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# UNIQUENESS OF MEROMORPHIC FUNCTIONS WITH REGARD TO MULTIPLICITY

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Abstract. In this paper, we investigate the uniqueness problem on meromorphic functions concerning differential polynomials sharing one value. A uniqueness result which related to multiplicity of meromorphic function is proved in this paper. By using the notion of multiplicity, our results will generalise and improve the result due to Chao Meng [J. Applied Math. Inform. 33 (2015), no. 5-6, pp. 475–484].

## 1. Introduction and main results

Let $f$ be a nonconstant meromorphic function in the whole complex plane. We shall use the following standard notations of the value distribution theory: $T(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $m(r,f)$, (see [3], [14], [15]). The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty, r \notin E$, where $E$ is a set of positive real number of finite linear measure, not necessarily the same at each occurrence. The notations $T(r)$ and $S(r)$ are defined, respectively, by

$$T(r) = max\left\{T(r,f), T(r,g)\right\}, \;\; S(r) = o(T(r)) \text{ as } r \to \infty, r \notin E,$$

for any two nonconstant meromorphic functions $f$ and $g$. A meromorphic function $a$ is called a small function with respect to $f$ provided that $T(r,a) = S(r,f)$. Moreover, $GCD(n_1, n_2, \ldots, n_k)$ denotes the greatest common divisor of positive integers $n_1, n_2, \ldots, n_k$.

Let $f$ and $g$ be two nonconstant meromorphic functions, and let $a \in \mathbb{C}$. We say that $f$ and $g$ share the value $a$ counting multiplicities $(CM)$, provided that $f - a$

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and $g-a$ have the same zeros with the same multiplicities. If $f-a$ and $g-a$ have the same zeros, then we say that $f$ and $g$ share $a$ ignoring multiplicities ($IM$). Similarly, we immediately get the definitions of $f$ and $g$ share $a$ $IM$ (or $CM$), where $a$ is a small function of $f$ and $g$. In addition, we also need the following notation, for any $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$,

$$\Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, a; f)}{T(r, f)}.$$

For a complex number $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer $k$, we denote by $E_k(a, f)$ the set of all $a$-points of $f$ where an $a$-point with multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. For a complex number $a \in \mathbb{C}\cup\{\infty\}$, if $E_k(a, f) = E_k(a, g)$, then we say that $f$ and $g$ share the value $a$ with weight $k$.

We write $f, g$ share $(a, k)$ to mean that $f, g$ share the value $a$ with weight $k$. Clearly if $f, g$ share $(a, k)$, then $f, g$ share $(a, p)$ for all integer $p$, $0 \leq p \leq k$. Also we note that $f, g$ share a value $a$ $IM$ or $CM$ if and only if $f, g$ share $(a, 0)$ or $(a, \infty)$, respectively. We call $f$ and $g$ share $(z, k)$ if $f - z$ and $g - z$ share $(0, k)$.

In 1997 Yang and Hua proved the following result.

**Theorem A** [13]**.** Let $f$ and $g$ be two nonconstant meromorphic functions, let $n \geq 11$ be an integer, and let $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value $a$ $CM$, then either $f = dg$ for some $(n + 1)$th root of unity $d$ or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where $c, c_1$ and $c_2$ are constants and satisfy $(c_1c_2)^{n+1}c^2 = -a^2$.

In 2001, Fang and Hong obtained the following result.

**Theorem B** [2]**.**Let $f$ and $g$ be two transcendental entire functions, and let $n \geq 11$ be an integer. If $f^n(f - 1)f'$ and $g^n(g - 1)g'$ share the value one $CM$, then $f \equiv g$.

In 2004, Lin and Yi extended the above theorem in view of the fixed-point. They proved the following result.

**Theorem C** [6]**.** Let $f$ and $g$ be two transcendental meromorphic functions, and let $n \geq 13$ be an integer. If $f^n(f - 1)^2f'$ and $g^n(g - 1)^2g'$ share $z$ $CM$, then $f \equiv g$.

