal{S}_p^*(\alpha) \) denote the class of \( p \)-valent strongly starlike functions of order \( \alpha \). Note that \( \mathcal{S}_p^*(1) \equiv \mathcal{S}_p^* \).

A function \( f \in \mathcal{A}_p \) is said to be \( p \)-valent strongly convex of order \( \alpha \), \( 0 < \alpha \leq 1 \), if
\[
\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\alpha \pi}{2}, \quad z \in \mathbb{E}, \tag{1.6}
\]
or, equivalently,

\[ 1 + \frac{zf''(z)}{f'(z)} \prec \left( \frac{1 + z}{1 - z} \right)^\alpha, \quad z \in \mathbb{E}. \]

Let \( \tilde{K}_p(\alpha) \) denote the class of \( p \)-valent strongly convex functions of order \( \alpha \). Note that \( \tilde{K}_p(1) \equiv K_p \).

A function \( f \in A_p \) is said to be \( p \)-valent parabolic starlike in \( \mathbb{E} \) if

\[ \Re \left( \frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - p \right|, \quad z \in \mathbb{E}. \tag{1.7} \]

Let \( S^p_\mathbb{P} \) denote the class of \( p \)-valent parabolic starlike functions. Write \( S^1_\mathbb{P} = S_\mathbb{P} \), the class of parabolic starlike functions.

A function \( f \in A_p \) is said to be uniformly \( p \)-valent convex in \( \mathbb{E} \) if

\[ \Re \left( 1 + \frac{zf''(z)}{f'(z)} f'(z) \right) > \left| \frac{zf''(z)}{f'(z)} - (p - 1) \right|, \quad z \in \mathbb{E}, \tag{1.8} \]

and is denoted by \( UCV_p \), the class of uniformly \( p \)-valent convex functions, and let \( UCV_1 = UCV \), the class of uniformly convex functions.

A function \( f \in A_p \) is said to be \( p \)-valent uniformly close-to-convex in \( \mathbb{E} \), if

\[ \Re \left( \frac{zf'(z)}{g(z)} \right) > \left| \frac{zf'(z)}{g(z)} - p \right|, \quad z \in \mathbb{E}, \tag{1.9} \]

for some \( g \in S^p_\mathbb{P} \). Let \( UCC_p \) denote the class of all such functions. Let \( UCC_1 = UCC \). Note that the function \( g(z) \equiv z^p \in S^p_\mathbb{P} \). Therefore, for \( g(z) \equiv z^p \), condition (1.9) becomes

\[ \Re \left( \frac{f'(z)}{z^{p-1}} \right) > \left| \frac{f'(z)}{z^{p-1}} - p \right|, \quad z \in \mathbb{E}. \tag{1.10} \]

Ronning [18] and Ma and Minda [15] studied the domain \( \Omega \), and the function \( q(z) \) is defined below:

\[ \Omega = \{ u + iv : u > \sqrt{(u - 1)^2 + v^2} \}. \]

Clearly the function

\[ q(z) = 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \tag{1.11} \]

maps the unit disk \( \mathbb{E} \) onto the domain \( \Omega \). Hence the conditions (1.7), (1.8), and (1.10) are equivalent to

\[ \frac{1}{p} \left( \frac{zf'(z)}{f(z)} \right) \prec q(z), \quad z \in \mathbb{E}, \]

\[ \frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec q(z), \quad z \in \mathbb{E}, \]

and

\[ \frac{f'(z)}{pz^{p-1}} \prec q(z), \]
respectively, where \( q(z) \) is given by (1.11).

For \( f \in \mathcal{A}_p \), we define the multiplier transformation \( I_p(n, \lambda) \) as

\[
I_p(n, \lambda)[f](z) = z^n \sum_{k=0}^{\infty} \left( \frac{k + \lambda}{p + \lambda} \right)^n a_k z^k, \quad \text{where } \lambda \geq 0, n \in \mathbb{Z}.
\]

The operator \( I_1(n, 0) \) is the well-known Sălăgean [19] derivative operator \( D^n f(z) = z + \sum_{k=0}^{\infty} k^n a_k z^k, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( f \in \mathcal{A} \). In 1992, Uralegaddi and Somanatha [25] investigated the operator \( I_1(n, 1) \). In 1993, Jung et al. [13] studied the transformation

\[
I_1(-1, \lambda)[f](z) = z + \sum_{k=2}^{\infty} \left( \frac{1 + \lambda}{k + \lambda} \right) a_k z^k, \quad \text{where } \lambda > -1, f \in \mathcal{A}.
\]

In 2003, Cho and Srivastava [9] and Cho and Kim [8] investigated the operator \( I_1(n, \lambda) \). In 2005, Aghalary et al. [1] studied the operator \( I_p(n, \lambda) \), whereas Uralegaddi and Somanatha [25] studied the operator \( I_1(n, 1) \). Recently, Billing [2, 3, 4, 5, 6], Singh et al. [20, 21], Brar and Billing [7] investigated the operator \( I_p(n, \lambda) \) and obtained certain sufficient conditions for starlike and convex functions.

Let \( S_n(\alpha) \) denote the class of functions \( f \in \mathcal{A} \) for which

\[
\Re \left( \frac{D^{n+1}[f](z)}{D^n[f](z)} \right) > \alpha, \quad z \in \mathbb{E}, 0 \leq \alpha < 1.
\]

In 1989, Owa, Shen and Obradović [17] studied this class and proved the following result.

**Theorem 1.1.** For \( n \in \mathbb{N}_0 \), if \( f \in \mathcal{A} \) satisfies

\[
\left| \frac{D^{n+1}[f](z)}{D^n[f](z)} - 1 \right|^{1-\beta} \left| \frac{D^{n+2}[f](z)}{D^{n+1}[f](z)} - 1 \right|^{\beta} < (1-\alpha)^{1-2\beta} \left( 1 - \frac{3}{2} \alpha + \alpha^2 \right)^{\beta}, \quad z \in \mathbb{E},
\]

for some \( \alpha (0 \leq \alpha \leq 1/2) \) and \( \beta (0 \leq \beta \leq 1) \), then \( \Re \left( \frac{D^{n+1}[f](z)}{D^n[f](z)} \right) > \alpha \); that is, \( f \in S_n(\alpha) \).

