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ON A CLASSIFICATION
OF ALMOST α-COSYMPLECTIC MANIFOLDS

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ABSTRACT. The object of the present paper is to study almost α-cosymplectic manifolds. We consider projectively flat, conformally flat, and concircularly flat almost α-cosymplectic manifolds (with the η-parallel tensor field ϕh) and get some new properties. We conclude the paper by giving an example of α-Kenmotsu manifold, which verifies our results.

1. INTRODUCTION

The theory of almost cosymplectic manifold was introduced by Goldberg and Yano in [5]. The products of almost Kaehler manifolds and the real $\mathbb{R}$ line or the circle S^1 are the simplest examples of almost cosymplectic manifolds. Topological and geometrical properties of almost cosymplectic manifolds have been studied by many mathematicians (see [4, 5, 6, 8, 9]).

Considering the recent stage of the developments in the theory, there is an impression that the geometers are focused on problems in almost contact metric geometry. Recently, a long awaited survey article, [3], concerning almost cosymplectic manifolds as Blair's monograph [1] about contact metric manifolds appeared.

Almost contact metric structure is given by a pair (η, Φ), where η is a 1-form, Φ is a 2-form, and $\eta \wedge \Phi^n$ is a volume element. It is well known that then there exists a unique vector field ξ, called the characteristic (Reeb) vector field, such that $i_\xi \eta = 1$ and $i_\xi \Phi = 0$. The Riemannian geometry appears if we try to introduce *a compatible* structure, which is a metric g and an affinor ϕ ($(1,1)$-tensor field),

such that

$$\Phi(X,Y) = g(\phi X, Y), \quad \phi^2 = -(Id - \eta \otimes \xi). \tag{1.1}$$

Then, the triple (ϕ, ξ, η) is called almost contact structure.

An almost contact metric manifold is called Einstein if its Ricci tensor S satisfies the condition

$$S(X,Y) = ag(X,Y).$$

Combining the assumption concerning the forms η and Φ, we obtain many different types of almost contact manifolds, for example, contact if η is contact form and $d\eta = \Phi$; almost cosymplectic if $d\eta = 0$ and $d\Phi = 0$; almost Kenmotsu if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$.

Classifications are obtained for contact metric, almost cosymplectic, almost α-Kenmotsu, and almost α-cosymplectic manifolds. Almost α-cosymplectic manifolds are studied in [8, 12, 13].

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional Riemannian manifold with metric g. The Ricci operator Q of (M, g) is defined by $g(QX, Y) = S(X, Y)$, where S denotes the Ricci tensor of type $(0, 2)$ on M. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by [14]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y] \tag{1.2}$$

for all $X, Y, Z \in T(M)$, where R is the curvature tensor and S is the Ricci tensor.

In fact M is projectively flat if and only if it is of constant curvature [17]. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

In Riemannian geometry, one of the basic interest is curvature properties and to what extend these determine the manifold itself. One of the important curvature properties is conformal flatness. The conformal (Weyl) curvature tensor is a measure of the curvature of spacetime and differs from the Riemannian curvature tensor. It is the traceless component of the Riemannian tensor, which has the same symmetries as the Riemannian tensor. The most important of its special property that it is invariant under conformal changes to the metric. Namely, if $g^* = kg$ for some positive scalar functions k, then the Weyl tensor satisfies the equation $W^* = W$. In other words, it is called conformal tensor. Weyl constructed a generalized curvature tensor of type $(1, 3)$ on a Riemannian manifold, which vanishes whenever the metric is (locally) conformally equivalent to a flat metric; for this reason he called it the conformal curvature tensor of the metric.

The Weyl conformal curvature tensor is defined by

$$
\begin{aligned}
C(X,Y)Z \;=\; & R(X,Y)Z \\
& -\frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\
& +\frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y],
\end{aligned} \tag{1.3}
$$

for all $X, Y, Z \in T(M)$, where R is the curvature tensor, S is the Ricci tensor, and $r = \operatorname{tr}(S)$ is scalar curvature [18].

A necessary condition for a Riemannian manifold to be conformally flat is that the Weyl curvature tensor vanish. The Weyl tensor vanish identically for two dimensional case. In dimensions greater than or equal four, it is generally nonzero. If the Weyl tensor vanishes in dimensions greater than or equal four, then the metric is locally conformally flat. So there exists a local coordinate system in which the metric is proportional to a constant tensor. For the dimensions greater than three, this condition is sufficient as well. But in dimension three the vanishing of the equation $c = 0$; that is,

$$
c(X,Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{2(2n-1)}[(\nabla_X r)Y - (\nabla_Y r)X],
$$

is a necessary and sufficient condition for the Riemannian manifold being conformally flat, where c is the divergence operator of C for all vector fields X and Y on M. It should be noted that if the manifold is conformally flat and of dimension greater than three, then $C = 0$ implies $c = 0$ [18].