In 2006, Lahiri and Pal proved the following result.

**Theorem D** [5]**.** Let $f$ and $g$ be two nonconstant meromorphic functions, and let $n \geq 14$ be an integer. If $E_{3)}\left(1, f^n(f^3 - 1)f'\right) = E_{3)}\left(1, g^n(g^3 - 1)g'\right)$, then $f \equiv g$.

In 2008, Chao Meng relaxed the nature of fixed point to $IM$ and proved the following result.

**Theorem E** [7]**.** Let $f$ and $g$ be two transcendental meromorphic functions, and let $n \geq 28$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share $z$ $IM$, then $f \equiv g$.

In 2009, Chao Meng relaxed nature of sharing value in the above theorem and proved the following result.

**Theorem F** [8]**.** Let $f$ and $g$ be two nonconstant meromorphic functions such that $f^n(f^3 - 1)f'$ and $g^n(g^3 - 1)g'$ share $(1, l)$, where $n$ is a positive integer such that $(n + 1)$ is not divisible by three. If

$(i) l = 2$ and $n \geq 14$; $(ii) l = 1$ and $n \geq 17$; $(iii) l = 0$ and $n \geq 35$, then $f \equiv g$.

In 2015, Chao Meng proved the following result.

**Theorem G** [9]**.** Let $f$ and $g$ be two nonconstant meromorphic functions, $n \geq 12$ a positive integer. If $f^n(f^3-1)f'$ and $g^n(g^3-1)g'$ share $(1,2)$ and $f$ and $g$ share $\infty$ $IM$, then $f \equiv g$.

In this paper, using the notion of multiplicity, we prove the following theorems.

**Theorem 1.1.** *Let $f$ and $g$ be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities atleast $s$, where $s$ is a positive integer. Let $n$ and $m$ be positive integers with $s(n-m-3) \geq 6$, and let $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$, where $a_0 \neq 0, a_1, a_2, \ldots, a_{m-1}, a_m \neq 0$ are complex constants. If $f^nP(f)f'$ and $g^nP(g)g'$ share $(1,2)$ and $f$ and $g$ share $\infty$ $IM$, then one of the following two cases holds:*
*(i)$f \equiv tg$ for a constant $t$ such that $t^d = 1$, where $d = GCD(n+m+1, \ldots, n+m+1-i, \ldots, n+1)$ and $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$;*
*(ii)$f$ and $g$ satisfy the algebraic equation $R(f,g) \equiv 0$, where*

$$R(w_1,w_2) = w_1^{n+1}\left(\frac{a_m w_1^m}{n+m+1} + \frac{a_{m-1}w_1^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}\right)$$
$$- w_2^{n+1}\left(\frac{a_m w_2^m}{n+m+1} + \frac{a_{m-1}w_2^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}\right).$$

**Theorem 1.2.** *Let $f$ and $g$ be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities atleast $s$, where $s$ is a positive integer. Let $n$and $m$ be positive integers with $s(n-m) > 3k+4+(k+2)s$ and $\Theta(\infty,f) + \Theta(\infty,g) > \frac{4}{n}$, and let $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$, where $a_0 \neq 0, a_1, a_2, \ldots, a_{m-1}, a_m \neq 0$ are complex constants. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $(1,2)$ and $f$ and $g$ share $\infty$ $IM$, then one of the following two cases holds:*
*(i)$f \equiv tg$ for a constant $t$ such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n+1, n)$ and $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$;*
*(ii)$f$ and $g$ satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(f,g) = f^nP(f) - g^nP(g)$.*

*Remark* 1.3. (1) We set $P(z) = (z-1)^m$. Then, with $a_m = 1$ and $a_0 = -1$ and under the condition (ii) of Theorem 1.1, Theorem 1.1 reduces to Theorem G, if $m = 3$ and $s = 1$.

