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ON A NEW SUBCLASS OF M-FOLD SYMMETRIC BIUNIVALENT FUNCTIONS EQUIPPED WITH SUBORDINATE CONDITIONS

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Abstract. In this paper, we introduce a new subclass of biunivalent function class Σ in which both $f(z)$ and $f^{-1}(z)$ are m-fold symmetric analytic functions. For functions of the subclass introduced in this paper, we obtain the coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ and also study the Fekete–Szegö functional estimate for this class. Consequences of the results are also discussed.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in C : |z| < 1\}$. Let S be the subclass of A consisting of functions, which are analytic and univalent in $\mathbb{U}$.

The Keobe one-quarter theorem [8] states that, the range of every function of the class S contains the disk $\{w : |w| < 1/4\}$. Therefore, every $f \in S$ has an inverse function f^{-1} satisfying

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f); r_0(f) \geq 1/4).$$

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The inverse of $f(z)$ has a series expansion in some disc about the origin of the form

$$f^{-1}(w) = w + A_2w^2 + A_3w^3 + \cdots . \tag{1.2}$$

A function $f(z)$, which is univalent in a neighborhood of the origin, and its inverse satisfy the condition $f(f^{-1}(w)) = w$
using (1.2) yields

$$w = f^{-1}(w) + a_2(f^{-1}(w))^2 + a_3(f^{-1}(w))^3 + \cdots , \tag{1.3}$$

and now using (1.3), we get the following result:

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.4}$$

An analytic function $f(z)$ is said to be biunivalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. The class of analytic biunivalent function in $\mathbb{U}$ is denoted by Σ.

For a brief history and interesting examples of functions in the class Σ; see the pioneering work on this subject by Srivastava et al. [18], which has apparently revived the study of biunivalent functions in recent years. From the work of Srivastava et al. [18], we choose to recall the following examples of functions in the class Σ:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of the biunivalent function class Σ. Such other common examples of functions in S as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ (see [18]).

If the function f and g are analytic in $\mathbb{U}$, then f is said to be subordinate to g, written as

$$f(z) \prec g(z) \qquad (z \in \mathbb{U})$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in \mathbb{U})$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

Lewin [11] studied the class of biunivalent functions, obtaining the bound 1.51 for the modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [6] conjectured that $|a_2| \leqq \sqrt{2}$ for $f \in \Sigma$. Later on, Netanyahu [14] showed that $\max|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Taha [7] introduced certain subclasses of the biunivalent function class Σ similar to the familiar subclasses $S^\star(\beta)$ and $K(\beta)$ of starlike and convex functions of order β $(0 \leqq \beta < 1)$ in $\mathbb{U}$, respectively (see [14]). The classes $S_\Sigma^\star(\beta)$ and $K_\Sigma(\beta)$ of bistarlike functions of order β in $\mathbb{U}$ and biconvex functions of order β in $\mathbb{U}$, corresponding to the function classes $S^\star(\beta)$ and $K(\beta)$, were also introduced analogously. For each of the function classes $S_\Sigma^\star(\beta)$ and $K_\Sigma(\beta)$, they found nonsharp estimates for the initial coefficients.

Recently, motivated substantially by the aforementioned pioneering work on this subject by Srivastava et al. [18], many authors investigated the coefficient bounds for various subclasses of biunivalent functions (see, for example, [1], [2], [3], [4], [10], [13], [15], [20], [22], and [23]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geqq 4$. The coefficient estimate problem for each of the coefficients

$$|a_n| \quad (n \in \mathbb{N} \setminus \{1, 2\}, \ \mathbb{N} = \{1, 2, 3, \ldots\})$$

is still an open problem.

For each function f in S, the function h given by

$$h(z) = \sqrt[m]{f(z^m)} \qquad (m \in \mathbb{N})$$

is univalent and maps the unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [16]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (m \in \mathbb{N}, \ z \in \mathbb{U}). \tag{1.5}$$