The concircular curvature tensor $\bar{C}$ of a $(2n+1)$-dimensional manifold is defined by

$$
\bar{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \tag{1.4}
$$

for all $X, Y, Z \in T(M)$, where R is the curvature tensor and $r = \operatorname{tr}(S)$ is scalar curvature [16, 17]. For $n \geq 1$, M is concircular flat if and only if the well known coincular curvature tensor $\bar{C}$ vanishes.

The paper is organized in the following way.

Section 2 is preliminary section. In this section, we remember basic properties of almost α-cosymplectic manifolds.

Section 3 is devoted to properties of almost α-cosymplectic manifolds with the η-parallel tensor field ϕh.

In Section 4, 5, and 6 we study, respectively, projectively flat, conformally flat and concircularly flat almost α-cosymplectic manifolds (with the η-parallel tensor field ϕh). We conclude the paper with an example on α-Kenmotsu manifold.

2. Preliminaries

An almost contact manifold is an odd-dimensional manifold M^{2n+1}, which carries a field ϕ of endomorphisms of the tangent spaces, a vector field ξ, called characteristic or Reeb vector field, and a 1-form η satisfying

$$
\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{2.1}
$$

where $I : TM^{2n+1} \to TM^{2n+1}$ is the identity mapping. From the definition it follows also that $\phi\xi = 0$, $\eta \circ \phi$ and that the $(1,1)$-tensor field ϕ has constant rank $2n$ (see [1]). An almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ is said to be normal when the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically, $[\phi, \phi]$ denoting the Nijenhuis tensor of ϕ. It is known that any almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any vector fields X, Y on M^{2n+1}. This metric g is called a compatible metric, and the manifold M^{2n+1} together with the structure $(M^{2n+1}, \phi, \xi, \eta, g)$ is called an almost contact metric manifold. As an immediate consequence of (2.2), one has $\eta = g(., \xi)$. The 2-form Φ of M^{2n+1} defined by

$$\Phi(X, Y) = g(\phi X, Y), \tag{2.3}$$

is called the fundamental 2-form of the almost contact metric manifold M^{2n+1}.

Almost contact metric manifolds such that both η and Φ are closed, are called almost cosymplectic manifolds and almost contact metric manifolds such that $d\eta = 0$, $d\Phi = 2\eta \wedge \Phi$, are almost Kenmotsu manifolds. Finally, a normal almost cosymplectic manifold is called a cosymplectic manifold and a normal almost Kenmotsu manifold is called a Kenmotsu manifold.

An almost contact metric manifold M^{2n+1} is said to be almost α-Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, and α is a nonzero real constant. Geometrical properties and examples of almost α-Kenmotsu manifolds are given in [7, 8, 10, 15]. If we join these two classes, we obtain a new notion of an almost α-cosymplectic manifold, which is defined by the following formula

$$d\eta = 0, \ d\Phi = 2\alpha\eta \wedge \Phi \tag{2.4}$$

for any real number α [8]. Obviously, a normal almost α-cosymplectic manifold is an α-cosymplectic manifold. An α-cosymplectic manifold is either cosymplectic manifold under the condition $\alpha = 0$ or α-Kenmotsu manifold $(\alpha \neq 0)$ for $\alpha \in \mathbb{R}$.

For an almost α-cosymplectic manifold, there exists an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$ such that $g(X_i, X_j) = 1$, $g(Y_i, Y_j) = 1$, and $Y_i = \phi X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a ϕ-basis.

We denote the distribution orthogonal to ξ by $\mathcal{D}$, that is $\mathcal{D} = ker(\eta) = \{X : \eta(X) = 0\}$, and let M^{2n+1} be an almost α-cosymplectic manifold with structure (ϕ, ξ, η, g). Since the 1-form is closed, we have $\mathcal{L}_\xi \eta = 0$ and $[X, \xi] \in \mathcal{D}$ for any $X \in \mathcal{D}$. The Levi–Civita connection satisfies $\nabla_\xi \xi = 0$ and $\nabla_\xi \phi \in \mathcal{D}$, which implies that $\nabla_\xi X \in \mathcal{D}$ for any $X \in \mathcal{D}$.