(2) Giving specific values for $s$ in Theorem 1.1, we get the following interesting cases:
*i*) If $s = 1$, then $n \geq m+9$.
*ii*) If $s = 2$, then $n \geq m+6$.
*iii*) If $s = 3$, then $n \geq m+5$.
We conclude that if $f$ and $g$ have zeros and poles of higher order multiplicity, then we can reduce the values of $n$.

(3) In order to discuss the case of sharing $IM$, we have to replace the sharing $(1,2)$ in the above Theorems 1.1 and 1.2 by $(1,0)$. Then we obtain the same result for the value of $n$ higher than the above mentioned Theorems 1.1 and 1.2.

The following example shows that $f$ and $g$ share $\infty$ $IM$, in Theorems 1.1 and 1.2 can not be removed.

**Example 1.4.** Let $P(z) = (z-1)^6(z+1)^6 z^{11}$, $f(z) = sinz$, $g(z) = cosz$, $k = 0$, and $s = 1$. It is easy to see that $n > m + 6$ and $P(f(z))f^n(z) = P(g(z))g^n(z)$. Therefore $P(f(z))f^n(z)$ and $P(g(z))g^n(z)$ share $1$ $CM$. It is also clear that though $f$ and $g$ satisfy $R(f,g) = 0$, where $R(w_1, w_2) = P(w_1)w_1(z) - P(w_2)w_2(z)$, we have $f \not\equiv tg$ for a constant $t$ satisfying $t^m = 1$, where $m \in Z^+$.

## 2. Some lemmas

For the proof of our main results, we need the following lemmas.

**Lemma 2.1.** [1] *If $F$ and $G$ share $(1,2)$ and $(\infty, k)$, where $0 \le k \le \infty$, then one of the following cases holds:*
*(1) $T(r,F) \le N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N_*}(r,\infty; F,G) + S(r,F) + S(r,G)$, the same inequality holds for $T(r,G)$;*
*(2) $F \equiv G$;*
*(3) $FG \equiv 1$.*

**Lemma 2.2.** [16] *Suppose that $f(z)$ is a nonconstant meromorphic function in the complex plane and that $k$ is a positive integer. Then*
$$N\left(r, \frac{1}{f^{(k)}}\right) \le N\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

**Lemma 2.3.** [17] *Let $f$ be a nonconstant meromorphic function, and let $p$ and $k$ be positive integers. Then*
$$N_p\left(r, \frac{1}{f^{(k)}}\right) \le T\left(r, f^{(k)}\right) - T(r,f) + N_{p+k}\left(r, \frac{1}{f}\right), \tag{2.1}$$
$$N_p\left(r, \frac{1}{f^{(k)}}\right) \le k\overline{N}(r,f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r,f). \tag{2.2}$$

**Lemma 2.4.** [12] *Let $f$ be a nonconstant meromorphic function, and let $a_1, a_2, \ldots,$ $a_n$ $(\neq 0)$ be finite complex numbers. Then*
$T\left(r, a_n f^n + a_{n-1}f^{n-1} + \cdots + a_1 f + a_0\right) = nT(r,f) + S(r,f).$

**Lemma 2.5.** [10] *Let $f$ and $g$ be two nonconstant meromorphic functions. Then $f^n P(f) f' \; g^n P(g) g' \neq 1$, where $n + m (\ge 6)$ is a positive integer.*

**Lemma 2.6.** [4] *Let $f$ and $g$ be two nonconstant meromorphic functions such that $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$ for all integers $n \ge 3$. Then $f^n(af+b) = g^n(ag+b)$ implies $f = g$, where $a$ and $b$ are two finite nonzero complex constants.*

**Lemma 2.7.** *Let $f$ and $g$ be two nonconstant meromorphic functions whose zeros and poles are of multiplicities at least $s$, where $s$ is a positive integer, and let $n$ and $k$ be positive integers. Let $F = [f^n P(f)]^{(k)}$ and $G = [g^n P(g)]^{(k)}$, where $p(z)$ is defined as in Theorem 1.1. If there exist two nonzero constants $b_1$ and $b_2$ such that $\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{G-b_1}\right)$ and $\overline{N}\left(r, \frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F-b_2}\right)$, then $(n-m)s \le 3k+3$.*