We denote the class of m-fold symmetric univalent functions by S_m, which are normalized by the above series expansion (1.5). In fact the functions in the class S are one fold symmetric (that is $m = 1$). Analogous to the concept of m-fold symmetric univalent functions, one can think of the concept of m-fold symmetric biunivalent function in a natural way. Each function f in the class Σ generates an m-fold symmetric biunivalent function for each positive integer m. The normalized form of f is given as (1.5) and f^{-1} is given by as follows:

$$\begin{aligned} g(w) =& w - a_{m+1} z^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ & - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots , \end{aligned} \tag{1.6}$$

where $f^{-1} = g$. We denote the class of m-fold symmetric biunivalent functions by Σ_m. For $m = 1$, the formula (1.6) coincides with the function (1.4) of the class Σ. Some examples of m-fold symmetric biunivalent functions are given here below:

$$\left(\frac{z^m}{1 - z^m}\right)^{\frac{1}{m}}, \qquad [-log(1 - z^m)]^{\frac{1}{m}}, \qquad \left[\frac{1}{2} log\left(\frac{1 + z^m}{1 - z^m}\right)^{\frac{1}{m}}\right].$$

Here in this paper, we also denote P the class of analytic functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots ,$$

such that

$$R(p(z)) > 0 \qquad (z \in \mathbb{U}).$$

In view of the work of Pommerenke [16] the m-fold symmetric function p in the class P is of the form

$$p(z) = 1 + c_m z^m + c_{2m} z^{2m} + c_{3m} z^{3m} + \cdots . \tag{1.7}$$

Let ϕ be an analytic function with positive real part in $\mathbb{U}$, with $\phi(0) = 1$ and $\phi'(0) > 0$. Also, let $\phi(\mathbb{U})$ be starlike with respect to one and symmetric with respect to the axis. Thus, ϕ has the Taylor series expansion

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \qquad (B_1 > 0). \tag{1.8}$$

Suppose that $u(z)$ and $v(w)$ are analytic in the unit disk $\mathbb{U}$ with $u(0) = v(0) = 0, |u(z)| < 1$, and $|v(w)| < 1$.
We suppose that

$$u(z) = b_m z^m + b_{2m} z^{2m} + b_{3m} z^{3m} + \cdots \qquad (|z| < 1) \tag{1.9}$$

and

$$v(w) = c_m z^m + c_{2m} z^{2m} + c_{3m} z^{3m} + \cdots \qquad (|w| < 1). \tag{1.10}$$

It is well known that

$$|b_m| \leq 1, |b_{2m}| \leq 1 - |b_m|^2, |c_m| \leq 1, |c_{2m}| \leq 1 - |c_m|^2. \tag{1.11}$$

By simple computations

$$\phi(u(z)) = 1 + B_1 b_m z^m + (B_1 b_{2m} + B_2 b_m^2) z^{2m} + \cdots \qquad (|z| < 1) \tag{1.12}$$

and

$$\phi(u(z)) = 1 + B_1 c_m z^m + (B_1 c_{2m} + B_2 c_m^2) z^{2m} + \cdots \qquad (|w| < 1). \tag{1.13}$$

Babalola [5] defined the class $L_\lambda(\beta)$ of λ-pseudo-starlike functions of order β as below.

Definition 1.1. Let $f \in A$; suppose that $0 \leq \beta < 1$ and that $\lambda \geq 1$ is real. Then $f(z) \in L_\lambda(\beta)$ of λ-pseudo-starlike functions of order β in the unit disk if and only if

$$Re \frac{z\,[f'(z)]^\lambda}{f(z)} > \beta.$$

Babalola [5] proved that, all pseudo-starlike functions are Bazilevic of type $(1-\frac{1}{\lambda})$ order $\beta^{\frac{1}{\lambda}}$ and univalent in open unit disk $\mathbb{U}$.

We now introduce the following subclass of m-fold symmetric biunivalent function class Σ_m.

Definition 1.2. A function $f \in \Sigma_m$ said to be in the class $S^\lambda_{\Sigma,m}(\phi)$, if the following subordination conditions hold:

$$\frac{z\,[f'(z)]^\lambda}{f(z)} \prec \phi(z) \tag{1.14}$$

and

$$\frac{w\,[g'(w)]^\lambda}{g(w)} \prec \phi(w),$$

where $g = f^{-1}$ and $\lambda \geq 1$.

For various special choices of the function ϕ and for the case when $m=1$, our function class $S^{\lambda}_{\Sigma,m}(\phi)$ reduces to the following known classes:

(1) Taking $m=1$, the function class is given by

$$S^{\lambda}_{\Sigma,m}(\phi) \equiv S^{\lambda}_{\Sigma,1}(\phi) \equiv S^{\lambda}_{\Sigma}(\phi).$$

(2) For $m=1$ and $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$ $(0<\alpha\le 1)$, the function class given by

$$S^{\lambda}_{\Sigma,m}(\phi) \equiv S^{\lambda}_{\Sigma,1}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$$

was studied by Joshi et al. [10].

