Khayyam J. Math. 4 (2018), no. 2, 198–213
DOI: 10.22034/kjm.2018.65161

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

SOLVABILITY OF NONLINEAR GOURSAT TYPE PROBLEM FOR HYPERBOLIC EQUATION WITH INTEGRAL CONDITION

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Communicated by C. Zhang

Abstract. This paper is concerned with the existence and uniqueness of a strong solution for linear problem by using a functional analysis method, which is based on an energy inequality and the density of the range of the operator generated by the problem. Applying an iterative process based on results obtained from the linear problem, we prove the existence and uniqueness of the weak generalized solution of nonlinear hyperbolic Goursat problem with integral condition.

1. Introduction

The Goursat problem arises in linear and nonlinear partial differential equations with mixed derivatives. The standard form of the Goursat problem is given by

$$\begin{aligned} u_{xt} &= f(x,t,u,u_x,u_t) \qquad 0 \le x \le a,\ 0 \le t \le T, \\ u(x,0) &= g\left(x\right), \qquad u(0,t) = h\left(t\right), \\ g\left(0\right) &= h\left(0\right) = u(0,0). \end{aligned}$$

The Goursat problem is named after the French mathematician Èdouard Goursat, where in [16] he studied a linear problem: $u_{xt} = a(x,t)u_x + b(x,t)u_t + c(x,t)u + f(x,t)$. The Goursat problem associated with hyperbolic partial differential equations arises in several areas of physics and engineering. Frisch and Chao [15], Cheung [14], Kaup and Newell [21], Ying and Wang, [27], Hillion [17],

Date: Received: 27 March 2018; Revised: 8 June 2018; Accepted: 28 June 2018.
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2010 *Mathematics Subject Classification.* 35.
Key words and phrases. Energy inequality, Goursat equation, nonlinear hyperbolic problems, integral condition, a priori estimate.

McClaughlin et al. [24] ,Chen and Li [13], and Kaup and Steudel [22] described in detail areas of applications, where a Goursat problem arises.

Cannon was the first who drew attention to these problems with an integral condition in [9], and the importance of the problems with integral conditions has been pointed out by Samarskii [25].

Mathematical modeling of problems with integral conditions is encountered in various applications in chemical engineering, thermoelasticity, underground water flow, plasma physics, and population dynamics.

More works related to these problems with an integral condition have been published, among them we cite the works of Kartynnik [20], Ionkin [18] Cannon and van der Hoek [11, 12], Yurchuk [28], Cannon, Esteva, and van der Hoek [10], Lin [23], Benouar and Yurchuk [1], Shi [26], Bouziani [2, 3, 4], Bouziani and Benouar [5, 6, 7], and Jumarhon and McKee [19]. Motivated by this, we study a nonlinear Goursat type problem for hyperbolic equation with integral condition, Our proof is based on a priori estimate and on the fact that the range of the operator generated by the considered problem is dense.

2. Formulation of the problem

In the rectangle $Q = (0, b) \times (0, T)$, with $T < \infty$, we consider the nonlinear Goursat hyperbolic equation:

$$\mathcal{L}u = \frac{\partial^2 u}{\partial x \partial t} + p(x,t)\frac{\partial u}{\partial t} + q(x,t)\frac{\partial u}{\partial x} = f\left(x, t, u, \frac{\partial u}{\partial x}\right), \qquad (x,t) \in \overline{Q}, \tag{2.1}$$

with the initial data

$$\ell u = u(x, 0) = \varphi(x), \qquad x \in [0, b], \tag{2.2}$$

and integral condition

$$\int_0^b u(x,t)\, dx = 0, \qquad t \in [0, T], \tag{2.3}$$

where $p(x,t)$ and $q(x,t)$ satisfy the conditions:

$$p, q \in C^2\left(\overline{Q}\right), \qquad c_0 \leq q(x,t) \leq c_1, \qquad \tfrac{\partial q(x,t)}{\partial t} \leq c_2, \qquad \tfrac{\partial q(x,t)}{\partial x} \leq c_3, \tag{2.4}$$

$$\text{and } 2\varepsilon < c_4 \leq \tfrac{\partial p(x,t)}{\partial t} \leq c_5, \qquad p(x,t) \leq c_6 \text{ where } \varepsilon << 1,\ (x,t) \in \overline{Q}, \tag{2.5}$$

where c_i, $i = \overline{0,6}$, and ε are positive constants. The functions f and φ are known functions.

We shall assume that there exists a positive constant L such that

$$|f(x,t,u_1,v_1) - f(x,t,u_2,v_2)| \leq L\left(|u_1 - u_2| + |v_1 - v_2|\right) \quad \text{for all } (x,t) \in Q. \tag{2.6}$$

This paper is organized as follows: In Section 3, we state and pose the linear problem associated to (2.1)–(2.3) and introduce the function spaces used throughout the paper as well. Then in Section 4, we prove the uniqueness of the solution of the linear problem. In Section 5, we show the existence of solutions. Finally, in Section 6, on the basis of the results obtained in Sections 4 and 5, and on the use of an iterative process, we prove the existence and uniqueness of the solution of the nonlinear problem (2.1)–(2.3).