*Proof.* By the second fundamental theorem of Nevanlinna's theory,

$$\begin{aligned} T(r,F) &\leq \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F-b_2}\right)+S(r,F) \\ &\leq \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,F). \end{aligned} \tag{2.3}$$

Combining (2.1), (2.2), (2.3), and Lemma 2.4, we get

$$\begin{aligned} (n+m)T(r,f) &\leq T(r,F)-\overline{N}\left(r,\frac{1}{F}\right)+N_{k+1}\left(r,\frac{1}{f^nP(f)}\right)+S(r,f) \\ &\leq \overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f^nP(f)}\right)+S(r,f) \\ &\leq N_{k+1}\left(r,\frac{1}{f^nP(f)}\right)+N_{k+1}\left(r,\frac{1}{g^nP(g)}\right)+k\overline{N}(r,g) \\ &\quad+\overline{N}(r,f)+S(r,f)+S(r,g) \\ &\leq \left(\frac{k+2}{s}+m\right)T(r,f)+\left(\frac{2k+1}{s}+m\right)T(r,g) \\ &\quad+S(r,f)+S(r,g) \\ &\leq \left(\frac{3k+3}{s}+2m\right)T(r)+S(r). \end{aligned} \tag{2.4}$$

Similarly, for the case of $g$,

$$(n+m)T(r,g) \leq \left(\frac{3k+3}{s}+2m\right)T(r)+S(r). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\left(n-m-\left(\frac{3k+3}{s}\right)\right)T(r) \leq S(r)$$

which gives $(n-m)s \leq 3k+3$. This completes the proof. □

**Lemma 2.8.** [11] *Let $f$ and $g$ be two nonconstant meromorphic functions, and let $n(\geq 1), k(\geq 1)$, and $m(\geq 1)$ be integers. Then* $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \neq 1$.

## 3. Poof of theorems

*Proof of Theorem 1.1.* Let

$$F=f^nP(f)f', \;\; G=g^nP(g)g', \tag{3.1}$$

$$F^*=\frac{a_m}{m+n+1}f^{m+n+1}+\frac{a_{m-1}}{m+n}f^{m+n}+\cdots+\frac{a_0}{n+1}f^{n+1}, \tag{3.2}$$

and $$G^*=\frac{a_m}{m+n+1}g^{m+n+1}+\frac{a_{m-1}}{m+n}g^{m+n}+\cdots+\frac{a_0}{n+1}g^{n+1}. \tag{3.3}$$

Thus we obtain that $F$ and $G$ share $(1,2)$. If possible, let the case 1 of Lemma 2.1; that is,

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_*(r,\infty;F,G) \\ + S(r,F) + S(r,G). \tag{3.4}$$

Moreover, by Lemma 2.4, we have

$$T(r,F^*) = (m+n+1)T(r,f) + S(r,f), \tag{3.5}$$

$$T(r,G^*) = (m+n+1)T(r,g) + S(r,g).$$

Since $(F^*)' = F$, we deduce

$$m\left(r,\frac{1}{F^*}\right) = m\left(r,\frac{1}{F}\right) + S(r,f).$$

By Nenanlinna's first fundamental theorem, we get

$$\begin{aligned} T(r,F^*) &\le \overline{N}(r,F) + N\left(r,\frac{1}{F^*}\right) - N\left(r,\frac{1}{F}\right) + S(r,f) \\ &\le T(r,F) + N\left(r,\frac{1}{f}\right) + \sum_{i=1}^{m} N\left(r,\frac{1}{f-b_i}\right) - \sum_{i=1}^{m} N\left(r,\frac{1}{f-c_i}\right) \\ &\quad - N\left(r,\frac{1}{f'}\right) + S(r,f), \end{aligned} \tag{3.6}$$

where $b_1, b_2, \ldots, b_m$ are roots of algebraic equation

$$\frac{a_m}{m+n+1}z^m + \frac{a_{m-1}}{m+n}z^{m-1} + \cdots + \frac{a_0}{n+1} = 0,$$

and $c_1, c_2, \ldots, c_m$ are roots of algebraic equation $a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0 = 0$.