(3) For $m=1$ and $\phi(z) = \left(\frac{1+(1-2\beta)z}{1-z}\right)$ $(0\le\beta<1)$, the function class given by

$$S^{\lambda}_{\Sigma,m}(\phi) \equiv S^{\lambda}_{\Sigma,1}\left(\frac{1+(1-2\beta)z}{1-z}\right)$$

was studied by Joshi et al. [10].

Motivated by the work of Ma and Minda [12] and Srivastava et al. [19], we introduce a new subclass of m-fold symmetric biunivalent functions. We obtain the coefficients bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ and also the Fekete–Szegö functional estimate for the subclass. The results improve the earlier results of Joshi et al. [10].

2. Coefficient estimates

We begin this section by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $S^{\lambda}_{\Sigma,m}(\phi)$ proposed by Definition 1.2.

Theorem 2.1. *Let the function f given by (1.5) be in the class $S^{\lambda}_{\Sigma,m}(\phi)$. Then*

$$|a_{m+1}| \le \frac{B_1\sqrt{2B_1}}{\sqrt{2\left(\lambda m+\lambda-1\right)^2 B_1 + \left|(m^2\lambda^2+\lambda m^2+2m\lambda^2-\lambda m+\lambda^2-2\lambda-m+1)B_1^2 - 2(\lambda m+\lambda-1)^2 B_2\right|}} \tag{2.1}$$

$$|a_{2m+1}| \le \begin{cases} \frac{B_1}{|2m\lambda+\lambda-1|}, & B_1 < \frac{2(\lambda m+\lambda-1)^2}{(m+1)|2m\lambda+\lambda-1|}, \\ \left((m+1) - \frac{2(m\lambda+\lambda-1)^2}{|2m\lambda+\lambda-1|B_1}\right) \frac{B_1^3}{2(m\lambda+\lambda-1)^2 B_1 + |(m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1)B_1^2 - 2(m\lambda+\lambda-1)^2 B_2|} \\ \quad + \frac{B_1}{|2\lambda m+\lambda-1|}, & B_1 \ge \frac{2(\lambda m+\lambda-1)^2}{(m+1)|2m\lambda+\lambda-1|}. \end{cases} \tag{2.2}$$

Proof. Let $f\in S^{\lambda}_{\Sigma,m}$ and $g=f^{-1}$. Then there are analysis functions $u:\mathbb{U}\to\mathbb{U}$ and $v:\mathbb{U}\to\mathbb{U}$, with

$$u(0)=v(0)=0,$$

satisfying the following conditions:

$$\frac{z[f'(z)]^{\lambda}}{f(z)} = \phi(u(z)) \tag{2.3}$$

and

$$\frac{w[g'(w)]^{\lambda}}{g(w)} = \phi(v(w)). \tag{2.4}$$

Comparing the corresponding coefficients of (2.3) and (2.4) yields

$$(m\lambda + \lambda - 1)a_{m+1} = B_1 b_m, \tag{2.5}$$

$$\left[\lambda(m+1)\left(\frac{(\lambda-1)(m+1)}{2} - 1\right) + 1\right] a_{m+1}^2 + (2m\lambda + \lambda - 1)a_{2m+1} = B_1 b_{2m} + B_2 b_m^2, \tag{2.6}$$

$$-(m\lambda + \lambda - 1)a_{m+1} = B_1 c_m, \tag{2.7}$$

and

$$\left[\lambda(m+1)\left(\frac{(\lambda-1)(m+1)}{2} + 2m\right) - m\right] a_{m+1}^2 - (2m\lambda+\lambda-1)a_{2m+1} = B_1 c_{2m} + B_2 c_m^2. \tag{2.8}$$

It implies from (2.5) and (2.7) that

$$c_m = -b_m. \tag{2.9}$$

By adding (2.6) and (2.8), further computation using (2.5) and (2.9) lead to

$$\left[(m^2\lambda^2 + \lambda m^2 + 2m\lambda^2 - \lambda m + \lambda^2 - 2\lambda - m + 1)B_1^2 - 2(\lambda m + \lambda - 1)^2 B_2\right] a_{m+1}^2 = B_1^3(b_{2m} + c_{2m}). \tag{2.10}$$