3. The linear problem

Let us, in this section, give the position of the linear problem and introduce the different function spaces needed to investigate the Goursat problem given by the hyperbolic equation:

$$\mathcal{L}u = \frac{\partial^2 u}{\partial x \partial t} + p(x,t)\frac{\partial u}{\partial t} + q(x,t)\frac{\partial u}{\partial x} = f(x,t), \qquad (x,t) \in \overline{Q}. \tag{3.1}$$

We employ certain function spaces to investigate our problem. Let $L^2(0,b)$ and $L^2(0,T;L^2(0,b))$ be the standard functional spaces. We denote by $C_0(0,b)$ the vector space of continuous functions with compact support in $(0,b)$. Since such functions are Lebesgue integrable with respect to dx, we can define on $C_0(0,b)$ the bilinear form given by

$$(u,w) = \int_0^b \Im_x^* u \cdot \Im_x^* w \; dx, \tag{3.2}$$

where

$$\Im_x^* u = \int_x^b u(\xi,t)\, d\xi.$$

The bilinear form (3.2) is considered as a scalar product on $C_0(0,b)$ for which $C_0(0,b)$ is not complete.

Definition 3.1. We denote by $B_2^1(0,b)$ a completion of $C_0(0,b)$ for the scalar product (3.2), which is denoted $(\cdot,\cdot)_{B_2^1(0,b)}$, called the Bouziani space or the space of square integrable primitive functions on $(0,b)$. By the norm of function u from $B_2^1(0,b)$, we understand the non-negative number:

$$\|u\|_{B_2^1(0,b)} = \sqrt{(u,u)_{B_2^1(0,b)}} = \|\Im_x^* u\|_{L^2(0,b)}. \tag{3.3}$$

For $u \in L^2(0,b)$, we have the elementary inequality

$$\|u\|_{B_2^1(0,b)} \le \frac{b}{\sqrt{2}}\|u\|_{L^2(0,b)}. \tag{3.4}$$

We denote by $L^2(0,T;B_2^1(0,b))$ the space of functions, which are square integrable in the Bochner sense, with the scalar product

$$(u,w)_{L^2\left(0,T;\, B_2^1(0,b)\right)} = \int_0^T (u(\bullet,t), w(\bullet,t))_{B_2^1(0,b)}\, dt. \tag{3.5}$$

Since the space $B_2^1(0,b)$ is a Hilbert space, it can be shown that $L^2(0,T;B_2^1(0,b))$ is a Hilbert space as well. The set of all continuous abstract functions in $[0,T]$ equipped with the norm

$$\sup_{0\le\tau\le T} \|u(\cdot,\tau)\|_{B_2^1(0,b)}, \tag{3.6}$$

is denoted by $C(0,T;B_2^1(0,b))$.

In this paper, we prove the existence and the uniqueness for a strong solution of the problem (3.2), (2.1)–(2.3) as a solution of the operator equation

$$Lu = \mathcal{F} \tag{3.7}$$

where $L = (\mathcal{L}, \ell)$, with domain of definition $D(L)$ consisting of functions $u \in L^2(Q)$ such that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, and $\frac{\partial^2 u}{\partial t \partial x} \in L^2(0,T; B_2^1(0,b))$ and u satisfies the condition (2.3) the operator L is considered from B to F, where B is the Banach space consisting of all functions $u(x,t)$ having a finite norm

$$\|u\|_B^2 = \left\| \frac{\partial u}{\partial t} \right\|_{L^2(0,T;\ L^2(0,b))}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2\left(0,T;\ B_2^1(0,b)\right)}^2 + \sup_{0 \le \tau \le T} \|u(\cdot,\tau)\|_{L^2(0,b)}^2,$$

and satisfying the condition (2.3) and F is the Hilbert space consisting of all elements $\mathcal{F} = (f, \varphi)$ for which the norm

$$\|\mathcal{F}\|_F^2 = \|f\|_{L^2(0,T;\ L^2(0,b))}^2 + \|\varphi\|_{L^2(0,b)}^2$$

is finite.

Then, we establish the energy inequality:

$$\|u\|_B \le c \|Lu\|_F,$$

and we show that the operator L has a closure $\overline{L}$.

Definition 3.2. A solution of the operator equation

$$\overline{L}u = \mathcal{F}$$

is called a strong solution of the problem (3.1), (2.1)–(2.3).

Since points of the graph $\overline{L}$ are limits of sequences of points of the graph of L, we can extend (3.7) to apply to strong solution by taking limits; that is,

$$\|u\|_B \le c \left\| \overline{L}u \right\|_F \qquad \forall u \in D(\overline{L}).$$

From this inequality we obtain the uniqueness of a strong solution if it exists, and the the equality of sets $R(\overline{L})$ and $\overline{R(L)}$. Thus, proving that the set $R(L)$ is dense in F.

4. An energy inequality and its consequences

Theorem 4.1. *For any function $u \in D(L)$, there exists a positive constant c, such that*

$$\|u\|_B \le c \|Lu\|_F. \tag{4.1}$$

Proof. Multiplying the equation (3.1) by the following Mu :

$$Mu = \Im_x^* u_t,$$

and integrating over Q^τ, where $Q^\tau = (0,b) \times (0,\tau)$, we get

$$\begin{aligned}
\int_{Q^\tau} \mathcal{L}u \cdot Mudxdt &= \int_{Q^\tau} \frac{\partial^2 u}{\partial x \partial t} \Im_x^* u_t dxdt + \int_{Q^\tau} p(x,t) \frac{\partial u}{\partial t} \Im_x^* u_t dxdt \\
&\quad + \int_{Q^\tau} q(x,t) \frac{\partial u}{\partial x} \Im_x^* u_t dxdt \\
&= \int_{Q^\tau} f(x,t) \Im_x^* u_t dxdt. \qquad (4.2)
\end{aligned}$$