Later on, Li and Owa [14] extended this result by proving the following result.

**Theorem 1.2.** For \( n \in \mathbb{N}_0 \), if \( f \in \mathcal{A} \) satisfies

\[
\left| \frac{D^{n+1}[f](z)}{D^n[f](z)} - 1 \right|^{\gamma} \left| \frac{D^{n+2}[f](z)}{D^{n+1}[f](z)} - 1 \right|^{\beta} < \begin{cases} (1-\alpha)^{(\frac{3}{2}-\alpha)\beta}, & 0 \leq \alpha \leq \frac{1}{2}, \\ 2^{\beta}(1-\alpha)^{\beta+\gamma}, & \frac{1}{2} \leq \alpha < 1, \end{cases}
\]

for some \( \alpha (0 \leq \alpha < 1), \beta \geq 0 \) and \( \gamma \geq 0 \) with \( \beta + \gamma > 0 \), then \( f \in S_n(\alpha), n \in \mathbb{N}_0 \).

Let \( S_n(p, \lambda, \alpha) \) denote the class of functions \( f \in \mathcal{A}_p \) for which

\[
\Re \left( \frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \right) > \frac{\alpha}{p}, \quad z \in \mathbb{E}, 0 \leq \alpha < p.
\]
In 2008, Singh et al. [20] investigated the above class and proved the following sufficient condition for a multivalent function to be a member of this class.

**Theorem 1.3.** Let \( f \in \mathcal{A}_p \) satisfy
\[
\left| \frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} - 1 \right|^\gamma \left| \frac{I_p(n+2, \lambda)[f](z)}{I_p(n+1, \lambda)[f](z)} - 1 \right|^\beta < M(p, \lambda, \alpha, \beta, \gamma), \quad z \in \mathbb{E},
\]
for some real numbers \( \alpha, \beta, \) and \( \gamma \) such that \( 0 \leq \alpha < p, \beta \geq 0, \gamma \geq 0, \beta + \gamma > 0; \) then \( f \in S_n(p, \lambda, \alpha), \) where \( n \in \mathbb{N}_0 \) and
\[
M(p, \lambda, \alpha, \beta, \gamma) = \begin{cases} (1 - \frac{\alpha}{p})^\gamma \left( 1 - \frac{\alpha}{p} + \frac{1}{2(p+\lambda)} \right)^\beta, & 0 \leq \alpha \leq \frac{p}{2}, \\ (1 - \frac{\alpha}{p})^{\gamma+\beta} \left( 1 + \frac{1}{(p+\lambda)} \right)^\beta, & \frac{p}{2} \leq \alpha < p. \end{cases}
\]

Recently, Billing [3, 5] also proved the following results and obtained sufficient conditions for starlikeness and convexity of univalent functions.

**Theorem 1.4.** Let \( \alpha, \beta \) be real numbers such that \( \alpha > \frac{2}{1-\beta}, 0 \leq \beta < 1, \) and let
\[
0 < M \equiv M(\alpha, \beta, \gamma, p) = \frac{(\alpha + p + \lambda)[\alpha(1-\beta)] - 2}{\alpha[1 + (1-\beta)(p+\lambda)]}.
\]
If \( f \in \mathcal{A}_p \) satisfies the differential inequality
\[
\left| (1 - \alpha) \frac{I_p(n, \lambda)f(z)}{z^p} + \alpha \frac{I_p(n+1, \lambda)f(z)}{z^p} - 1 \right| < M(\alpha, \beta, \gamma, p),
\]
then
\[
\Re \left( \frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \right) > \beta, \quad z \in \mathbb{E}.
\]

**Theorem 1.5.** Let \( \alpha \) be a nonzero complex number such that \( \Re(\alpha) > 0, \) and let \( h(z) \) be analytic and convex in \( \mathbb{E} \) with \( h(0) = 1. \) If \( f \in \mathcal{A}_p \) satisfies
\[
\frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \left\{ 1 + \alpha \left( \frac{I_p(n+2, \lambda)[f](z)}{I_p(n+1, \lambda)[f](z)} - \frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \right) \right\} < \frac{p + \lambda}{\alpha} \int_0^1 t^{\frac{\mu+\lambda}{\alpha} - 1} h(zt)dt,
\]
then
\[
\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} < h(z), \quad z \in \mathbb{E}.
\]

For more references see [10, 11, 12, 22, 23, 24]. In the present paper, we study a differential subordination involving multiplier transformation \( I_p(n, \lambda) \) defined above. In particular cases to our main result, we obtain sufficient conditions for starlikeness, convexity, and close-to-convexity of multivalent and univalent analytic functions in a parabolic region.

To prove our main results, we shall use the following lemma of Miller and Mocanu ([16, p.132]).
Lemma 1.6. Let $q$ be univalent in $E$, and let $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(E)$ with $\phi(w) \neq 0$, when $w \in q(E)$. Set $Q(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$, and suppose that either

(i) $h$ is convex, or
(ii) $Q$ is starlike.

In addition, assume that

(iii) $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for all $z$ in $E$.

If $p$ is analytic in $E$ with $p(0) = q(0)$, $p(E) \subset D$, and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)], \quad z \in E,$$

then $p(z) \prec q(z)$ and $q$ is the best dominant.

2. Main results

Theorem 2.1. Let $q, q(z) \neq 0$ be a univalent function in $E$ such that

(i) $\Re\left[1 + \frac{zq''(z)}{q'(z)}\right] > 0$,
(ii) $\Re\left[1 + \frac{zq''(z)}{q'(z)} + \frac{p + \lambda}{\alpha}\right] > 0$.

If $f \in A_p$ satisfies

$$\frac{I_p(n + 1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \left[1 + \alpha\left(\frac{I_p(n + 2, \lambda)[f](z)}{I_p(n + 1, \lambda)[f](z)} - \frac{I_p(n + 1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)}\right)\right] \prec q(z) + \frac{\alpha}{p + \lambda} zq'(z),$$

(2.1)

then

$$\frac{I_p(n + 1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \prec q(z), \quad z \in E,$$

and $q(z)$ is the best dominant, where $\lambda \geq 0, n \in \mathbb{N}_0$, and $\alpha$ is a nonzero complex number.

