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GENERALIZED RICCI SOLITONS ON TRANS-SASAKIAN MANIFOLDS

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Abstract. The object of the present research is to shows that a trans-Sasakian manifold, which also satisfies the Ricci soliton and generalized Ricci soliton equation, satisfying some conditions, is necessarily the Einstein manifold.

1. Introduction

In 1982, Hamilton [8] introduced that the Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric; that is, they are stationary points of the Ricci flow given by

$$\frac{\partial g}{\partial t} = -2Ric(g). \tag{1.1}$$

A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by

$$L_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where S is the Ricci tensor, L_V is the Lie derivative along the vector field V on M, and λ is a real scalar. Ricci soliton is said to be shrinking, steady, or expanding according as $\lambda < 0, \lambda = 0$, or $\lambda > 0$, respectively.

If the vector field V is the gradient of a potential function -ψ, then g is called a gradient Ricci soliton and equation (1.2) assumes the form $Hess\psi = S + \lambda g$.

On the other hand, the roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations. This subject has manifold

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connections with the other fields of pure mathematics and substantial applications in applied areas such as mechanics, optics, phase space of dynamical system, thermodynamics, and control theory.

The importance of Ricci soliton comes from the facts that they are corresponding to self-similar solutions of the Ricci flow and at the same time they are natural generalizations of Einstein metrics. The concept is named after Gregorio Ricci–Curbastro. Ricci flow solutions are invariant under diffeomorphisms and scaling, so one is led to consider solutions that evolve exactly in these ways. A metric g_0 on a smooth manifold M is a Ricci soliton if there exist a function $\sigma(t)$ and a family of diffeomorphisms $\{\eta(t)\} \subset Diff(M)$ such that

$$g(t) = \sigma(t)\eta(t)^* g_0$$

is a solution of the Ricci flow. In this expression, $\eta(t)^* g_0$ refers to the pullback of the metric g_0 by the diffeomorphism $\eta(t)$. Equivalently, a metric g_0 is a Ricci soliton if and only if it satisfies equation (1.2), which is a generalization of the Einstein condition for the metrics

$$Ric(g_0) = \lambda g_0.$$

Some generalizations, like, gradient Ricci solitons [2], quasi Einstein manifolds [3], and generalized quasi Einstein manifolds [4], play an important role in solutions of geometric flows and describe the local structure of certain manifolds. In 2016, Nurowski and Randall [4] introduced the notion of generalized Ricci soliton as a class of overdetermined system of equations

$$\mathcal{L}_X g + 2aX^\sharp \otimes X^\sharp - 2bS - 2\lambda g = 0, \tag{1.3}$$

where $\mathcal{L}_X g$ and $X^\sharp$ denote, respectively, the Lie derivative of the metric g in the directions of vector field X and the canonical one-form associated to X, and some real constants a, b, and λ.

In 1985, Oubina [15] introduced a new class of almost contact metric manifolds known as trans-Sasakian manifolds. This class consists of both Sasakian and Kenmotsu structures. The above manifolds are studied by several authors, like, Blair [1], Marrero [14], and Kenmotsu [11]. In 1925, Levy [13] obtained the necessary and sufficient conditions for the existence of such tensors. later, Sharma [18] initiated the study of Ricci solitons in almost contact Riemannian geometry. Followed by Tripathi [21], Nagaraja et al. [16], Turan [22], and others extensively studied Ricci solitons in almost contact metric manifold. Therefore, motivated by these studies in the present paper, author studies the generalized Ricci soliton in trans-Sasakian manifolds. Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures, and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. In [19] Siddiqi also studied some properties of conformal η-Ricci solitons in δ-Lorentzian trans-Sasakian manifolds which is closely related with this paper.

The main result of this paper is Theorem 3.1, in which we have to find the conditions for an n-dimensional trans-Sasakian manifold admitting generalized Ricci soliton will be an Einstein manifold.

2. Preliminaries

A differentiable manifold M is said to be an almost contact metric manifold equipped with almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1, 1)$ tensor field ϕ, a vector field ξ, a one-form η, and an indefinite metric g such that

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad \eta(X) = g(X, \xi) \tag{2.2}$$

for all $X, Y \in M$, $\xi \in \Gamma(TM)$ and one-form $\eta \in \Gamma(\bar{T}M)$.