Standard integration by parts each term in (4.2) by using the condition (2.3), we obtain

$$\begin{aligned}
\int_{Q^\tau} \frac{\partial^2 u}{\partial x \partial t} \Im_x^* u_t dxdt &= \int_0^\tau \frac{\partial u}{\partial t} \Im_x^* u_t \Big|_{x=0}^{x=b} dt + \int_{Q^\tau} \left(\frac{\partial u}{\partial t} \right)^2 dxdt \\
&= \int_{Q^\tau} \left(\frac{\partial u}{\partial t} \right)^2 dxdt, \qquad (4.3)
\end{aligned}$$

$$\begin{aligned}
\int_{Q^\tau} p(x,t) \frac{\partial u}{\partial t} \Im_x^* u_t dxdt &= -\frac{1}{2} \int_0^\tau p(x,t) \left(\Im_x^* u_t \right)^2 \Big|_{x=0}^{x=b} dt \\
&\quad + \frac{1}{2} \int_{Q^\tau} \frac{\partial p(x,t)}{\partial x} \left(\Im_x^* u_t \right)^2 dxdt \\
&= \frac{1}{2} \int_{Q^\tau} \frac{\partial p(x,t)}{\partial x} \left(\Im_x^* u_t \right)^2 dxdt, \qquad (4.4)
\end{aligned}$$

$$\begin{aligned}
\int_{Q^\tau} q(x,t) \frac{\partial u}{\partial x} \Im_x^* u_t dxdt &= \int_0^\tau q(x,t) u \Im_x^* u_t \Big|_{x=0}^{x=b} dt + \int_{Q^\tau} q(x,t) u u_t dxdt \\
&\quad - \int_{Q^\tau} \frac{\partial q(x,t)}{\partial x} u \Im_x^* u_t dxdt \\
&= \frac{1}{2} \int_0^b q(x,\tau) u^2 dx - \frac{1}{2} \int_0^b q(x,0) \varphi^2 dx \\
&\quad - \frac{1}{2} \int_{Q^\tau} \frac{\partial q(x,t)}{\partial t} u^2 dxdt \\
&\quad - \int_{Q^\tau} \frac{\partial q(x,t)}{\partial x} u \Im_x^* u_t dxdt. \qquad (4.5)
\end{aligned}$$

By virtue of the Cauchy inequality with ε,

$$ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2, \qquad a, b \in \mathbb{R},$$

we obtain

$$\int_{Q^\tau} f(x,t) \Im_x^* u_t dxdt \leq \frac{1}{2\varepsilon} \int_{Q^\tau} f^2 dxdt + \frac{\varepsilon}{2} \int_{Q^\tau} \left(\Im_x^* u_t \right)^2 dxdt, \qquad (4.6)$$

and

$$\int_{Q^\tau} \frac{\partial q(x,t)}{\partial x} u \Im_x^* u_t dxdt \leq \frac{1}{2\varepsilon} \int_{Q^\tau} \left(\frac{\partial q(x,t)}{\partial x} u \right)^2 dxdt + \frac{\varepsilon}{2} \int_{Q^\tau} \left(\Im_x^* u_t \right)^2 dxdt. \quad (4.7)$$

Substituting (4.3)–(4.5) into (4.2), and according to (4.6) and (4.7) and the condition (2.5) we get:

$$\begin{aligned}
&\int_{Q^\tau} \left(\frac{\partial u}{\partial t} \right)^2 dxdt + \frac{c_4}{2} \int_{Q^\tau} \left(\Im_x^* u_t \right)^2 dxdt + \frac{c_0}{2} \int_0^b u(x,\tau)^2 dx \\
&\leq \frac{1}{2\varepsilon} \int_{Q^\tau} f^2 dxdt + \varepsilon \int_{Q^\tau} \left(\Im_x^* u_t \right)^2 dxdt \\
&+ \frac{c_1}{2} \int_0^b \varphi^2 dx + \left(\frac{c_2^2}{2\varepsilon} + \frac{c_3}{2} \right) \int_{Q^\tau} u^2 dxdt.
\end{aligned}$$

By using Lemma 1 in [8] we obtain

$$\begin{aligned}
&\int_{Q^\tau} \left(\frac{\partial u}{\partial t} \right)^2 dxdt + \int_{Q^\tau} \left(\Im_x^* u_t \right)^2 dxdt + \int_0^b u(x,\tau)^2 dx \\
&\leq k \left(\int_{Q^\tau} f^2 dxdt + \int_0^b \varphi^2 dx \right) \qquad (4.8)
\end{aligned}$$

where

$$k = \frac{\max\left(\frac{c_1}{2}, \frac{1}{2\varepsilon} \right)}{\min\left(1, \frac{c_4}{2} - \varepsilon, \frac{c_0}{2} \right)} \exp\left(\left(\frac{c_2^2}{2\varepsilon} + \frac{c_3}{2} \right) T \right).$$

The right-hand side of (4.8) is independent of τ, hence replacing the left-hand side by its upper bound with respect to τ from 0 to T, we obtain the desired inequality, where $c = (k)^{\frac{1}{2}}$. □

Proposition 4.2. *The operator L from B to F admits a closure.*

Proof. Suppose that $\{u_n\} \in D(L)$ is a sequence such that

$$u_n \to 0 \qquad \text{in } B \qquad (4.9)$$

and

$$Lu_n \to (f, \varphi) \qquad \text{in } F; \qquad (4.10)$$

we must show that $f \equiv 0$ and $\varphi \equiv 0$.