Proof. On writing $\frac{I_p(n + 1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} = u(z)$, in (2.1), we obtain

$$u(z) + \frac{\alpha}{p + \lambda} zu'(z) \prec q(z) + \frac{\alpha}{p + \lambda} zq'(z).$$

Let us define the functions $\theta$ and $\phi$ as follows:

$$\theta(w) = w$$

and

$$\phi(w) = \frac{\alpha}{p + \lambda}.$$

Clearly, both the functions $\theta$ and $\phi$ are analytic in $D = \mathbb{C}$ and $\phi(w) \neq 0$ in $D$. Therefore,

$$Q(z) = \phi(q(z))zq'(z) = \frac{\alpha}{p + \lambda} zq'(z).$$
and

\[ h(z) = \theta(q(z)) + Q(z) = q(z) + \frac{\alpha}{p + \lambda} z q'(z). \]

On differentiating, we obtain

\[ \frac{z Q'(z)}{Q(z)} = 1 + \frac{z q''(z)}{q'(z)} \]

and

\[ \frac{z h'(z)}{Q(z)} = 1 + \frac{z q''(z)}{q'(z)} + \frac{p + \lambda}{\alpha}. \]

In view of the given conditions, we see that Q is starlike and \( \Re \left( \frac{z h'(z)}{Q(z)} \right) > 0 \).

Therefore, the proof, now follows from Lemma 1.6. \( \square \)

**Theorem 2.2.** Let \( q, q(z) \neq 0 \) be a univalent function in \( E \) such that

(i) \( \Re \left[ 1 + \frac{z q''(z)}{q'(z)} \right] > 0, \)

(ii) \( \Re \left[ 1 + \frac{z q''(z)}{q'(z)} + \frac{p + \lambda}{\alpha} \right] > 0. \)

If \( f \in A \) satisfies

\[ (1 - \alpha) \frac{I_p(n, \lambda)[f](z)}{z^p} + \alpha \frac{I_p(n + 1, \lambda)[f](z)}{z^p} < q(z) + \frac{\alpha}{p + \lambda} z q'(z), \]

then

\[ \frac{I_p(n, \lambda)[f](z)}{z^p} < q(z), \quad z \in E, \]

and \( q(z) \) is the best dominant, where \( \lambda \geq 0, n \in \mathbb{N}_0, \) and \( \alpha \) is a nonzero complex number.

**Proof.** On writing \( \frac{I_p(n, \lambda)[f](z)}{z^p} = u(z) \). Equation 2.2, reduces to

\[ u(z) + \frac{\alpha}{p + \lambda} z u'(z) < q(z) + \frac{\alpha}{p + \lambda} z q'(z). \]

Further, the proof follows on the same lines as in Theorem 2.1. \( \square \)

Setting \( \lambda = 0 \) and \( p = 1 \) in Theorem 2.1 and Theorem 2.2, we, respectively, obtain the next two results for Salagean operator.

**Theorem 2.3.** Let \( q, q(z) \neq 0 \) be a univalent function in \( E \) such that

(i) \( \Re \left[ 1 + \frac{z q''(z)}{q'(z)} \right] > 0, \)

(ii) \( \Re \left[ 1 + \frac{z q''(z)}{q'(z)} + \frac{1}{\alpha} \right] > 0. \)

If \( f \in A \) satisfies

\[ \frac{D^{n+1}[f](z)}{D^n[f](z)} \left[ 1 + \alpha \left( \frac{D^{n+2}[f](z)}{D^{n+1}[f](z)} - \frac{D^{n+1}[f](z)}{D^n[f](z)} \right) \right] < q(z) + \alpha z q'(z), \]

then

\[ \frac{D^{n+1}[f](z)}{D^n[f](z)} < q(z), \quad z \in E, \]
and \( q(z) \) is the best dominant, where \( n \in \mathbb{N}_0 \) and \( \alpha \) is a nonzero complex number.

**Theorem 2.4.** Let \( q, q(z) \neq 0 \) be a univalent function in \( E \) such that

(i) \( \Re \left[ 1 + \frac{z q''(z)}{q'(z)} \right] > 0, \)

(ii) \( \Re \left[ 1 + \frac{z q''(z)}{q'(z)} + \frac{1}{\alpha} \right] > 0. \)

If \( f \in A \) satisfies

\[
(1 - \alpha) \frac{D^n[f](z)}{z} + \frac{\alpha D^{n+1}[f](z)}{z} < q(z) + \alpha z q'(z),
\]

then

\[
\frac{D^n[f](z)}{z} < q(z), z \in E,
\]

and \( q(z) \) is the best dominant, where \( n \in \mathbb{N}_0 \) and \( \alpha \) is a nonzero complex number.

### 3. Applications

**Remark 3.1.** When we select the dominant \( q(z) = 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \) in Theorems 2.1 and 2.2, a little calculation yields that

\[
1 + \frac{z q''(z)}{q'(z)} = \frac{1 + z}{2(1 - z)} + \frac{\sqrt{z}}{(1 - z) \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)} + \frac{p + \lambda}{\alpha}.
\]

For a positive real number \( \alpha \), we note that \( q(z) \) satisfies the conditions (i) and (ii) of Theorems 2.1 and 2.2. Therefore, we immediately conclude the following results, respectively.

**Theorem 3.2.** Let \( \alpha \) be a positive real number. If \( f \in A_p \) satisfies

\[
\frac{I_p(n + 1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \left[ 1 + \alpha \left( \frac{I_p(n + 2, \lambda)[f](z)}{I_p(n + 1, \lambda)[f](z)} - \frac{I_p(n + 1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \right) \right]
\]

\[
< 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2
\]

\[
+ \frac{\alpha}{p + \lambda} \left( \frac{4 \sqrt{z}}{\pi^2(1 - z) \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)} \right),
\]

then

\[
\frac{I_p(n + 1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in E,
\]

where \( \lambda \geq 0, n \in \mathbb{N}_0 \).
Theorem 3.3. Let $\alpha$ be a positive real number. If $f \in A_p$ satisfies

\[
(1-\alpha) \frac{I_p(n, \lambda)[f](z)}{z^p} + \alpha \frac{I_p(n + 1, \lambda)[f](z)}{z^p}
\]

\[
< 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 + \frac{\alpha}{p + \lambda} \left( \frac{4\sqrt{z}}{\pi^2(1-z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right),
\]

then

\[
\frac{I_p(n, \lambda)[f](z)}{z^p} < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in E,
\]

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.2, we have the following result.