In [21], Tanno classified the connected almost contact metric manifold. For such a manifold, the sectional curvature of the plane section containing ξ is constant, say c. He showed that they can be divided into three classes. (1) homogeneous normal contact Riemannian manifolds with $c > 0$. Other two classes can be seen in Tanno [20].

In [9], Grey and Harvella introduced the classification of almost Hermitian manifolds; there appears a class W_4 of Hermitian manifolds which is closely related to the conformal Kaehler manifolds. The class $C_6 \oplus C_5$ (*see* [1], [5], [14]) coincides with the class of trans-Sasakian structure of type (α, β). In fact, the local nature of the two subclasses, namely C_6 and C_5 of trans-Sasakian structures are characterized completely.

An almost contact metric structure on M is called a trans-Sasakian (*see* [17], [14]) if $(M \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi(X) - f\xi, \eta(X)\frac{d}{dt}\right)$$

for all vector fields X on M and smooth functions f on $M \times R$ and G is the product metric on $M \times R$. This may be expressed by the condition

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.3}$$

where α and β are some scalar functions. We note that trans-Sasakian structure of type $(0, 0)$, $(\alpha, 0)$, and $(0, \beta)$ are the cosymplectic, α-Sasakian, and β-Kenmotsu manifold, respectively. In particular, if $\alpha = 1, \beta = 0$ and $\alpha = 0, \beta = 1$, then a trans-Sasakian manifold reduces to a Sasakian and Kenmotsu manifolds, respectively. From (1.3), it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta[X - \eta(X)]\xi \tag{2.4}$$

and

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta[g(X, Y) - \eta(X)\eta(Y)]. \tag{2.5}$$

for any vector fields X and Y on M, ∇ denotes the Levi–Civita connection with respect to g, α and β are smooth functions on M. The existence of condition (1.3) is ensured by the above discussion.

The Riemannian curvature tensor R with respect to Levi–Civita connections ∇ and the Ricci tensor S of a Riemannian manifold M are defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.6}$$

$$S(X,Y)=\sum_{i=1}^{n} g(R(X,e_i)e_i,Y) \tag{2.7}$$

for $X,Y,Z\in\Gamma(TM)$, where ∇ is with respect to the Riemannian metric g and $\{e_1,e_2,\ldots,e_i\}$, where $1\le i\le n$ is the orthonormal frame.

Given a smooth function ψ on M, the gradient of ψ is defined by

$$g(grad\psi,X)=X(\psi), \tag{2.8}$$

and the *Hessian* of ψ is defined by

$$(Hess\psi)(X,Y)=g(\nabla_X grad\psi,Y), \tag{2.9}$$

where $X,Y\in\Gamma(TM)$. For $X\in\Gamma(TM)$, we define $X^\sharp\in\Gamma(\bar{T}M)$ by

$$X^\sharp(Y)=g(X,Y). \tag{2.10}$$

The generalized Ricci soliton equation in Riemannian manifold M is defined in [16] by

$$\mathcal{L}_X g=-2aX^\sharp\odot X^\sharp+2bS+2\lambda g, \tag{2.11}$$

where $X\in\Gamma(TM)$ and $\mathcal{L}_X g$ is the Lie-derivative of g along X given by

$$(\mathcal{L}_X g)(Y,Z)=g(\nabla_Y X,Z)+g(\nabla_Z X,Y) \tag{2.12}$$

for all $Y,Z\in\Gamma(TM)$, and $a,b,\lambda\in R$. Equation (2.11) is a generalization of

(1) Killing's equation ($a=b=\lambda=0$),
(2) equation for homotheties ($a=b=0$),
(3) Ricci soliton ($a=0,b=-1$),
(4) case of Einstein–Weyl ($a=1,b=\frac{-1}{n-2}$),
(5) metric projective structures with skew-symmetric Ricci tensor in projective class ($a=1,b=\frac{-1}{n-2},\lambda=0$),
(6) vacuum near-horizon geometry equation ($a=1,b=\frac{1}{2}$) (*see* [6], [10], [12]).