According to (4.9), we get

$$u_n \to 0 \qquad \text{in } D'(Q).$$

By virtue of the continuity of derivation of $D'(Q)$ in $D'(Q)$, we deduce that

$$\mathcal{L} u_n \to 0 \qquad \text{in } D'(Q). \qquad (4.11)$$

Further, according to (4.10) we have

$$\mathcal{L} u_n \to f \qquad \text{in } L^2(Q); \qquad (4.12)$$

thus we have

$$\mathcal{L} u_n \to f \qquad \text{in } D'(Q). \qquad (4.13)$$

Then by of the uniqueness of the limit in $D'(Q)$, we see that $f \equiv 0$. On the other hand, (4.10) implies that

$$\ell u_n \to \varphi \qquad \text{in } L^2(0,b). \tag{4.14}$$

Moreover, since by virtue of (4.9) and the fact that

$$\int_0^b (\ell u_n)^2 \, dx \leq \|u_n\|_B^2 \qquad \forall n,$$

we have

$$\ell u_n \to 0 \qquad \text{in } L^2(0,b). \tag{4.15}$$

Now the uniqueness of the limit in $L^2(0,b)$ implies that $\varphi \equiv 0$. □

Theorem 4.1 is valid for strong solution; that is, we have the inequality

$$\|u\|_B \leq c \left\| \overline{L} u \right\|_F \qquad \forall u \in D(\overline{L}). \tag{4.16}$$

Hence we obtain the following result.

Corollary 4.3. *A strong solution of the problem* (3.1), (2.2)–(2.3) *is unique if it exists and depends continuously on* $\mathcal{F} = (f,\varphi) \in F$.

Corollary 4.4. *The range* $R(\overline{L})$ *of the operator* $\overline{L}$ *is closed in* F, *and* $R(\overline{L}) = \overline{R(L)}$.

5. Existence of solutions

To show the existence of solutions, we prove that $R(L)$ is dense in F for all $u \in D(L)$ and for arbitrary $\mathcal{F} = (f,\varphi) \in F$.

Theorem 5.1. *Suppose the conditions of theorem 4.1 are satisfied. Then the problem* (3.1), (2.2)–(2.3) *admits a unique strong solution* $u = \overline{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$.

Proof. First we prove that $R(L)$ is dense in F , for the special case, where $D(L)$ is reduced to $D_0(L)$, where $D_0(L) = \{u,\ u \in D(L) : \ell u = 0\}$. □

Proposition 5.2. *Let the conditions of theorem 5.1 be satisfied. If, for* $\omega \in L^2(Q)$ *and for all* $u \in D_0(L)$, *we have*

$$\int_Q \mathcal{L}u.\omega \; dxdt = 0, \tag{5.1}$$

then ω *vanishes almost everywhere in* Q.

Proof. The scalar product of F is defined by

$$(Lu,\omega)_F = \int_Q \mathcal{L}u.\omega dxdt \tag{5.2}$$

the equality (5.1) can be written as follows:

$$\int_Q \frac{\partial^2 u}{\partial x \partial t}\omega dxdt = -\int_Q \left(p(x,t)\frac{\partial u}{\partial t} + q(x,t)\frac{\partial u}{\partial x} \right) \omega dxdt. \tag{5.3}$$

If we put

$$u(x,t) = \begin{cases} \Im_x^\star z & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases}$$

such that $z, \frac{\partial z}{\partial t}, \frac{\partial \Im_x^\star z}{\partial t} \in L^2(Q)$, then z satisfies the condition (2.3). As a result of (5.3), we obtain the equality

$$\int_Q \frac{\partial z}{\partial t}\omega dxdt = -\int_Q \left(p(x,t)\frac{\partial \Im_x^\star z}{\partial t} + q(x,t)z\right)\omega dxdt = 0. \tag{5.4}$$

In terms of the given function ω, and from the equality (5.4) we give the function ω in terms of z as follows:

$$\omega = \frac{\partial z}{\partial t}, \text{ so } \omega \in L^2(Q), \tag{5.5}$$

and z satisfies the same condition of the function u and

$$z(x,T) = 0. \tag{5.6}$$

Replacing ω in (5.4) by its representation (5.5), we obtain

$$\int_Q \left(\frac{\partial z}{\partial t}\right)^2 dxdt = -\int_Q \left(p(x,t)\frac{\partial(\Im_x^\star z)}{\partial t}\frac{\partial z}{\partial t} + q(x,t)z\frac{\partial z}{\partial t}\right) dxdt. \tag{5.7}$$

Integrating by parts each term in the right-hand side of (5.7) with respect to x and t by taking the conditions of the function z yields

$$-\int_Q p(x,t)\frac{\partial(\Im_x^\star z)}{\partial t}\frac{\partial z}{\partial t}dxdt = -\frac{1}{2}\int_Q \frac{\partial p(x,t)}{\partial x}\left(\Im_x^\star \frac{\partial z}{\partial t}\right)^2 dxdt, \tag{5.8}$$

$$\begin{aligned} -\int_Q q(x,t)z\frac{\partial z}{\partial t}dxdt &= \frac{1}{2}\int_Q \frac{\partial q(x,t)}{\partial t}z^2 dxdt - \frac{1}{2}\int_0^\alpha q(x,t)\, z^2\Big|_{t=0}^{t=T} dx \\ &= -\frac{1}{2}\int_Q \frac{\partial q(x,t)}{\partial t}z^2 dxdt + \frac{1}{2}\int_0^b q(x,T)\, z(x,T)^2 dx \\ &\quad -\frac{1}{2}\int_0^b q(x,t)\, z(x,0)^2 dx. \end{aligned} \tag{5.9}$$

We combining (5.8) and (5.9) in (5.7), we get

$$\begin{aligned} \int_Q \left(\frac{\partial z}{\partial t}\right)^2 dxdt &= -\frac{1}{2}\int_Q \frac{\partial p(x,t)}{\partial x}\left(\Im_x^\star \frac{\partial z}{\partial t}\right)^2 dxdt - \frac{1}{2}\int_Q \frac{\partial q(x,t)}{\partial t}z^2 dxdt \\ &\quad -\frac{1}{2}\int_0^b q(x,t)\, z(x,0)^2 dx, \end{aligned} \tag{5.10}$$

and thus $z = 0$ in Q; then $\omega = 0$ in Q. This proves Proposition 5.2. □

Theorem 5.3. *Suppose that the conditions of Theorem 4.1 are satisfied. Then the problem* (3.1), (2.2)–(2.3) *admits a unique strong solution* $u = \overline{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$.