Corollary 3.4. Let $\alpha$ be a positive real number. If $f \in A_p$ satisfies

\[
\frac{zf'(z)}{pf(z)} \left[ 1 + \alpha \left( 1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)} \right) \right] < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2
\]

\[
+ \frac{4\alpha\sqrt{z}}{p\pi^2(1-z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right), \quad z \in E,
\]

then $f \in S_p^n$.

Setting $p = 1$ in above corollary, we get the following.

Corollary 3.5. Let $\alpha$ be a positive real number. If $f \in A$ satisfies

\[
\frac{zf'(z)}{f(z)} \left[ 1 + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right] < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 + \frac{4\alpha\sqrt{z}}{\pi^2(1-z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right), \quad z \in E,
\]

then $f \in S_p$.

Setting $\lambda = 0, n = 1$ in Theorem 3.2, we obtain the following corollary.

Corollary 3.6. Let $\alpha$ be a positive real number. If $f \in A_p$ satisfies

\[
\frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \left[ 1 + \alpha \left( 2zf''(z) + z^2f'''(z) - \frac{zf''(z)}{f'(z)} \right) \right]
\]

\[
< 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 + \frac{4\alpha\sqrt{z}}{p\pi^2(1-z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right), \quad z \in E,
\]

then $f \in UCV_p$.

Setting $p = 1$ in above corollary, we have the following result.

Corollary 3.7. Let $\alpha$ be a positive real number. If $f \in A$ satisfies

\[
\left( 1 + \frac{zf''(z)}{f'(z)} \right) \left[ 1 + \alpha \left( 2zf''(z) + z^2f'''(z) - \frac{zf''(z)}{f'(z)} \right) \right]
\]

\[
< 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 + \frac{4\alpha\sqrt{z}}{\pi^2(1-z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right), \quad z \in E,
\]

then $f \in UCV$. 
Corollary 3.8. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A}_p \) satisfies
\[
(1-\alpha) \frac{f'(z)}{pz^{p-1}} + \frac{\alpha}{p^2} \left[ \frac{f'(z)}{z^{p-1}} + \frac{f''(z)}{z^{p-2}} \right] \prec 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2 + \frac{4\alpha \sqrt{z}}{p\pi^2(1-z)} \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right), \quad z \in \mathbb{E},
\]
then \( f \in \mathcal{UCC}_p \).

Taking \( p = 1 \) in above corollary, we get the following result.

Corollary 3.9. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A} \) satisfies
\[
f'(z) + \alpha z f''(z) \prec 1 + \frac{(1-2\beta)z}{1-z}, \quad 0 \leq \beta < 1, \quad z \in \mathbb{E},
\]
then \( f \in \mathcal{UCC} \).

Remark 3.10. When we select the dominant \( q(z) = \frac{1+(1-2\beta)z}{1-z}, 0 \leq \beta < 1 \) in Theorems 2.1 and 2.2, a little calculation yields that
\[
1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{1-z},
\]
\[
1 + \frac{zq''(z)}{q'(z)} + \frac{p+\lambda}{\alpha} = \frac{1+z}{1-z} + \frac{p+\lambda}{\alpha}.
\]
For a positive real number \( \alpha \), we see that \( q(z) \) satisfies the conditions (i) and (ii) of Theorems 2.1 and 2.2. Therefore, we get the following results.

Theorem 3.11. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A}_p \) satisfies
\[
\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \left[ 1 + \alpha \left( \frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)} - \frac{I_p(n,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \right) \right] \prec \frac{1+(1-2\beta)z}{1-z} + \frac{2\alpha(1-\beta)z}{(p+\lambda)(1-z)^2},
\]
then
\[
\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \frac{1+(1-2\beta)z}{1-z}, \quad 0 \leq \beta < 1, \quad z \in \mathbb{E},
\]
where \( \lambda \geq 0, n \in \mathbb{N}_0 \).

Theorem 3.12. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A}_p \) satisfies
\[
(1-\alpha) \frac{I_p(n,\lambda)[f](z)}{z^p} + \alpha \frac{I_p(n+1,\lambda)[f](z)}{z^p} \prec \frac{1+(1-2\beta)z}{1-z} + \frac{2\alpha(1-\beta)z}{(p+\lambda)(1-z)^2},
\]
then
\[ \frac{I_p(n, \lambda)[f](z)}{z^p} < \frac{1 + (1 - 2\beta)z}{1 - z}, \quad 0 \leq \beta < 1, z \in \mathbb{E}, \]
where \( \lambda \geq 0, n \in \mathbb{N}_0. \)

Setting \( \lambda = n = 0 \) in Theorem 3.11, we obtain the following criterion for starlikeness.

**Corollary 3.13.** Let \( \alpha \) be a positive real number. If \( f \in A_p \) satisfies
\[ \frac{zf'(z)}{pf(z)} \left[ 1 + \frac{\alpha}{p} \left( 1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)} \right) \right] < \frac{1 + (1 - 2\beta)z}{1 - z} + \frac{2\alpha(1 - \beta)z}{p(1 - z)^2}, \quad 0 \leq \beta < 1, z \in \mathbb{E}, \]
then \( \frac{zf'(z)}{pf(z)} \left[ 1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)} \right] < 1 + (1 - 2\beta)\frac{z}{1 - z}; \) that is, \( f \in S^*_p(\gamma), \) where \( \gamma = p\beta < p. \)

Setting \( p = 1 \) in above corollary we have the following.

**Corollary 3.14.** Let \( \alpha \) be a positive real number. If \( f \in A \) satisfies
\[ \frac{zf'(z)}{f(z)} \left[ 1 + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)} \right) \right] < \frac{1 + (1 - 2\beta)z}{1 - z} + \frac{2\alpha(1 - \beta)z}{(1 - z)^2}, \quad 0 \leq \beta < 1, z \in \mathbb{E}, \]
then \( \frac{zf'(z)}{f(z)} \left[ 1 + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)} \right) \right] < 1 + (1 - 2\beta)\frac{z}{1 - z}; \) that is, \( f \in S^*(\beta). \)

Setting \( \lambda = 0, n = 1 \) in Theorem 3.11, we obtain the following result of convexity.