Equation (2.11), is also a generalization of Einstein manifolds [4]. Note that, if $X=grad\psi$, where $\psi\in C^\infty(M)$, the generalized Ricci soliton equation is given by

$$Hess\psi=-ad\psi\odot d\psi+bS+\lambda g. \tag{2.13}$$

3. Main results

In an n-dimensional trans-Sasakian manifold M, we have the following relations:

$$R(X,Y)\xi=(\alpha^2-\beta^2)[\eta(Y)X-\eta(X)Y]+2\alpha\beta[\eta(Y)\phi X-\eta(X)\phi Y] \tag{3.1}$$
$$+[(Y\alpha)\phi X-(X\alpha)\phi Y+(Y\beta)\phi^2X-(X\beta)\phi^2Y],$$

$$S(X,\xi)=[(n-1)(\alpha^2-\beta^2)-(\xi\beta)]\eta(X)-((\phi X)\alpha)-(n-2)(X\beta), \tag{3.2}$$

$$Q\xi=(n-1)(\alpha^2-\beta^2)-(\xi\beta))\xi+\phi(grad\alpha)-2n(grad\beta), \tag{3.3}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.
Further, for a trans-Sasakian manifold, we have

$$\phi(grad\alpha) = 2n(grad\beta), \tag{3.4}$$

and

$$2\alpha\beta + (\xi\alpha) = 0. \tag{3.5}$$

Using (3.4) and (3.5), for constants α and β, we have

$$R(\xi,X)Y = (\alpha^2 - \beta^2)[g(X,Y)\xi - \eta(Y)X], \tag{3.6}$$

$$R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y], \tag{3.7}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \beta^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{3.8}$$

$$S(X,\xi) = [((n-1)(\alpha^2 - \beta^2)]\eta(X), \tag{3.9}$$

$$Q\xi = [(n-1)(\alpha^2 - \beta^2)]\xi. \tag{3.10}$$

An important consequence of (2.4) is that ξ is a geodesic vector field; that is,

$$\nabla_\xi\xi = 0. \tag{3.11}$$

For an arbitrary vector field X, we have that

$$d\eta(\xi,X) = 0. \tag{3.12}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (3.7), we have

$$K_\xi = g(R(\xi,X),\xi,X) = (\alpha^2 - \beta^2). \tag{3.13}$$

It follows from (3.13) that the ξ-sectional curvature does not depend on X.

Theorem 3.1. *Let M be a trans-Sasakian manifold of dimension n and let it satisfy the generalized Ricci soliton (2.13) with condition $a[\lambda+(n-1)b(\alpha^2-\beta^2)] \neq -1$; then ψ is a constant function. Furthermore, if $b \neq 0$, then M is an Einstein manifold.*

From Theorem 3.1, we get the following remarks.

Remark 3.2. let M be a trans-Sasakian manifold which satisfies the gradient Ricci soliton equation $Hess\psi = -S + \lambda g$; then ψ is a constant function and M is an Einstein manifold.

Remark 3.3. In a trans-Sasakian manifold M, there is no nonconstant smooth function ψ such that $Hess\psi = \lambda g$ for some constant λ.

For the proof of Theorem 3.1, first we need to prove the following lemmas.

Lemma 3.4. *Let M be a trans-Sasakian manifold. Then we have*

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = (\alpha^2 - \beta^2)g(X,Y) + g(\nabla_\xi\nabla_\xi X, Y) + Yg(\nabla_\xi X,\xi), \tag{3.14}$$

where $X, Y \in \Gamma(TM)$ and Y is orthogonal to ξ.

Proof. From the property of Lie-derivative, we note that

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = \xi((\mathcal{L}_X g)(Y,\xi)) - (\mathcal{L}_X g)(\mathcal{L}_\xi Y,\xi) - (\mathcal{L}_X g)(Y,\mathcal{L}_\xi \xi); \tag{3.15}$$

since $\mathcal{L}_\xi Y = [\xi, Y]$, $\mathcal{L}_\xi \xi = [\xi,\xi]$, by using (2.12) and (3.15), we have

$$\begin{aligned}(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) &= \xi g(\nabla_Y X,\xi) + \xi g(\nabla_\xi X, Y) - g(\nabla_{[\xi,Y]}X,\xi) \\ &\quad - g(\nabla_\xi X,[\xi,Y]) \\ &= g(\nabla_\xi\nabla_Y X,\xi) + g(\nabla_Y X,\nabla_\xi\xi) + g(\nabla_\xi\nabla_\xi X,Y) + g(\nabla_\xi X,\nabla_\xi Y) \\ &\quad - g(\nabla_\xi X,\nabla_\xi Y) - g(\nabla_{[\xi,Y]}X,\xi) + g(\nabla_\xi X.\nabla_Y\xi),\end{aligned} \tag{3.16}$$