Proof. First we prove that $R(L)$ is dense in F , for the special case, where $D(L)$ is reduced to $D_0(L)$, where $D_0(L) = \{u,\ u \in D(L) : \ell u = 0\}$. □

We return to the proof of Theorem 5.1. We have already noted that it is sufficient to prove that the set $R(L)$ dense in F.

The rest of proof of Theorem 5.1. Suppose that, for some $W = (\omega, \omega_0) \in R(L)^{\perp}$ and for all $u \in D(L)$, it holds

$$(Lu, \omega)_F = \int_Q \mathcal{L}u.\omega dxdt + \int_0^1 \ell u.\omega_0 dx = 0. \tag{5.11}$$

Then we must prove that $W = 0$. Putting $u \in D_0(L)$ in (5.11), we have

$$\int_Q \mathcal{L}u.\omega dxdt = 0, \quad u \in D_0(L).$$

Hence Proposition 5.2 implies that $\omega = 0$. Thus (5.11) takes the form

$$\int_0^1 (\ell u)(\omega_0)\, dx = 0, \quad u \in D(L). \tag{5.12}$$

Since the range of the trace operator ℓ is dense in the Hilbert F space with the norm

$$\left(\int_0^1 (\ell u)^2\, dx\right)^{\frac{1}{2}},$$

the equality (5.12) implies that $\omega_0 = 0$ (we recall satisfies a compatibility conditions). Hence $W = 0$. This completes the proof of Theorem 5.1. □

6. The nonlinear problem

This section is consecrated to the proof of the existence, uniqueness, and continuous dependence of the solution on the data of the problem (2.1)–(2.3). Let us consider the following auxiliary problem with homogeneous equation:

$$\mathcal{L}u = \frac{\partial^2 w}{\partial x \partial t} + p(x,t)\frac{\partial w}{\partial t} + q(x,t)\frac{\partial w}{\partial x} = 0, \qquad (x,t) \in \overline{Q}, \tag{6.1}$$

$$\ell w = w(x,0) = \varphi(x), \qquad x \in [0,b], \tag{6.2}$$

$$\int_0^b w(x,t)\, dx = 0, \qquad t \in [0,T], \tag{6.3}$$

If u is a solution of problem (2.1)–(2.3) and w is a solution of problem (6.1)–(6.3), then $y = u - w$ satisfies

$$\mathcal{L}u = \frac{\partial^2 y}{\partial x \partial t} + p(x,t)\frac{\partial y}{\partial t} + q(x,t)\frac{\partial y}{\partial x} = G\left(x,t,y,\frac{\partial y}{\partial x}\right), \qquad (x,t) \in \overline{Q}, \tag{6.4}$$

$$\ell y = y(x,0) = 0, \qquad x \in [0,b], \tag{6.5}$$

$$\int_0^b y(x,t)\, dx = 0, \qquad t \in [0,T], \tag{6.6}$$

where $G\left(x,t,y,\frac{\partial y}{\partial x}\right) = f\left(x,t,y+w,\frac{\partial y}{\partial x} + \frac{\partial w}{\partial x}\right)$, where the function G satisfies the condition (2.6); that is, there exists a positive constant L such that

$$|f(x,t,u_1,v_1) - f(x,t,u_2,v_2)| \le L\left(|u_1 - u_2| + |v_1 - v_2|\right) \quad \text{for all } (x,t) \in Q, \tag{6.7}$$

According to results of the previous section, we deduce that the problem (6.1)–(6.3) admits a unique solution that depends continuously upon the initial condition (6.2). Then, we shall prove that problem (6.4)–(6.6) has a unique weak solution.

Firstly, we precise the concept of the solution that we are considering. Let $v = v(x,t)$ be any function from $\widetilde{C^1}(Q)$, the space of functions v belonging to $C^1(Q)$, having $\frac{\partial^2 v}{\partial x \partial t}$ continuous in Q.

We shall compute the integral $\int_Q G\Im_x^\star v dxdt$; for this we assume that $y, v \in \widetilde{C^1}(Q)$, $y(x,0) = 0$, $v(x,T) = 0$, and $\int_Q y(x,t)dx = \int_Q v(x,t)dx = 0$. By using conditions on y and v, we have

$$\int_Q \frac{\partial^2 y}{\partial x \partial t} \Im_x^* v dxdt = \int_Q \frac{\partial y}{\partial t} v dxdt, \tag{6.8}$$

$$\int_Q p(x,t) \frac{\partial y}{\partial t} \Im_x^* v dxdt = -\int_Q v\left(\Im_x^* p_t y\right) dxdt + \int_Q \frac{\partial v}{\partial t}\left(\Im_x^* p y\right) dxdt, \tag{6.9}$$

$$\int_\Omega q(x,t) \frac{\partial y}{\partial x} \Im_x^* v dxdt = -\int_Q yv\Im_x^* q dxdt, \tag{6.10}$$

$$\int_Q G\Im_x^\star v dxdt = -\int_Q v\Im_x^* G dxdt. \tag{6.11}$$

It then follows from (6.8)–(6.11) that

$$A(y,v) = -\int_Q v\Im_x^* G dxdt, \tag{6.12}$$

where

$$A(y,v) = \int_Q \frac{\partial y}{\partial t} v dxdt - \int_Q v\left(\Im_x^* p_t y\right) dxdt + \int_Q \frac{\partial v}{\partial t}\left(\Im_x^* p y\right) dxdt - \int_Q yv\Im_x^* q dxdt.$$

Definition 6.1. A function $y \in L^2(0,T;\ H^1(0,b))$ is called a weak solution of problem (6.4)–(6.6), if (6.12) holds.