It follows from (3.4) and (3.6) that

$$\begin{aligned} T(r,F^*) \le\; & N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_*(r,\infty;F,G) \\ & + N\left(r,\frac{1}{f}\right) + \sum_{i=1}^{m} N\left(r,\frac{1}{f-b_i}\right) - \sum_{i=1}^{m} N\left(r,\frac{1}{f-c_i}\right) - N\left(r,\frac{1}{f'}\right) \\ & + S(r,f). \end{aligned} \tag{3.7}$$

It follows from (3.1) that

$$N_2\left(r,\frac{1}{F}\right) + \overline{N}(r,F) \le 2\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-c_1}\right) + \cdots + N\left(r,\frac{1}{f-c_m}\right) \\ + N\left(r,\frac{1}{f'}\right) + \overline{N}(r,f). \tag{3.8}$$

$$N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,G)\leq 2\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-c_1}\right)+\cdots+N\left(r,\frac{1}{g-c_m}\right)$$
$$+N\left(r,\frac{1}{g'}\right)+\overline{N}(r,g). \tag{3.9}$$

From (3.7), (3.8), and (3.9), we obtain

$$T(r,F^*)\leq 2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-b_1}\right)+\cdots+N\left(r,\frac{1}{f-b_m}\right)+N\left(r,\frac{1}{f}\right)$$
$$+2\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-c_1}\right)+\cdots+N\left(r,\frac{1}{g-c_m}\right)+N\left(r,\frac{1}{g'}\right)$$
$$+\overline{N_*}(r,\infty;F,G)+\overline{N}(r,f)+\overline{N}(r,g)+S(r,f)+S(r,g). \tag{3.10}$$

By Lemma 2.2, we have

$$N\left(r,\frac{1}{g'}\right)\leq N\left(r,\frac{1}{g}\right)+\overline{N}(r,g)+S(r,f). \tag{3.11}$$

Also, we have

$$\overline{N}(r,f)+\overline{N}(r,g)+\overline{N_*}(r,\infty;F,G)\leq N(r,f)+N(r,g). \tag{3.12}$$

By our assumption, zeros and poles of $f$ and $g$ are of multiplicities atleast $s$; we have

$$\overline{N}(r,g)\leq\frac{1}{s}N(r,g)\leq\frac{1}{s}T(r,g), \tag{3.13}$$

$$\overline{N}(r,\frac{1}{g})\leq\frac{1}{s}N(r,\frac{1}{g})\leq\frac{1}{s}T(r,g). \tag{3.14}$$

We deduce from (3.10)–(3.14) that

$$(n+m+1)T(r,f)\leq\left(\frac{2}{s}+m+2\right)T(r,f)+\left(\frac{3}{s}+m+2\right)T(r,g) \tag{3.15}$$
$$+S(r,f)+S(r,g),$$

$$\left(n-\frac{2}{s}-1\right)T(r,f)\leq\left(\frac{3}{s}+m+2\right)T(r,g)+S(r,f)+S(r,g). \tag{3.16}$$

Similarly,

$$\left(n-\frac{2}{s}-1\right)T(r,g)\leq\left(\frac{3}{s}+m+2\right)T(r,f)+S(r,f)+S(r,g). \tag{3.17}$$

From (3.15) and (3.17), we deduce that $(n-m-3)s\leq 5$, which contradicts the assumption $(n-m-3)s\geq 6$.