Using (2.9) and (2.10), together with (1.11), yield

$$|(m^2\lambda^2 + \lambda m^2 + 2m\lambda^2 - \lambda m + \lambda^2 - 2\lambda - m + 1)B_1^2 - 2(\lambda m + \lambda - 1)^2 B_2||a_{m+1}|^2 \leq 2B_1^3(1 - |b_m|^2). \tag{2.11}$$

Equations (2.5) and (2.11) give the desired estimate on $|a_{m+1}|$ as asserted in (2.1). By subtracting (2.8) from (2.6), we obtain

$$2(2m\lambda + \lambda - 1)a_{2m+1} = (m+1)(2m\lambda + \lambda - 1)a_{m+1}^2 + B_1(b_{2m} - c_{2m}). \tag{2.12}$$

From (1.11), (2.5), (2.8), and (2.12), it follows that

$$\begin{aligned}
|a_{2m+1}| &\leq \frac{(m+1)}{2}|a_{m+1}|^2 + \frac{B_1}{2\,|2m\lambda + \lambda - 1|}(|b_{2m}| + |c_{2m}|) \\
&\leq \frac{(m+1)}{2}|a_{m+1}|^2 + \frac{B_1}{|2m\lambda + \lambda - 1|}(1 - |b_m|^2) \\
&= \left(\frac{m+1}{2} - \frac{(m\lambda + \lambda - 1)^2}{|2m\lambda + \lambda - 1|\, B_1}\right)|a_{m+1}|^2 + \frac{B_1}{|2m\lambda + \lambda - 1|},
\end{aligned}$$

which implies the assertion (2.2). □

For the case of one-fold symmetric function, Theorem 2.1 reduces to Corollary 2.2 below.

Corollary 2.2. *Let the function f given by (1.5) be in the class $S_{\Sigma}^{\lambda}(\phi)$. Then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{(2\lambda-1)\left[(2\lambda-1)B_1 + |\lambda B_1^2 - (2\lambda-1)B_2|\right]}} \tag{2.13}$$

and

$$|a_3| \le \begin{cases} \frac{B_1}{3\lambda-1}, & B_1 < \frac{(2\lambda-1)^2}{3\lambda-1}, \\ \left(1 - \frac{(2\lambda-1)^2}{(3\lambda-1)B_1}\right) \frac{B_1^3}{(2\lambda-1)^2B_1 + |(2\lambda^2-\lambda)B_1^2 - (2\lambda-1)^2B_2|} + \frac{B_1}{3\lambda-1}, & B_1 \ge \frac{(2\lambda-1)^2}{3\lambda-1}. \end{cases} \tag{2.14}$$

Remark 2.3. For $f \in S_{\Sigma}^{\lambda}(\phi)$, the function ϕ is given by

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \qquad (0 < \alpha \le 1),$$

and so $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$. Hence Corollary 2.2 reduces to an improved results of Joshi et al. [10].
On the other hand when

$$\phi(z) = \frac{1 + (1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \qquad (0 \le \beta < 1),$$

$B_1 = B_2 = 2(1-\beta)$, and thus Corollary 2.2 reduces to the improved results of Joshi et al. [10].

For the case of one-fold symmetric functions with $\lambda = 1$, the class reduces to the strongly starlike functions; the function ϕ is given by

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z + \cdots \qquad (0 < \alpha \le 1), \tag{2.15}$$

which gives

$$B_1 = 2\alpha \text{ and } B_2 = 2\alpha^2.$$

Hence, Theorem 2.1 gives the following corollary.

Corollary 2.4. *Let the function f given by (1.5) be in the class $S_{\Sigma,1}^{1}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{1+\alpha}} \tag{2.16}$$

and

$$|a_3| \le \begin{cases} \alpha, & 0 < \alpha \le \frac{1}{4}, \\ \frac{5\alpha^2}{1+\alpha}, & \frac{1}{4} < \alpha \le 1. \end{cases} \tag{2.17}$$