**Corollary 3.15.** Let \( \alpha \) be a positive real number. If \( f \in A_p \) satisfies
\[ \frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \left[ 1 + \frac{\alpha}{p} \left( 2zf'(z) + z^2zf''(z) \right) \frac{zf''(z)}{f'(z)} - \frac{zf''(z)}{f'(z)} \right] < \frac{1 + (1 - 2\beta)z}{1 - z} + \frac{2\alpha(1 - \beta)z}{p(1 - z)^2}, \quad 0 \leq \beta < 1, z \in \mathbb{E}, \]
then \( \frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < 1 + (1 - 2\beta)\frac{z}{1 - z}; \) that is, \( f \in K_p(\gamma), \) where \( \gamma = p\beta < p. \)

Selecting \( p = 1 \) in above corollary, we obtain:

**Corollary 3.16.** Let \( \alpha \) be a positive real number. If \( f \in A \) satisfies
\[ \left( 1 + \frac{zf''(z)}{f'(z)} \right) \left[ 1 + \alpha \left( 2zf''(z) + z^2zf''(z) \right) \frac{zf''(z)}{f'(z)} - \frac{zf''(z)}{f'(z)} \right] < \frac{1 + (1 - 2\beta)z}{1 - z} + \frac{2\alpha(1 - \beta)z}{(1 - z)^2}, \quad 0 \leq \beta < 1, z \in \mathbb{E}, \]
then \( 1 + \frac{zf''(z)}{f'(z)} < 1 + (1 - 2\beta)\frac{z}{1 - z}; \) that is, \( f \in K(\beta). \)
Setting $\lambda = 0, n = 1$, and $\beta = 0$ in Theorem 3.12, we obtain the following result of close-to-convexity.

**Corollary 3.17.** Let $\alpha$ be a positive real number. If $f \in A_p$ satisfies
\[
(1 - \alpha) \frac{f'(z)}{p z^{p-1}} + \frac{\alpha}{p^2} \left[ \frac{f''(z)}{z^{p-1}} + \frac{f'(z)}{z^{p-2}} \right] < \frac{1 + z}{1 - z} + \frac{2 \alpha z}{p(1 - z)^2}, \quad z \in \mathbb{E},
\]
then $f \in C_p$.

Setting $p = 1$ in above result, we have the following.

**Corollary 3.18.** Let $\alpha$ be a positive real number. If $f \in A$ satisfies
\[
f'(z) + \alpha zf''(z) < 1 + z - z + 2 \alpha z (1 - z)^2, \quad z \in \mathbb{E},
\]
then $f \in C$.

**Remark 3.19.** When we select the dominant $q(z) = e^z$ in Theorems 2.1 and 2.2, a little calculation yields that
\[
1 + \frac{z q''(z)}{q'(z)} = 1 + z,
\]
\[
1 + \frac{z q''(z)}{q'(z)} + \frac{p + \lambda}{\alpha} = 1 + z + \frac{p + \lambda}{\alpha}.
\]

For a positive real number $\alpha$, we note that $q(z)$ satisfies the conditions (i) and (ii) of Theorems 2.1 and 2.2. Therefore, we get the following results.

**Theorem 3.20.** Let $\alpha$ be a positive real number. If $f \in A_p$ satisfies
\[
\frac{I_p(n + 1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \left[ 1 + \alpha \left( \frac{I_p(n + 2, \lambda)[f](z)}{I_p(n + 1, \lambda)[f](z)} \right) \right] < e^z + \frac{\alpha}{p + \lambda} z e^z,
\]
then
\[
\frac{I_p(n + 1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} < e^z, \quad z \in \mathbb{E},
\]
where $\lambda \geq 0, n \in \mathbb{N}_0$.

**Theorem 3.21.** Let $\alpha$ be a positive real number. If $f \in A_p$ satisfies
\[
(1 - \alpha) \frac{I_p(n, \lambda)[f](z)}{z^p} + \frac{\alpha}{p^2} \frac{I_p(n + 1, \lambda)[f](z)}{z^p} < e^z + \frac{\alpha}{p + \lambda} z e^z,
\]
then
\[
\frac{I_p(n, \lambda)[f](z)}{z^p} < e^z, \quad z \in \mathbb{E},
\]
where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.20, we obtain the following criterion for starlike-ness.
Corollary 3.22. Let \( \alpha \) be a positive real number. If \( f \in A_p \) satisfies
\[
\frac{zf'(z)}{pf(z)} \left[ 1 + \frac{\alpha}{p} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right] \prec e^z + \frac{\alpha ze^z}{p}, \quad z \in \mathbb{E},
\]
then \( \frac{zf'(z)}{pf(z)} \prec e^z \); that is, \( f \in S_p^* \).

Setting \( p = 1 \) in above corollary, we have the following.

Corollary 3.23. Let \( \alpha \) be a positive real number. If \( f \in A \) satisfies
\[
\frac{zf'(z)}{f(z)} \left[ 1 + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right] \prec e^z + \alpha ze^z, \quad z \in \mathbb{E},
\]
then \( \frac{zf'(z)}{f(z)} \prec e^z \); that is, \( f \in S^* \).

Setting \( \lambda = 0, n = 1 \) in Theorem 3.20, we obtain the following result of convexity.

Corollary 3.24. Let \( \alpha \) be a positive real number. If \( f \in A_p \) satisfies
\[
\frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \left[ 1 + \frac{\alpha}{p} \left( 2zf''(z) + \frac{zf''(z)}{f'(z)} + z^2f''(z) - \frac{zf'(z)}{f'(z)} \right) \right] \prec e^z + \frac{\alpha ze^z}{p}, \quad z \in \mathbb{E},
\]
then \( \frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec e^z \); that is, \( f \in K_p \).

Selecting \( p = 1 \) in above corollary, we obtain the following.

Corollary 3.25. Let \( \alpha \) be a positive real number. If \( f \in A \) satisfies
\[
\left( 1 + \frac{zf''(z)}{f'(z)} \right) \left[ 1 + \alpha \left( 2zf''(z) + \frac{zf''(z)}{f'(z)} + z^2f''(z) - \frac{zf'(z)}{f'(z)} \right) \right] \prec e^z + \alpha ze^z, \quad z \in \mathbb{E},
\]
then \( 1 + \frac{zf''(z)}{f'(z)} \prec e^z \); that is, \( f \in K \).