Let us construct an iteration sequence in the following way: Starting with $y^{(0)} = 0$, the sequence $(\{y^{(n)}\}_{n\in N}$ is defined as follows: given the element $y^{(n-1)}$, then, for $n = 1, 2, \ldots$, solve the problem:

$$\frac{\partial^2 y^{(n)}}{\partial x \partial t} + p(x,t)\frac{\partial y^{(n)}}{\partial t} + q(x,t)\frac{\partial y^{(n)}}{\partial x} = G\left(x, t, y^{(n-1)}, \frac{\partial y^{(n-1)}}{\partial x}\right), \quad (x,t) \in \overline{Q}, \tag{6.13}$$

$$y^{(n)}(x,0) = 0, \qquad x \in [0,b], \tag{6.14}$$

$$\int_0^b y^{(n)}(x,t)\,dx = 0, \qquad t \in [0,T], \tag{6.15}$$

Theorem 5.1 asserts that, for fixed n, each problem (6.13)–(6.15) has a unique solution $y^{(n)}(x,t)$. If we set $Z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t)$, then we have the new problem

$$\frac{\partial^2 Z^{(n)}}{\partial x \partial t} + p(x,t)\frac{\partial Z^{(n)}}{\partial t} + q(x,t)\frac{\partial Z^{(n)}}{\partial x} = \theta^{(n-1)}(x,t), \qquad (x,t) \in \overline{Q}, \tag{6.16}$$

$$Z^{(n)}(x,0) = 0, \qquad x \in [0,b], \tag{6.17}$$

$$\int_0^b Z^{(n)}(x,t)\,dx = 0, \qquad t \in [0,T], \tag{6.18}$$

where

$$\theta^{(n-1)}(x,t) = G\left(x,t,y^{(n)},\frac{\partial y^{(n)}}{\partial x}\right) - G\left(x,t,y^{(n-1)},\frac{\partial y^{(n-1)}}{\partial x}\right).$$

Lemma 6.2. *Assume that condition (6.7) holds; then, for the linearized problem (6.16)–(6.18), we have the a priori estimate*

$$\left\|Z^{(n)}\right\|_{L^2(0,T;\ H^1(0,b))} \leq \widetilde{K}\left\|Z^{(n-1)}\right\|_{L^2(0,T;\ H^1(0,b))}, \tag{6.19}$$

where $\widetilde{K}$ is a positive constant given by

$$\widetilde{K} = \sqrt{\frac{2k^* L^2 T}{\min(1,T)}}$$

and

$$k^* = \frac{\left(\frac{1}{2}+\frac{1}{2\varepsilon}\right)}{\min\left(\frac{1}{2},\frac{c_4}{2}-\varepsilon,\frac{c_0}{2}\right)}\exp\left(\max\left(\frac{c_2^2}{2\varepsilon}+\frac{c_3}{2},\frac{1}{2}+\frac{1}{2}c_6^2+c_1\right)T\right).$$

Proof. Multiplying equation (6.16) by $\Im_x^* Z_t^{(n)}$, and integrating over Q^τ, where $Q^\tau = (0,b)\times(0,\tau)$, we get

$$\int_{Q^\tau}\frac{\partial^2 Z^{(n)}}{\partial x\partial t}\Im_x^*\frac{\partial Z^{(n)}}{\partial t}dxdt + \int_\Omega p(x,t)\frac{\partial Z^{(n)}}{\partial t}\Im_x^*\frac{\partial Z^{(n)}}{\partial t}dxdt$$
$$+\int_{Q^\tau} q(x,t)\frac{\partial Z^{(n)}}{\partial x}\Im_x^*\frac{\partial Z^{(n)}}{\partial t}dxdt = \int_{Q^\tau}\theta^{(n-1)}(x,t)\,\Im_x^*\frac{\partial Z^{(n)}}{\partial t}dxdt. \tag{6.20}$$

Standard integration by parts each term in (6.20) with use the condition (6.18) and (2.5) and follows the same method in Section 4, we have

$$\int_{Q^\tau}\left(\frac{\partial Z^{(n)}}{\partial t}\right)^2 dxdt + \frac{c_4}{2}\int_Q\left(\Im_x^*\frac{\partial Z^{(n)}}{\partial t}\right)^2 dxdt + \frac{c_0}{2}\int_0^b Z^{(n)}(x,\tau)^2\,dx$$
$$\leq \frac{1}{2\varepsilon}\int_{Q^\tau}\left(\theta^{(n-1)}\right)^2 dxdt + \varepsilon\int_{Q^\tau}\left(\Im_x^*\frac{\partial Z^{(n)}}{\partial t}\right)^2 dxdt$$
$$+\left(\frac{c_2^2}{2\varepsilon}+\frac{c_3}{2}\right)\int_Q\left(Z^{(n)}\right)^2 dxdt.$$

Again, Multiplying equation (6.16) by $\frac{\partial Z^{(n)}}{\partial x}$ and integrating over Q^{τ}, we get

$$\begin{aligned}
&\frac{1}{2}\int_0^b \left(\frac{\partial Z^{(n)}}{\partial x}\right)^2 dxdt \\
&\quad \leq \frac{1}{2}\int_{Q^{\tau}} \left(\theta^{(n-1)}\right)^2 dxdt + \frac{1}{2}\int_{Q^{\tau}} \left(\frac{\partial Z^{(n)}}{\partial t}\right)^2 dxdt \\
&\qquad + \left(\frac{1}{2}+\frac{1}{2}c_6^2+c_1\right)\int_{Q^{\tau}} \left(\frac{\partial Z^{(n)}}{\partial x}\right)^2 dxdt.
\end{aligned}$$