Moreover, an almost α-cosymplectic manifold satisfies the following equations, (see [8]):

$$\nabla_X \xi = -\alpha\phi^2 X - \phi h X = -\mathcal{A}, \tag{2.5}$$

$$(h \circ \phi)X + (\phi \circ h)X = 0, \quad (\mathcal{A} \circ \phi)X + (\phi \circ \mathcal{A})X = -2\alpha\phi X, \tag{2.6}$$

$$(\nabla_X \eta)Y = \alpha[g(X, Y) - \eta(X)\eta(Y)] + g(\phi Y, hX), \tag{2.7}$$

$$\text{tr}(\mathcal{A}\phi) = \text{tr}(\phi\mathcal{A}) = 0, \ \text{tr}(h\phi) = \text{tr}(\phi h) = 0, \tag{2.8}$$

$$\text{tr}(\mathcal{A}) = -2\alpha n, \ \text{tr}(h) = 0 \tag{2.9}$$

for any vector fields X and Y on M^{2n+1}.

From [8, Lemma 2.2], we have

$$(\nabla_{\phi X}\phi)\phi Y + (\nabla_X\phi)Y = -\alpha\eta(Y)\phi X - 2\alpha g(X,\phi Y)\xi - \eta(Y)hX,$$

for any vector fields X and Y on M^{2n+1}.

Lemma 2.1. [12]. *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be an almost α-cosymplectic manifold. Then, for any* $X, Y \in \chi(M^{2n+1})$,

$$\begin{aligned}
R(X,Y)\xi &= \alpha^2(\eta(X)Y - \eta(Y)X) - \alpha(\eta(X)\phi hY - \eta(Y)\phi hX) \\
&\quad +(\nabla_Y\phi h)X - (\nabla_X\phi h)Y, &\text{(2.10)} \\
lX &= R(X,\xi)\xi = \alpha^2\phi^2 X + 2\alpha\phi hX - h^2 X + \phi(\nabla_\xi h)X, &\text{(2.11)} \\
lX - \phi l\phi X &= 2[\alpha^2\phi^2 X - h^2 X], &\text{(2.12)} \\
(\nabla_\xi h)X &= -\phi lX - \alpha^2\phi X - 2\alpha hX - \phi h^2 X, &\text{(2.13)} \\
S(X,\xi) &= -2n\alpha^2\eta(X) - g(div(\phi h), X), &\text{(2.14)} \\
S(\xi,\xi) &= -[2n\alpha^2 + \text{tr}h^2]. &\text{(2.15)}
\end{aligned}$$

3. Almost α-cosymplectic manifolds with the η-parallel tensor field ϕh

For any vector field X on M^{2n+1}, we can take $X = X^T + \eta(X)\xi$, X^T is tangentially part of X, and $\eta(X)\xi$ is the normal part of X. We say that any symmetric $(1,1)$-type tensor field B on a Riemannian manifold (M, g) is said to be a η-parallel tensor if it satisfies the equation

$$g((\nabla_{X^T}B)Y^T, Z^T) = 0$$

for all tangent vectors X^T, Y^T, and Z^T orthogonal to ξ [2].

Proposition 3.1. [13]. *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be an almost α-cosymplectic manifold. If the tensor field ϕh is η-parallel, then we have*

$$\begin{aligned}
(\nabla_X\phi h)Y &= \eta(X)[lY - \alpha^2\phi^2 Y - 2\alpha\phi hY + h^2 Y] \\
&\quad -\eta(Y)[\alpha\phi hX - h^2 X] - g(Y, \alpha\phi hX - h^2 X)\xi &\text{(3.1)}
\end{aligned}$$

for all vector fields X and Y on M.

Proposition 3.2. [13]. *An almost α-cosymplectic manifold with the η-parallel tensor field ϕh satisfies the following relation*

$$R(X,Y)\xi = \eta(Y)lX - \eta(X)lY, \qquad \text{(3.2)}$$

where $l = R(.,\xi)\xi$ is the Jacobi operator with respect to the characteristic vector field ξ.