Setting \( \lambda = 0, n = 1 \) in Theorem 3.21, we get the following result of close-to-convexity.

Corollary 3.26. Let \( \alpha \) be a positive real number. If \( f \in A_p \) satisfies
\[
(1 - \alpha)\frac{f'(z)}{pz^{p-1}} + \frac{\alpha}{p^2} \left[ \frac{f'(z)}{z^{p-1}} + \frac{f''(z)}{z^{p-2}} \right] \prec e^z + \frac{\alpha ze^z}{p}, \quad z \in \mathbb{E},
\]
then \( f \in C_p \).

Setting \( p = 1 \) in above corollary, we get the following.

Corollary 3.27. Let \( \alpha \) be a positive real number. If \( f \in A \) satisfies
\[
f'(z) + \alpha zf''(z) \prec e^z + \alpha ze^z, \quad z \in \mathbb{E},
\]
then \( f \in C \).
Remark 3.28. When we select the dominant \( q(z) = \frac{\alpha'(1 - z)}{\alpha' - z}, 1 \leq \alpha' < \frac{3}{2} \) in Theorems 2.1 and 2.2, a little calculation yields that
\[
1 + \frac{zq''(z)}{q'(z)} = \frac{\alpha' + z}{\alpha' - z},
\]
\[
1 + \frac{zq''(z)}{q'(z)} + \frac{p + \lambda}{\alpha} = \frac{\alpha' + z}{\alpha' - z} + \frac{p + \lambda}{\alpha}.
\]

For a positive real number \( \alpha \), we see that \( q(z) \) satisfies the conditions (i) and (ii) of Theorems 2.1 and 2.2. Therefore, we get the following results.

Theorem 3.29. Let \( \alpha \) be a positive real number. If \( f \in A_p \) satisfies
\[
I_p(n + 1, \lambda)[f](z) - I_p(n, \lambda)[f](z) - I_p(n + 1, \lambda)[f](z)\]
\[
\leq \frac{\alpha'(1 - z)}{\alpha' - z} + \frac{\alpha}{p} \frac{(\alpha' - \alpha^2)z}{(\alpha' - z)^2},
\]
then
\[
\frac{I_p(n + 1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \leq \frac{\alpha'(1 - z)}{\alpha' - z}, \quad 1 \leq \alpha' < \frac{3}{2}, z \in \mathbb{E},
\]
where \( \lambda \geq 0, n \in \mathbb{N}_0 \).

Theorem 3.30. Let \( \alpha \) be a positive real number. If \( f \in A_p \) satisfies
\[
(1 - \alpha) \frac{I_p(n, \lambda)[f](z)}{z^p} + \alpha \frac{I_p(n + 1, \lambda)[f](z)}{z^p}
\]
\[
\leq \frac{\alpha'(1 - z)}{\alpha' - z} + \frac{\alpha}{p} \frac{(\alpha' - \alpha^2)z}{(\alpha' - z)^2},
\]
then
\[
\frac{I_p(n, \lambda)[f](z)}{z^p} \leq \frac{\alpha'(1 - z)}{\alpha' - z}, \quad 1 \leq \alpha' < \frac{3}{2}, z \in \mathbb{E},
\]
where \( \lambda \geq 0, n \in \mathbb{N}_0 \).

Setting \( \lambda = n = 0 \) in Theorem 3.29, we obtain the following criterion for starlikeness.

Corollary 3.31. Let \( \alpha \) be a positive real number. If \( f \in A_p \) satisfies
\[
\frac{zf'(z)}{pf'(z)} \left[ 1 + \frac{\alpha}{p} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right] \leq \frac{\alpha'(1 - z)}{\alpha' - z}
\]
\[
+ \frac{\alpha}{p} \frac{(\alpha' - \alpha^2)z}{(\alpha' - z)^2}, 1 \leq \alpha' < \frac{3}{2}, z \in \mathbb{E},
\]
then
\[
\frac{zf'(z)}{pf'(z)} \leq \frac{\alpha'(1 - z)}{\alpha' - z}; \quad \text{that is,} \quad f \in S_p^*.
\]

Setting \( p = 1 \) in above corollary, we have the following.
Corollary 3.32. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A} \) satisfies
\[
\frac{zf'(z)}{f(z)} \left[ 1 + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right] < \frac{\alpha'(1-z)}{\alpha' - z} + \alpha \frac{(\alpha' - \alpha'^2)z}{(\alpha' - z)^2}, \quad 1 \leq \alpha' < \frac{3}{2}, \quad z \in \mathbb{E},
\]
then \( \frac{zf'(z)}{f(z)} < \frac{\alpha'(1-z)}{\alpha' - z} \); that is, \( f \in \mathcal{S}^* \).

Setting \( \lambda = 0, n = 1 \) in Theorem 3.29, we obtain the following result of convexity.

Corollary 3.33. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A}_p \) satisfies
\[
\frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \left[ 1 + \alpha \left( \frac{2zf''(z) + z^2f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)} \right) \right] < \frac{\alpha'(1-z)}{\alpha' - z} + \frac{\alpha(\alpha' - \alpha'^2)z}{p(\alpha' - z)^2}, \quad 1 \leq \alpha' < \frac{3}{2}, \quad z \in \mathbb{E},
\]
then \( \frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{\alpha'(1-z)}{\alpha' - z} \); that is, \( f \in \mathcal{K}_p \).

Selecting \( p = 1 \) in above corollary, we obtain the following.

Corollary 3.34. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A} \) satisfies
\[
\left( 1 + \frac{zf''(z)}{f'(z)} \right) \left[ 1 + \alpha \left( \frac{2zf''(z) + z^2f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)} \right) \right] < \frac{\alpha'(1-z)}{\alpha' - z} + \frac{\alpha(\alpha' - \alpha'^2)z}{(\alpha' - z)^2}, \quad 1 \leq \alpha' < \frac{3}{2}, \quad z \in \mathbb{E},
\]
then \( \frac{zf''(z)}{f'(z)} < \frac{\alpha'(1-z)}{\alpha' - z} \); that is, \( f \in \mathcal{K} \).