Now, combining the last two previous inequalities, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{Q^{\tau}} \left(\frac{\partial Z^{(n)}}{\partial t}\right)^2 dxdt + \left(\frac{c_4}{2}-\varepsilon\right)\int_{Q^{\tau}} \left(\Im_x^*\frac{\partial Z^{(n)}}{\partial t}\right)^2 dxdt \\
&+\frac{c_0}{2}\int_0^b Z^{(n)}\left(x,\tau\right)^2 dx + \frac{1}{2}\int_0^b \left(\frac{\partial Z^{(n)}\left(x,\tau\right)}{\partial x}\right)^2 dx \\
&\leq \left(\frac{c_2^2}{2\varepsilon}+\frac{c_3}{2}\right)\int_{Q^{\tau}} \left(Z^{(n)}\right)^2 dxdt + \left(\frac{1}{2}+\frac{1}{2}c_6^2+c_1\right)\int_{Q^{\tau}} \left(\frac{\partial Z^{(n)}}{\partial x}\right)^2 dxdt \\
&\quad + \left(\frac{1}{2}+\frac{1}{2\varepsilon}\right)\int_{Q^{\tau}} \left(\theta^{(n-1)}\right)^2 dxdt,
\end{aligned}$$

Then, applying Gronwall's lemma implies that

$$\begin{aligned}
&\int_{Q^{\tau}} \left(\frac{\partial Z^{(n)}}{\partial t}\right)^2 dxdt + \int_{Q^{\tau}} \left(\Im_x^*\frac{\partial Z^{(n)}}{\partial t}\right)^2 dxdt + \int_0^b \left(Z^{(n)}\right)^2 dx \\
&\qquad + \int_0^b \left(\frac{\partial Z^{(n)}}{\partial x}\right)^2 dxdt \leq k^*\left(\int_{Q^{\tau}} \left(\theta^{(n-1)}\right)^2 dxdt\right), \qquad (6.21)
\end{aligned}$$

where

$$k^* = \frac{\left(\frac{1}{2}+\frac{1}{2\varepsilon}\right)}{\min\left(\frac{1}{2},\frac{c_4}{2}-\varepsilon,\frac{c_0}{2}\right)}\exp\left(\max\left(\frac{c_2^2}{2\varepsilon}+\frac{c_3}{2},\frac{1}{2}+\frac{1}{2}c_6^2+c_1\right)T\right).$$

By virtue of condition (6.7), we obtain

$$\begin{aligned}
\left(\int_Q \left(\theta^{(n-1)}\right)^2 dxdt\right) &\leq L^2\int_Q \left(\left|Z^{(n-1)}\left(x,t\right)\right| + \left|\frac{\partial Z^{(n-1)}\left(x,t\right)}{\partial x}\right|\right)^2 dxdt \\
&\leq 2L^2\int_0^T \left(\left\|Z^{(n-1)}\left(\bullet,t\right)\right\|_{L^2(0,b)}^2\right. \\
&\qquad\qquad \left. + \left\|\frac{\partial Z^{(n-1)}\left(\bullet,t\right)}{\partial x}\right\|_{L^2(0,b)}^2\right) dt. \qquad (6.22)
\end{aligned}$$

Substituting (6.22) into (6.21), we get

$$\int_{Q^\tau}\left(\frac{\partial Z^{(n)}}{\partial t}\right)^2 dxdt+\int_{Q^\tau}\left(\Im_x^*\frac{\partial Z^{(n)}}{\partial t}\right)^2 dxdt+\left\|Z^{(n)}\left(\bullet,\tau\right)\right\|^2_{H^1(0,b)}$$
$$\leq 2k^*L^2\int_0^T\left(\left\|Z^{(n-1)}\left(\bullet,t\right)\right\|^2_{L^2(0,b)}+\left\|\frac{\partial Z^{(n-1)}\left(\bullet,t\right)}{\partial x}\right\|^2_{L^2(0,b)}\right)dt,$$

so, we obtain

$$\int_{Q^\tau}\left(\frac{\partial Z^{(n)}}{\partial t}\right)^2 dxdt+\int_{Q^\tau}\left(\Im_x^*\frac{\partial Z^{(n)}}{\partial t}\right)^2 dxdt+\left\|Z^{(n)}\left(\bullet,\tau\right)\right\|^2_{H^1(0,b)}$$
$$\leq 2k^*L^2\int_0^T\left\|Z^{(n-1)}\left(\bullet,t\right)\right\|^2_{H^1(0,b)}dt, \tag{6.23}$$

After discarding the second term on the left-hand side of (6.23) and integrating the resulted inequality over the interval $(0,T)$, we obtain

$$\left\|Z^{(n)}\right\|_{L^2(0,T;\ H^1(0,b))}\leq\frac{2k^*L^2T}{\min\left(1,T\right)}\left\|Z^{(n-1)}\right\|_{L^2(0,T;\ H^1(0,b))}.$$

Then, we obtain the desired inequality (6.19). □

From the criteria of convergence of series, we see that the series $\sum_{n=1}^{\infty}Z^{(n)}$ converges if $\frac{2k^*L^2T}{\min(1,T)}<1$,; that is, if $L<\sqrt{\frac{\min(1,T)}{2k^*T}}$. Since $Z^{(n)}(x,t)=y^{(n+1)}(x,t)-y^{(n)}(x,t)$, then it follows that the sequence $(y^{(n)})_{n\in N}$ defined by

$$y^{(n)}(x,t)=\sum_{i=0}^{n-1}Z^{(i)}+y^{(0)}(x,t),$$

converges to an element $y\in L^2\left(0,T;\ H^1(0,b)\right)$.