For the case of one-fold symmetric functions with $\lambda = 1$, the class reduces to the strongly starlike functions, and the function ϕ is given by

$$\phi(z) = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \qquad (0 \le \beta < 1);$$

so that

$$B_1 = B_2 = 2(1-\beta).$$

Corollary 2.5. *Let the function f given by (1.5) be in the class $S^{1}_{\Sigma,1}\left(\frac{1+(1-2\beta)z}{1-z}\right)$. Then*

$$|a_2| \le \frac{2(1-\beta)}{\sqrt{1+|1-2\beta|}} \tag{2.18}$$

and

$$|a_3| \le \begin{cases} \frac{5-6\beta}{2}, & 0 \le \beta < \frac{3}{4}, \\ 1-\beta, & \frac{3}{4} \le \beta < 1. \end{cases} \tag{2.19}$$

3. Fekete–Szegö problem

The classical Fekete–Szegö inequality, presented by means of Loewner's method, for the coefficients of $f \in S$, is

$$\left|a_3 - \mu a_2^2\right| \le 1 + 2\exp(-2\mu/(1-\mu)) \quad \text{for } \mu \in [0,1)\,.$$

As $\mu \to 1^-$, we have the elementary inequality $|a_3 - a_2^2| \le 1$. Moreover, the coefficient functional

$$\Phi_\mu(f) = a_3 - \mu a_2^2$$

on the normalized analytic functions f in the unit disk $\mathbb{U}$ plays an important role in function theory. The problem of maximizing the absolute value of the functional $\Phi_\mu(f)$ is called the Fekete–Szegö problem, see [9].

In this section, we aim to provide Fekete-Szegö inequalities for functions in the class $S^{\lambda}_{\Sigma,m}(\phi)$. These inequalities are given in the following theorem.

Theorem 3.1. *Let the function $f(z)$, given by (1.5), be in the class $S^{\lambda}_{\Sigma,m}(\phi)$. Then*

$$\left|a_{2m+1} - \mu a_{m+1}^2\right| \le \begin{cases} \dfrac{B_1}{|2m\lambda+\lambda-1|}, & 0 \le |h(\mu)| < \frac{1}{2|2m\lambda+\lambda-1|} \\ 2B_1\,|h(\mu)|\,, & |h(\mu)| \ge \frac{1}{|2m\lambda+\lambda-1|}, \end{cases} \tag{3.1}$$

where

$$h(\mu) = \frac{B_1^2(m+1-2\mu)}{2\left[(m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1)B_1^2 - 2(m\lambda+\lambda-1)^2B_2\right]}.$$

Proof. From the equation (2.10), we get

$$a_{m+1}^2 = \frac{B_1^3(b_{2m}+c_{2m})}{(m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1)B_1^2 - 2(m\lambda+\lambda-1)^2B_2}. \tag{3.2}$$

By subtracting (2.6) from (2.8), we get

$$a_{2m+1} = \frac{(m+1)}{2}a_{m+1}^2 + \frac{B_1(b_{2m}-c_{2m})}{2(2m\lambda+\lambda-1)}. \tag{3.3}$$

From equations (3.2) and (3.3), we obtain

$$a_{2m+1} - \mu a_{m+1}^2 = B_1\left[\left(h(\mu) + \frac{1}{2(2m\lambda+\lambda-1)}\right)b_{2m} + \left(h(\mu) - \frac{1}{2(2m\lambda+\lambda-1)}\right)c_{2m}\right],$$

where

$$h(\mu)=\frac{B_1^2(m+1-2\mu)}{2\left[(m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1)B_1^2-2(m\lambda+\lambda-1)^2B_2\right]}.$$

All B_i are real and $B_1>0$, which implies the assertion equation (3.1). $\square$

For the case of one-fold symmetric functions, Theorem 3.1 reduces to the following Corollary 3.2.

Corollary 3.2. *Let the function f given by (1.5) be in the class $S_{\Sigma,1}^{\lambda}(\phi)$. Then*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{B_1}{3\lambda-1}, & 0\leq|h(\mu)|<\frac{1}{2(3\lambda-1)},\\ \\ 2B_1|h(\mu), & |h(\mu)|\geq\frac{1}{2(3\lambda-1)},\end{cases}$$

where

$$h(\mu)=\frac{B_1^2(1-\mu)}{2(2\lambda-1)\left[\lambda B_1^2-(2\lambda-1)B_2\right]}.$$

Taking $\mu=1$ and $\mu=0$ in Theorem 3.1, we have the following corollaries.