**Case 2:** Suppose $FG\equiv 1$, by Lemma 2.5, we get a contradiction.

**Case 3:** If $F\equiv G$, that is

$$F^*=G^*+c,$$

where $c$ is a constant, then it follows that

$$T(r,f)=T(r,g)+S(r,f). \tag{3.18}$$

Suppose that $c\neq 0$, by (3.2), (3.3), (3.5), (3.13), (3.14), (3.18), the second fundamental theorem, and lemma 2.4 we have

$$\begin{aligned}
T(r,G^*) &\leq \overline{N}\left(r,\frac{1}{G^*}\right)+\overline{N}\left(r,\frac{1}{G^*+c}\right)+\overline{N}(r,G^*)+S(r,g)\\
(n+m+1)T(r,g) &\leq \overline{N}\left(r,\frac{1}{G^*}\right)+\overline{N}\left(r,\frac{1}{F^*}\right)+\overline{N}(r,G^*)+S(r,g)\\
&\leq \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^m+\cdots+\frac{m+n+1}{n+1}\frac{a_0}{a_m}}\right)+\overline{N}\left(r,\frac{1}{f}\right)\\
&\quad+\overline{N}\left(r,\frac{1}{f^m+\cdots+\frac{m+n+1}{n+1}\frac{a_0}{a_m}}\right)+\overline{N}(r,g)+S(r,g)\\
&\leq \left(\frac{3}{s}+2m\right)T(r,f)+S(r,f)+S(r,g),
\end{aligned} \tag{3.19}$$

which contradicts our assumption $(n-m-3)s\geq 6$. Therefore $F^*=G^*$ that is,

$$\begin{aligned}
&f^{n+1}\left(\frac{a_mf^m}{m+n+1}+\frac{a_{m-1}f^{m-1}}{m+n}+\cdots+\frac{a_0}{n+1}\right)\\
&\qquad=g^{n+1}\left(\frac{a_mg^m}{m+n+1}+\frac{a_{m-1}g^{m-1}}{m+n}+\cdots+\frac{a_0}{n+1}\right).
\end{aligned} \tag{3.20}$$

Let $h=\frac{f}{g}$. If $h$ is constant, then, substituting $f=gh$ into (3.20), we deduce,

$$\frac{a_mg^{m+n+1}\left(h^{m+n+1}-1\right)}{m+n+1}+\frac{a_{m-1}g^{m+n}\left(h^{m+n}-1\right)}{m+n}+\cdots+\frac{a_0g^{n+1}\left(h^{n+1}-1\right)}{n+1}=0, \tag{3.21}$$

which implies $h^d=1$, where $d=(n+m+1,n+m,\ldots,n+m+1-i,\ldots,n+1), a_{m-i}\neq 0$ for some $i=0,1,\ldots,m$.

Thus $f=tg$, for a constant $t$, such that $t^d=1$, where $d=(n+m+1,n+m,\ldots,n+m+1-i,\ldots,n+1), a_{m-i}\neq 0$, for some $i=0,1,\ldots,m$.

If $h$ is not constant, then, by (3.21), $f$ and $g$ satisfy the algebraic equation $R(f,g)\equiv 0$, where

$$\begin{aligned}
R(w_1,w_2)=w_1^{n+1}&\left(\frac{a_mw_1^m}{n+m+1}+\frac{a_{m-1}w_1^{m-1}}{n+m}+\cdots+\frac{a_0}{n+1}\right)\\
&-w_2^{n+1}\left(\frac{a_mw_2^m}{n+m+1}+\frac{a_{m-1}w_2^{m-1}}{n+m}+\cdots+\frac{a_0}{n+1}\right).
\end{aligned}$$

This completes the proof of Theorem 1.1. □

*Proof of Theorem 1.2.* Let

$$F=[f^nP(f)]^{(k)},\ \ G=[g^nP(g)]^{(k)}, \tag{3.22}$$

$$F^* = \frac{a_m(n+m)!}{(n+m-k+1)!}f^{n+m-k+1} + \frac{a_{m-1}(n+m-1)!}{(n+m-k)!}f^{n+m-k} + \cdots + \frac{a_0\, n!}{(n-k+1)!}f^{n-k+1}, \tag{3.23}$$

and
$$G^* = \frac{a_m(n+m)!}{(n+m-k+1)!}g^{n+m-k+1} + \frac{a_{m-1}(n+m-1)!}{(n+m-k)!}g^{n+m-k} + \cdots + \frac{a_0\, n!}{(n-k+1)!}g^{n-k+1}. \tag{3.24}$$