We must show that the limit function y is a solution of the problem under study. To do this, we will show that y verifies the condition (6.12) as mentioned in Definition 6.1. So, we consider the weak formulation of problem (6.13)–(6.15)

$$A\left(y,v\right)=-\int_Q v\Im_x^*G\left(x,t,y^{(n-1)},\frac{\partial y^{(n-1)}}{\partial x}\right)dxdt. \tag{6.24}$$

From (6.24) we have

$$\begin{aligned}A\left(y^{(n)}-y,v\right)+A\left(y,v\right) &= -\int_Q v\left(\Im_x^*G\left(x,t,y^{(n-1)},\frac{\partial y^{(n-1)}}{\partial x}\right)\right.\\ &\qquad\left.-\Im_x^*G\left(x,t,y,\frac{\partial y}{\partial x}\right)\right)dxdt\\ &\quad-\int_Q v\Im_x^*G\left(x,t,y,\frac{\partial y}{\partial x}\right)dxdt.\end{aligned} \tag{6.25}$$

However, we apply the Cauchy–Schwarz inequality, and we get

$$\begin{aligned} A\left(y^{(n)}-y,v\right) &= \int_Q \left(\frac{\partial y^{(n)}}{\partial t}-\frac{\partial y}{\partial t}\right) v dxdt - \int_Q v\left(\left(\Im_x^* p_t y^{(n)}\right)-\left(\Im_x^* p_t y\right)\right) dxdt \\ &\quad + \int_Q \frac{\partial v}{\partial t}\left(\left(\Im_x^* p y^{(n)}\right)-\left(\Im_x^* p y\right)\right) dxdt \\ &\quad - \int_Q \left(y^{(n)}-y\right) v \Im_x^* q dxdt \\ &\leq c_7 \left\|y^{(n)}-y\right\|_{L^2(0,T;\ H^1(0,b))} \|v\|_{L^2(0,T;\ H^1(0,b))}, \end{aligned} \tag{6.26}$$

where

$$c_7 = \max\left(1, \left(\frac{c_5 b}{\sqrt{2}}+c_1 b\right), \frac{c_6 b}{\sqrt{2}}\right).$$

and we have:

$$-\int_Q v\left(\Im_x^* G\left(x,t,y^{(n-1)},\frac{\partial y^{(n-1)}}{\partial x}\right)-\Im_x^* G\left(x,t,y,\frac{\partial y}{\partial x}\right)\right) dxdt$$
$$\leq \frac{bL}{\sqrt{2}} \left\|y^{(n)}-y\right\|_{L^2(0,T;\ H^1(0,b))} \|v\|_{L^2(0,T;\ L^2(0,b))}. \tag{6.27}$$

Taking into account (6.26) and (6.27) and passing to the limit in (6.25) as $n \to \infty$, we obtain

$$A(y,v) = -\int_Q v \Im_x^* G\left(x,t,y,\frac{\partial y}{\partial x}\right) dxdt.$$

Therefore, we have established the following result.

Theorem 6.3. *Assume that conditions* (2.5) *and* (6.7) *are hold, and that*

$$L < \sqrt{\frac{\min(1,T)}{2kT}};$$

then problem (6.4)–(6.6) *admits a weak solution in* $L^2(0,T;\ H^1(0,b))$.

It remains to prove that problem (6.4)–(6.6) admits a unique solution.

Theorem 6.4. *Under the conditions* (2.5) *and* (6.7) *the solution of the problem* (6.4)–(6.6) *is unique.*

Proof. Suppose that y_1 and y_2 in $L^2(0,T;\ H^1(0,b))$ are two solution of (6.4)–(6.6); then $Z = y_1 - y_2$ satisfies $Z \in L^2(0,T;\ H^1(0,b))$ and

$$\frac{\partial^2 Z}{\partial x \partial t} + p(x,t)\frac{\partial Z}{\partial t} + q(x,t)\frac{\partial Z}{\partial x} = \psi(x,t), \qquad (x,t) \in \overline{Q}, \tag{6.28}$$

$$Z(x,0) = 0, \qquad x \in [0,b], \tag{6.29}$$

$$\int_0^b Z(x,t)\,dx = 0, \qquad t \in [0,T], \tag{6.30}$$

where

$$\psi(x,t) = G\left(x,t,y_1,\frac{\partial y_1}{\partial x}\right) - G\left(x,t,y_2,\frac{\partial y_2}{\partial x}\right).$$

Taking the inner product in $L^2(0,T;\ L^2(0,b))$ of equation (6.28) and $\Im_x^* Z_t$ and following the same procedure done in establishing the proof of Lemma 6.2, we get

$$\|Z\|_{L^2(0,T;\ H^1(0,b))} \leq \widetilde{K}\,\|Z\|_{L^2(0,T;\ H^1(0,b))}\,, \tag{6.31}$$

where $\widetilde{K}$ is the same constant of Lemma 6.2.

Since $\widetilde{K} < 1$, then from (6.31) that

$$\left(1-\widetilde{K}\right)\|Z\|_{L^2(0,T;\ H^1(0,b))} \leq 0,$$

from which we conclude that $y_1 = y_2$ in $L^2(0,T;\ H^1(0,b))$. □

Remark 6.5. Since w is a strong solution of the linear problem (6.1)–(6.3), and $y = u - w$ is a weak solution of the non-linear problem (6.4)–(6.6). Then $u \in L^2(0,T;\ H^1(0,b))$ is a weak solution of the main nonlinear problem (2.1)–(2.3).

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