From (2.4), we get $\nabla_\xi\xi = \phi\xi = 0$; so that we get

$$\begin{aligned}(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) &= g(\nabla_\xi\nabla_Y X,\xi) + g(\nabla_\xi\nabla_\xi X,Y) - g(\nabla_{[\xi,Y]}X,\xi) \\ &\quad + Yg(\nabla_\xi X,\xi) - g(\nabla_Y\nabla_\xi X,\xi);\end{aligned} \tag{3.17}$$

using (3.6) and (3.17), we obtain

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = g(R(\xi,Y)X,\xi) + g(\nabla_\xi\nabla_\xi X,Y) + Yg(\nabla_\xi X,\xi). \tag{3.18}$$

Now from (3.6), with $g(Y,\xi)=0$, we get

$$g(R(\xi,Y)X,\xi) = g(R(Y,\xi)\xi,X) = (\alpha^2-\beta^2)g(X,Y). \tag{3.19}$$

the lemma follows from (3.17) and (3.18).

□

Now, we have another useful lemma.

Lemma 3.5. *Let M be a Riemannian manifold, and let $\psi \in C^\infty(M)$. Then we have*

$$(\mathcal{L}_\xi(d\psi\odot d\psi))(Y,\xi) = Y(\xi(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)), \tag{3.20}$$

where $\xi, Y \in \Gamma(TM)$.

Proof. We calculate

$$\begin{aligned}(\mathcal{L}_\xi(d\psi\odot d\psi))(Y,\xi) &= \xi(Y(\psi)\xi(\psi) - [\xi,Y](\psi)\xi(\psi) - Y(\psi)[\xi,\xi](\psi) \\ &= \xi(Y(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)) - [\xi,Y](\psi)\xi(\psi);\end{aligned}$$

since $[\xi,Y](\psi) = \xi(Y(\psi)) - Y(\xi(\psi))$, we get

$$\begin{aligned}(\mathcal{L}_\xi(d\psi\odot d\psi))(Y,\xi) &= [\xi,Y](\psi)\xi(\psi) + Y(\xi(\psi))\xi(\psi) \\ &\quad + Y(\psi)\xi(\xi(\psi)) - [\xi,Y](\psi)\xi(\psi) \\ &= Y(\xi(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)).\end{aligned}$$

□

Lemma 3.6. *Let M be a trans-Sasakian manifold of dimension n, which satisfies the generalized Ricci soliton equation (2.13). Then we have*

$$\nabla_\xi\ grad\psi = [\lambda + b(n-1)(\alpha^2-\beta^2)]\xi - a\xi(\psi)grad\psi. \tag{3.21}$$

Proof. Let $Y \in \Gamma(TM)$. From the definition of Ricci curvature S (2.7) and the curvature condition (3.7), we have

$$\begin{aligned} S(\xi, Y) &= g(R(\xi, e_i)e_i, Y) \\ &= g(R(e_i, Y)\xi, e_i) \\ &= (\alpha^2 - \beta^2)[\eta(Y)g(e_i, e_i) - \eta(e_i)g(Y, e_i) \\ &= (\alpha^2 - \beta^2)[n\eta(Y) - \eta(Y)] \\ &= (n-1)(\alpha^2 - \beta^2)\eta(Y) \\ &= (n-1)(\alpha^2 - \beta^2)g(\xi, Y), \end{aligned}$$

where $\{e_1, e_2, \ldots, e_i\}$, and $1 \leq i \leq n$ is an orthonormal frame on M implies that

$$\begin{aligned} \lambda g(\xi, Y) + bS(\xi, Y) &= \lambda g(\xi, Y) + b(n-1)(\alpha^2 - \beta^2)g(\xi, Y) \\ &= [\lambda + b(n-1)(\alpha^2 - \beta^2)]g(\xi, Y). \end{aligned} \tag{3.22}$$

From (2.13) and (3.22), we obtain

$$\begin{aligned} (Hess\psi)(\xi, Y) &= -a\xi(\psi)(Y)(\psi) + [\lambda + b(n-1)(\alpha^2 - \beta^2)]g(\xi, Y) \\ &= -a\xi(\psi)g(grad\psi, Y) + [\lambda + b(n-1)(\alpha^2 - \beta^2)]g(\xi, Y); \end{aligned} \tag{3.23}$$

the lemma follows from equation (3.23) and the definition of *Hessian* (2.7). ☐

Now, with help of Lemmas 3.4, 3.5, and 3.6, we can prove Theorem 3.1.