Corollary 3.3. *Let the function f given by (1.5) be in the class $S_{\Sigma,m}^{\lambda}(\phi)$. Then*

$$\left|a_{2m+1}-a_{m+1}^2\right|\leq$$

$$\begin{cases}\dfrac{B_1}{|2m\lambda+\lambda-1|}, & \frac{B_2}{B_1^2}\in(-\infty,\rho_1)\cup(\rho_2,\infty),\\ \\ \frac{B_1^3(m-1)}{\left|(m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1)B_1^2-2(m\lambda+\lambda-1)^2B_2\right|}, & \\ \\ \frac{B_2}{B_1^2}\in\left(\rho_1,\frac{m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1}{2(m\lambda+\lambda-1)^2}\right)\cup\left(\frac{m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1}{2(m\lambda+\lambda-1)^2},\rho_2\right), &\end{cases}$$

where

$$\rho_1=\frac{m^2\lambda^2-m^2\lambda+2m\lambda^2+\lambda^2-\lambda}{2(m\lambda+\lambda-1)^2}$$

and

$$\rho_2=\frac{m^2\lambda^2+3m^2\lambda+2m\lambda^2-2m\lambda+\lambda^2-3\lambda-2m+2}{2(m\lambda+\lambda-1)^2}.$$

For the case of one-fold symmetric functions, Corollary 3.3 reduces to the following corollary.

Corollary 3.4. *Let the function f given by (1.5) be in the class $S_{\Sigma,1}^{\lambda}(\phi)$. Then*

$$\left|a_3-a_2^2\right|\leq\frac{B_1}{3\lambda-1}.$$

Also, letting $\lambda=1$, we obtain

$$\left|a_3-a_2^2\right|\leq\frac{B_1}{2}.$$

Corollary 3.5. *Let the function f given by (1.5) be in the class $S_{\Sigma,m}^{\lambda}(\phi)$. Then*

$$|a_{2m+1}| \leq$$

$$\begin{cases} \dfrac{B_1}{|2m\lambda+\lambda-1|}, & \frac{B_2}{B_1^2} \in (-\infty, \sigma_1) \cup (\sigma_2, \infty), \\ \frac{B_1^3(m+1)}{\left|(m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1)B_1^2-2(m\lambda+\lambda-1)^2B_2\right|}, & \\ \frac{B_2}{B_1^2} \in \left(\sigma_1, \frac{m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1}{2(m\lambda+\lambda-1)^2}\right) \cup \left(\frac{m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1}{2(m\lambda+\lambda-1)^2}, \sigma_2\right), & \end{cases}$$

where

$$\sigma_1 = \frac{m^2\lambda^2 - m^2\lambda + 2m\lambda^2 - 4m\lambda + \lambda^2 - 3\lambda + 2}{2(m\lambda+\lambda-1)^2}$$

and

$$\sigma_2 = \frac{m^2\lambda^2 + 3m^2\lambda + 2m\lambda^2 + 2m\lambda + \lambda^2 + \lambda - 2m}{2(m\lambda+\lambda-1)^2}.$$

Corollary 3.6. *Let the function f given by (1.5) be in the class $S_{\Sigma,1}^{\lambda}(\phi)$. Then*

$$|a_3| \leq \begin{cases} \frac{B_1}{3\lambda-1}, & \frac{B_2}{B_1^2} \in \left(-\infty, \frac{2\lambda^2-4\lambda+1}{(2\lambda-1)^2}\right) \cup \left(\frac{2\lambda^2+2\lambda-1}{(2\lambda-1)^2}, \infty\right), \\ \frac{B_1^3}{(2\lambda-1)[\lambda B_1^2-(2\lambda-1)B_2]}, & \frac{B_2}{B_1^2} \in \left(\frac{2\lambda^2-4\lambda+1}{(2\lambda-1)^2}, \frac{\lambda}{2\lambda-1}\right) \cup \left(\frac{\lambda}{2\lambda-1}, \frac{2\lambda^2+2\lambda-1}{(2\lambda-1)^2}\right). \end{cases}$$

For the cases of one-fold symmetric functions and $\lambda = 1$, Corollary 3.6 reduces to the following corollary.

Corollary 3.7 (see [21]). *Let the function f given by (1.5) be in the class $S_{\Sigma,1}^{1}(\phi)$. Then*

$$|a_3| \leq \begin{cases} \frac{B_1}{2}, & \frac{B_2}{B_1^2} \in (-\infty, -1) \cup (3, \infty), \\ \frac{B_1^3}{B_1^2-B_2}, & \frac{B_2}{B_1^2} \in (-1, 1) \cup (1, 3). \end{cases}$$

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