Thus we obtain that $F$ and $G$ share $(1,2)$. If possible, let the case 1 of Lemma 2.1; that is,

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N_*}(r,\infty;F,G) + S(r,F) + S(r,G). \tag{3.25}$$

Moreover, by Lemma 2.4, we have

$$T(r,F^*) = (n+m-k+1)T(r,f) + S(r,f), \tag{3.26}$$

$$T(r,G^*) = (n+m-k+1)T(r,g) + S(r,g). \tag{3.27}$$

Since $(F^*)' = F$, we deduce

$$m\left(r,\frac{1}{F^*}\right) = m\left(r,\frac{1}{F}\right) + S(r,f). \tag{3.28}$$

By Nenanlinna's first fundamental theorem, we get

$$\begin{aligned} T(r,F^*) &\le \overline{N}(r,F) + N\left(r,\frac{1}{F^*}\right) - N\left(r,\frac{1}{F}\right) + S(r,f) \\ &\le T(r,F) + N\left(r,\frac{1}{f}\right) + \sum_{i=1}^{m} N\left(r,\frac{1}{f-b_i}\right) - \sum_{i=1}^{m} N\left(r,\frac{1}{f-c_i}\right) \\ &\quad - k\overline{N}(r,f) + S(r,f), \end{aligned} \tag{3.29}$$

where $b_1, b_2, \ldots, b_m$ are roots of algebraic equation

$$\frac{a_m(n+m)!}{(n+m-k+1)!}z^m + \frac{a_{m-1}(n+m-1)!}{(n+m-k)!}z^{m-1} + \cdots + \frac{a_0 n!}{(n-k+1)!} = 0,$$

and $c_1, c_2, \ldots, c_m$ are roots of algebraic equation $a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0 = 0$.

It follows from (3.25) and (3.29) that

$$\begin{aligned}T(r,F^*) \leq\ & N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N_*}(r,\infty;F,G)\\ & +N\left(r,\frac{1}{f}\right)+\sum_{i=1}^{m}N\left(r,\frac{1}{f-b_i}\right)-\sum_{i=1}^{m}N\left(r,\frac{1}{f-c_i}\right)-k\overline{N}(r,f)\\ & +S(r,f).\end{aligned} \tag{3.30}$$

It follows from (3.22) that

$$\begin{aligned}N_2\left(r,\frac{1}{F}\right)+\overline{N}(r,F) &= N_2\left(r,\frac{1}{[f^nP(f)]^{(k)}}\right)+\overline{N}(r,F)\\ &\leq (k+2)\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-c_1}\right)+\cdots+N\left(r,\frac{1}{f-c_m}\right)\\ &\quad +(k+1)\overline{N}(r,f),\end{aligned} \tag{3.31}$$

$$\begin{aligned}N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,G) &= N_2\left(r,\frac{1}{[g^nP(g)]^{(k)}}\right)+\overline{N}(r,G)\\ &\leq (k+2)\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-c_1}\right)+\cdots+N\left(r,\frac{1}{g-c_m}\right)\\ &\quad +(k+1)\overline{N}(r,g).\end{aligned} \tag{3.32}$$

From (3.30), (3.31), and (3.32), we obtain

$$\begin{aligned}T(r,F^*) \leq\ & (k+2)\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-b_1}\right)+\cdots+N\left(r,\frac{1}{f-b_m}\right)\\ & +k\overline{N}(r,f)+\overline{N}(r,f)+(k+2)\overline{N}\left(r,\frac{1}{g}\right)+k\overline{N}(r,g)+\overline{N}(r,g)\\ & +N\left(r,\frac{1}{g-c_1}\right)+\cdots+N\left(r,\frac{1}{g-c_m}\right)+\overline{N_*}(r,\infty;F,G)+N\left(r,\frac{1}{f}\right)\\ & -k\overline{N}(r,f)+S(r,f)+S(r,g).\end{aligned} \tag{3.33}$$