Proof of Theorem 3.1. Let $Y \in \Gamma(TM)$ be such that $g(\xi, Y) = 0$; from Lemma 3.4, with $X = grad\ \psi$, we have

$$2(\mathcal{L}_\xi(Hess\psi))(Y, \xi) = Y(\psi) + g(\nabla_\xi \nabla_\xi\ grad\psi, Y) + Yg(\nabla_\xi\ grad\psi, \xi); \tag{3.24}$$

from Lemma 3.6 and equation (3.24), we get

$$\begin{aligned} 2(\mathcal{L}_\xi(Hess\psi))(Y, \xi) = Y(\psi) &+ [\lambda + b(n-1)(\alpha^2 - \beta^2)]g(\nabla_\xi \xi, Y) \\ &- ag(\nabla_\xi(\xi(\psi)grad\ \psi), Y) \\ &+ [\lambda + b(n-1)(\alpha^2 - \beta^2)]Yg(\xi, \xi) - aY(\xi(\psi)^2). \end{aligned} \tag{3.25}$$

Since $\nabla_\xi \xi = 0$ and $g(\xi, \xi) = 1$, from equation (3.25), we obtain

$$\begin{aligned} 2(\mathcal{L}_\xi(Hess\psi))(Y, \xi) = Y(\psi) &- a\xi(\xi(\psi))Y(\psi) - a\xi(\psi)g(\nabla_\xi\ grad\psi, Y) \\ &- 2a\xi(\psi)Y(\xi(\psi)). \end{aligned} \tag{3.26}$$

From Lemma 3.6 and equation (3.26) and since $g(\xi, Y) = 0$, we have

$$\begin{aligned} 2(\mathcal{L}_\xi(Hess\psi))(Y, \xi) = Y(\psi) &- a\xi(\xi(\psi))Y(\psi) + a^2\xi(\psi)^2Y(\psi) \\ &- 2a\xi(\psi)Y(\xi(\psi)). \end{aligned} \tag{3.27}$$

Note that, from (2.11) and (2.12), we have $\mathcal{L}_\xi g = 0$, which is a Killing vector field; it implies that $\mathcal{L}_\xi S = 0$; taking the Lie derivative of the generalized Ricci soliton equation (2.13) yields

$$\begin{aligned} (\alpha^2 - \beta^2)Y(\psi) - a\xi(\xi(\psi))Y(\psi) + a^2\xi(\psi)^2Y(\psi) - 2a\xi(\psi)Y(\xi(\psi)) \\ = -2aY(\xi(\psi))\xi(\psi) - 2aY(\psi)\xi(\xi(\psi)), \end{aligned} \tag{3.28}$$

which is equivalent to

$$Y(\psi)[(\alpha^2-\beta^2)+a\xi(\xi(\psi))+a^2\xi(\psi)^2]=0; \tag{3.29}$$

according to Lemma 3.6, we have

$$\begin{aligned} a\xi(\xi(\psi)) &= a\xi g(\xi, grad\ \psi) \\ &= ag(\xi, \nabla_\xi\ grad\psi) \\ &= a[\lambda+b(n-1)(\alpha^2-\beta^2)]-a^2\xi(\psi)^2, \end{aligned} \tag{3.30}$$

by equations (3.29) and (3.30), we obtain

$$Y(\psi)[1+a(\lambda+b(n-1)(\alpha^2-\beta^2))]=0; \tag{3.31}$$

since $a[\lambda+b(n-1)(\alpha^2-\beta^2)]\neq -1$, we find that $Y(\psi)=0$; that is, $grad\psi$ is parallel to ξ. Hence $grad\ \psi=0$ as $D=ker\eta$ is not integrable any where, which means ψ is a constant function. □

Now, for particular values of α and β, we have following cases:

- For $\alpha=0$ or $(\beta=1)$, we can state

  Corollary 3.7. *Let M be a β-Kenmotsu (or Kenmotsu) manifold of dimension n, and it satisfies the generalized Ricci soliton (2.13) with condition $a[\lambda-(n-1)b\beta^2)]\neq -1$; then ψ is a constant function. Furthermore, if $b\neq 0$, then M is an Einstein manifold.*

- For $\beta=0$, or $(\alpha=1)$ we can state

  Corollary 3.8. *Let M be a α-Sasakian (or Sasakian) manifold of dimension n, and it satisfies the generalized Ricci soliton (2.13) with condition $a[\lambda+(n-1)b\alpha^2)]\neq -1$; then ψ is a constant function. Furthermore, if $b\neq 0$, then M is an Einstein manifold.*

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