Setting \( \lambda = 0, n = 1 \) in Theorem 3.30, we have the following result of close-to-convexity.

Corollary 3.35. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A}_p \) satisfies
\[
(1 - \alpha) \frac{f'(z)}{zp^{p-1}} + \alpha \frac{f''(z)}{zp^{p-2}} \left[ \frac{f'(z)}{zp-1} + \frac{f''(z)}{zp-2} \right] < \frac{\alpha'(1-z)}{\alpha' - z} + \frac{\alpha(\alpha' - \alpha'^2)z}{p(\alpha' - z)^2}, \quad 1 \leq \alpha' < \frac{3}{2}, \quad z \in \mathbb{E},
\]
then \( f \in \mathcal{C}_p \).

Taking \( p = 1 \) in above corollary, we have:

Corollary 3.36. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A} \) satisfies
\[
f'(z) + azf''(z) < \frac{\alpha'(1-z)}{\alpha' - z} + \frac{\alpha(\alpha' - \alpha'^2)z}{(\alpha' - z)^2}, \quad 1 \leq \alpha' < \frac{3}{2}, \quad z \in \mathbb{E},
\]
then \( f \in \mathcal{C} \).
Remark 3.37. When we select the dominant \( q(z) = 1 + az, 0 \leq a < 1 \), in Theorems 2.1 and 2.2, a little calculation yields that
\[
1 + \frac{zq''(z)}{q'(z)} = 1,
\]
\[
1 + \frac{zq''(z)}{q'(z)} + \frac{p + \lambda}{\alpha} = 1 + \frac{p + \lambda}{\alpha}.
\]
For a positive real number \( \alpha \), we note that \( q(z) \) satisfies the conditions (i) and (ii) of Theorems 2.1 and 2.2. Therefore, we get the following results.

Theorem 3.38. Let \( \alpha \) be a positive real number. If \( f \in A_p \) satisfies
\[
\frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \left[ 1 + \alpha \left( \frac{I_p(n+2, \lambda)[f](z)}{I_p(n+1, \lambda)[f](z)} - \frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \right) \right] < 1 + az + \frac{\alpha}{p+\lambda}az, \quad 0 \leq a < 1,
\]
then
\[
\frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} < 1 + az, \quad z \in \mathbb{E},
\]
where \( \lambda \geq 0, n \in \mathbb{N}_0 \).

Theorem 3.39. Let \( \alpha \) be a positive real number. If \( f \in A_p \) satisfies
\[
(1 - \alpha) \frac{I_p(n, \lambda)[f](z)}{z^p} + \alpha \frac{I_p(n+1, \lambda)[f](z)}{z^p} < 1 + az + \frac{\alpha}{p+\lambda}az, \quad 0 \leq a < 1,
\]
then
\[
\frac{I_p(n, \lambda)[f](z)}{z^p} < 1 + az, \quad z \in \mathbb{E},
\]
where \( \lambda \geq 0, n \in \mathbb{N}_0 \).

Setting \( \lambda = n = 0 \) in Theorem 3.38, we obtain the following criterion for starlikeness.

Corollary 3.40. Let \( \alpha \) be a positive real number. If \( f \in A_p \) satisfies
\[
\frac{zf'(z)}{pf(z)} \left[ 1 + \frac{\alpha}{p} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right] < 1 + (1 + \frac{\alpha}{p})az, \quad 0 \leq a < 1, z \in \mathbb{E},
\]
then
\[
\frac{zf'(z)}{pf(z)} < 1 + az; \text{ that is, } f \in S_p^*.
\]
Setting \( p = 1 \) in above corollary, we have the following.

Corollary 3.41. Let \( \alpha \) be a positive real number. If \( f \in A \) satisfies
\[
\frac{zf'(z)}{f(z)} \left[ 1 + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right] < 1 + (1 + \alpha)az, \quad 0 \leq a < 1, z \in \mathbb{E},
\]
then
\[
\frac{zf'(z)}{f(z)} < 1 + az; \text{ that is, } f \in S^*.
\]
Setting \( \lambda = 0, n = 1 \) in Theorem 3.38, we obtain the following result of convexity.
Corollary 3.42. Let $\alpha$ be a positive real number. If $f \in A_p$ satisfies
\[
\frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \left[ 1 + \frac{\alpha}{p} \left( \frac{2zf''(z) + z^2f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)} \right) \right] < 1 + \left( 1 + \frac{\alpha}{p} \right) az, \quad 0 \leq a < 1, z \in \mathbb{E},
\]
then $\frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < 1 + az$; that is, $f \in K_p$.

Selecting $p = 1$ in above corollary, we obtain the following.

Corollary 3.43. Let $\alpha$ be a positive real number. If $f \in A$ satisfies
\[
\left( 1 + \frac{zf''(z)}{f'(z)} \right) \left[ 1 + \alpha \left( \frac{2zf''(z) + z^2f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)} \right) \right] < 1 + (1 + \alpha)az, \quad 0 \leq a < 1,
\]
then $1 + \frac{zf''(z)}{f'(z)} < 1 + az, z \in \mathbb{E}$.

For $\lambda = 0, n = 1$, Theorem 3.39 gives the following sufficient condition for close-to-convexity.

Corollary 3.44. Let $\alpha$ be a positive real number. If $f \in A_p$ satisfies
\[
(1 - \alpha) \frac{f'(z)}{pz^{p-1}} + \frac{\alpha}{p^2} \left[ \frac{f'(z)}{z^{p-1}} + \frac{f''(z)}{z^{p-2}} \right] < 1 + \left( 1 + \frac{\alpha}{p} \right) az, \quad 0 \leq a < 1, z \in \mathbb{E},
\]
then $f \in C_p$.

Setting $p = 1$ in above corollary, we have the following sufficient condition for close-to-convexity.

Corollary 3.45. Let $\alpha$ be a positive real number. If $f \in A$ satisfies
\[
f'(z) + \alpha zf''(z) < 1 + (1 + \alpha)az, \quad 0 \leq a < 1, z \in \mathbb{E},
\]
then $f \in C$.