Also, we have

$$\overline{N}(r,f)+\overline{N}(r,g)+\overline{N_*}(r,\infty;F,G)\leq N(r,f)+N(r,g). \tag{3.34}$$

By our assumption, zeros and poles of $f$ and $g$ are of multiplicities atleast $s$; we have

$$\overline{N}(r,g)\leq \frac{1}{s}N(r,g)\leq \frac{1}{s}T(r,g) \tag{3.35}$$

$$\overline{N}(r,\frac{1}{g})\leq \frac{1}{s}N(r,\frac{1}{g})\leq \frac{1}{s}T(r,g). \tag{3.36}$$

We deduce from (3.33)–(3.36) that

$$(n+m-k+1)T(r,f) \le \left(\frac{k+2}{s}+2+m\right)T(r,f)+\left(\frac{2k+2}{s}+1+m\right)T(r,g) + S(r,f)+S(r,g)$$

$$\left(n-1-k-\frac{(k+2)}{s}\right)T(r,f) \le \left(\frac{2k+2}{s}+1+m\right)T(r,g)+S(r,f)+S(r,g). \tag{3.37}$$

Similarly,

$$\left(n-1-k-\frac{(k+2)}{s}\right)T(r,g) \le \left(\frac{2k+2}{s}+1+m\right)T(r,f)+S(r,f)+S(r,g). \tag{3.38}$$

From (3.37) and (3.38), we deduce that $(n-m)s \le 3k+4+(k+2)s$, which contradicts the assumption $(n-m)s > 3k+4+(k+2)s$.

**Case 2:** Suppose that $FG \equiv 1$; by Lemma 2.8, we get a contradiction.

**Case 3:** If $F \equiv G$, then this implies that

$$[f^nP(f)]^{(k)} = [g^nP(g)]^{(k)}. \tag{3.39}$$

Integrating for (3.39), we have

$$[f^nP(f)]^{(k-1)} = [g^nP(g)]^{(k-1)} + b_{k-1}, \tag{3.40}$$

where $b_{k-1}$ is constant. If $b_{k-1} \ne 0$, we obtain $(n-m)s \le 3k+3 < 3k+4+(k+2)s$ by Lemma 2.7. This is a contradiction with our assumption that $(n-m)s > 3k+4+(k+2)s$. Thus $b_{k-1} = 0$. By repeating $k$ times,

$$f^nP(f) = g^nP(g). \tag{3.41}$$

If $m = 1$ in (3.41), then $f = g$ by Lemma 2.6. Suppose that $m \ge 2$ and $h = \frac{f}{g}$. If $h$ is constant, putting $f = gh$ in (3.41), we get

$$a_mg^{n+m}\left(h^{n+m}-1\right)+a_{m-1}g^{n+m-1}\left(h^{n+m-1}-1\right)+\cdots+a_0g^n\left(h^n-1\right) = 0, \tag{3.42}$$

which implies $h^d = 1$, where $d = GCD(n+m,\dots,n+m-i,\dots,n+1,n)$. Hence $f \equiv tg$ for a constant $t$ such that $t^d = 1$, $d = GCD(n+m,\dots,n+m-i,\dots,n+1,n)$, $i = 0,1,\dots,m$.

If $h$ is not constant, then we can see that $f$ and $g$ satisfy the algebraic equation $R(f,g) = 0$, by (3.42), where $R(f,g) = f^nP(f) - g^nP(g)$.
This completes the proof of Theorem 1.2. □

## 4. Open problems.

Following questions are posed from the results:
**Question 4.1.** Can $n$ in Theorems 1.1–1.2 be still reduced?
**Question 4.2.** Is it possible to replace the nonconstant meromorphic functions by nonconstant entire functions?
**Question 4.3.** What can be said about if the sharing one value is replaced by a small function?
**Question 4.4.** Can $(1,2)$ shared value be replaced by $(1,l)$ $(l \ge 0)$ shared value

in Theorems 1.1–1.2?

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