Remark 3.46. When we select the dominant $q(z) = \left( \frac{1 + z}{1 - z} \right)^\gamma, 0 < \gamma \leq 1$ in Theorems 2.1 and 2.2, a little calculation yields that
\[
1 + \frac{zq''(z)}{q'(z)} = \frac{1 + z^2 + 2\gamma z}{1 - z^2},
\]
\[
1 + \frac{zq''(z)}{q'(z)} + \frac{p + \lambda}{\alpha} = \frac{1 + z^2 + 2\gamma z}{1 - z^2} + \frac{p + \lambda}{\alpha}.
\]

For a positive real number $\alpha$, we note that $q(z)$ satisfies the conditions (i) and (ii) of Theorems 2.1 and 2.2. Therefore, we get the following results.
Theorem 3.47. Let $\alpha$ be a positive real number. If $f \in A_p$ satisfies
\[
\frac{I_p(n + 1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \left[ 1 + \alpha \left( \frac{I_p(n + 2, \lambda)[f](z)}{I_p(n + 1, \lambda)[f](z)} - \frac{I_p(n + 1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \right) \right] \\
< \left( \frac{1 + z}{1 - z} \right) ^\gamma \left[ 1 + \frac{2\alpha \gamma z}{(p + \lambda)(1 - z^2)} \right], \quad 0 < \gamma \leq 1,
\]
then
\[
\frac{I_p(n + 1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \prec \left( \frac{1 + z}{1 - z} \right) ^\gamma, \quad z \in \mathbb{E},
\]
where $\lambda \geq 0, n \in \mathbb{N}_0$.

Theorem 3.48. Let $\alpha$ be a positive real number. If $f \in A_p$ satisfies
\[
(1 - \alpha) \frac{I_p(n, \lambda)[f](z)}{z^p} + \alpha \frac{I_p(n + 1, \lambda)[f](z)}{z^p} \\
< \left( \frac{1 + z}{1 - z} \right) ^\gamma \left[ 1 + \frac{2\alpha \gamma z}{(p + \lambda)(1 - z^2)} \right], \quad 0 < \gamma \leq 1,
\]
then
\[
\frac{I_p(n, \lambda)[f](z)}{z^p} \prec \left( \frac{1 + z}{1 - z} \right) ^\gamma, \quad z \in \mathbb{E},
\]
where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Theorem 3.47, we obtain the following criterion for strongly starlikeness.

Corollary 3.49. Let $\alpha$ be a positive real number. If $f \in A_p$ satisfies
\[
\frac{zf'(z)}{pf(z)} \left[ 1 + \alpha \left( \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right] \\
< \left( \frac{1 + z}{1 - z} \right) ^\gamma \left[ 1 + \frac{2\alpha \gamma z}{p(1 - z^2)} \right], \quad 0 < \gamma \leq 1, z \in \mathbb{E},
\]
then
\[
\frac{zf'(z)}{pf(z)} \prec \left( \frac{1 + z}{1 - z} \right) ^\gamma; \text{ that is, } f \in \tilde{S}_p(\gamma).
\]

Setting $p = 1$ in above corollary, we obtain the following.

Corollary 3.50. Let $\alpha$ be a positive real number. If $f \in A$ satisfies
\[
\frac{zf'(z)}{f(z)} \left[ 1 + \alpha \left( \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right] \\
< \left( \frac{1 + z}{1 - z} \right) ^\gamma \left[ 1 + \frac{2\alpha \gamma z}{1 - z^2} \right], \quad 0 < \gamma \leq 1, z \in \mathbb{E},
\]
then
\[
\frac{zf'(z)}{f(z)} \prec \left( \frac{1 + z}{1 - z} \right) ^\gamma; \text{ that is, } f \in \tilde{S}(\gamma).
\]

Setting $\lambda = 0, n = 1$ in Theorem 3.47, we obtain the following result of convexity.
Corollary 3.51. Let $\alpha$ be a positive real number. If $f \in A_p$ satisfies
\[
\frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \left[ 1 + \frac{\alpha}{p} \left( \frac{2zf''(z) + z^2f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)} \right) \right]
\prec \left( \frac{1+z}{1-z} \right)^\gamma \left[ 1 + \frac{2\alpha\gamma z}{p(1-z^2)} \right], \quad 0 < \gamma \leq 1, z \in \mathbb{E},
\]
then $\frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec \left( \frac{1+z}{1-z} \right)^\gamma$; that is, $f \in \tilde{K}_p(\gamma)$.

Selecting $p = 1$ in above corollary, we obtain the following.

Corollary 3.52. Let $\alpha$ be a positive real number. If $f \in A$ satisfies
\[
\left( 1 + \frac{zf''(z)}{f'(z)} \right) \left[ 1 + \alpha \left( \frac{2zf''(z) + z^2f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)} \right) \right]
\prec \left( \frac{1+z}{1-z} \right)^\gamma \left[ 1 + \frac{2\alpha\gamma z}{1-z^2} \right], \quad 0 < \gamma \leq 1, z \in \mathbb{E},
\]
then $1 + \frac{zf''(z)}{f'(z)} \prec \left( \frac{1+z}{1-z} \right)^\gamma$; that is, $f \in \tilde{K}(\gamma)$.

Setting $\lambda = 0, n = 1$ in Theorem 3.48, we get the following result of close-to-convex functions.

Corollary 3.53. Let $\alpha$ be a positive real number. If $f \in A_p$ satisfies
\[
(1 - \alpha) \frac{f'(z)}{p^p z^{p-1}} + \frac{\alpha}{p^p} \left[ \frac{f'(z)}{z^{p-1}} + \frac{f''(z)}{z^{p-2}} \right]
\prec \left( \frac{1+z}{1-z} \right)^\gamma \left[ 1 + \frac{2\alpha\gamma z}{p(1-z^2)} \right], \quad 0 < \gamma \leq 1, z \in \mathbb{E},
\]
then $f \in C_p$.

Setting $p = 1$ in above result, we have the following.

Corollary 3.54. Let $\alpha$ be a positive real number. If $f \in A$ satisfies
\[
f'(z) + \alpha zf''(z) \prec \left( \frac{1+z}{1-z} \right)^\gamma \left[ 1 + \frac{2\alpha\gamma z}{1-z^2} \right], \quad 0 < \gamma \leq 1, z \in \mathbb{E},
\]
then $f \in C$.

Acknowledgement. We are thankful to the reviewers for their valuable comments.
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