Theorem 3.3. [13]. *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be an almost α-cosymplectic manifold. If the tensor field ϕh is η-parallel, then ξ is the eigenvector of Ricci operator on M^{2n+1}.*

4. Projectively flat almost α-cosymplectic manifolds (with the η-parallel tensor field ϕh)

Theorem 4.1. *A projectively flat almost α-cosymplectic manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ has a scalar curvature*

$$r = 2n\mathrm{tr}(\phi(\nabla_\xi h)) + S(\xi, \xi)(1 + 2n). \qquad (4.1)$$

Proof. Let us suppose that almost α-cosymplectic manifold is projectively flat. If we take the inner product of (1.2) with W, we get

$$g(R(X,Y)Z,W) = \frac{1}{2n}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)].$$

By setting $W = X = \xi$ in the last equation and using (2.11) and (2.14), we obtain

$$S(Y,Z) = -2n\left(\begin{array}{c} \alpha^2 g(Y,Z) + 2\alpha g(\phi Y, hZ) + g(hZ, hY) \\ +g((\nabla_\xi h)Z, \phi Y) + \frac{1}{2n}\eta(Y)g(div(\phi h), Z) \end{array} \right). \qquad (4.2)$$

Considering the ϕ-basis and and putting $Y = Z = e_i$ in (4.2), we get

$$\sum_{i=1}^{2n+1} S(e_i, e_i) = \sum_{i=1}^{2n+1} -2n\left(\begin{array}{c} \alpha^2 g(e_i, e_i) + 2\alpha g(\phi e_i, he_i) + g(he_i, he_i) \\ +g((\nabla_\xi h)e_i, \phi e_i) + \frac{1}{2n}\eta(e_i)g(div(\phi h), e_i) \end{array} \right)$$

$$r = -2n\left(\begin{array}{c} \alpha^2(2n+1) - 2\alpha\mathrm{tr}(\phi h) + \mathrm{tr}(h^2) \\ -\mathrm{tr}(\phi(\nabla_\xi h)) + \frac{1}{2n}g(div(\phi h), \xi) \end{array} \right).$$

Then by (2.8), (2.14), and (2.15), we obtain (4.1). $\square$

Theorem 4.2. *A projectively flat α-cosymplectic manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is an Einstein manifold.*

Proof. If we take $h = 0$ in the proof of Theorem 4.1, we obtain $S(Y,Z) = -2n\alpha^2 g(Y,Z)$. This means manifold is Einstein. $\square$

Theorem 4.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an projectively flat almost α-cosymplectic manifold with the η-parallel tensor field ϕh. Then $r = \mathrm{tr}(l)(2n+1)$.*

Proof. Let us suppose that $(M^{2n+1}, \phi, \xi, \eta, g)$ is projectively flat almost α-cosymplectic manifold with the η-parallel tensor field ϕh. If we take the inner product of (1.2) with W, we get

$$g(R(X,Y)Z,W) = \frac{1}{2n}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)].$$

By setting $W = X = \xi$ in the last equation and using Proposition 3.2 and Theorem 3.3, we obtain

$$g(R(Y,\xi)\xi, Z) = \frac{1}{2n}[S(Y,Z) - \eta(Y)S(\xi, Z)]$$

$$g(lY,Z) = \frac{1}{2n}[S(Y,Z) - \eta(Y)\eta(Z)\mathrm{tr}(l)].$$

If we set $Y = Z = e_i$ in the last equation, we complete the proof. $\square$

5. CONFORMALLY FLAT ALMOST α-COSYMPLECTIC MANIFOLDS

Theorem 5.1. *A conformally flat almost α-cosymplectic manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ satisfies the following:*

$$0 = \mathrm{tr}(\phi(\nabla_\xi h)). \tag{5.1}$$

Proof. Let us suppose that almost α-cosymplectic manifold is conformally flat. If we take the inner product of (1.3) with W, we get

$$g(R(X,Y)Z,W) = \frac{1}{2n-1}\left(\begin{array}{c} g(Y,Z)g(QX,W) - g(X,Z)g(QY,W) \\ +S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \end{array} \right)$$
$$- \frac{r}{2n(2n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

By setting $W = X = \xi$ in the last equation and using (2.11) and (2.14) we obtain

$$\begin{aligned}
S(Y,Z) = {}& (2n-1)(-\alpha^2 g(Y,Z) + \alpha^2\eta(Y)\eta(Z) \\
& +2\alpha g(\phi hY, Z) - g(h^2 Y, Z) + g(\phi(\nabla_\xi h)Y, Z)) \\
& -g(Y,Z)(-2n\alpha^2 - \mathrm{tr}(h^2) + \eta(Z)(-2n\alpha^2\eta(Y) - g(div(\phi h), Y)) \\
& +\eta(Y)(-2n\alpha^2\eta(Z) - (div(\phi h))Z) \\
& +\frac{r}{2n}(g(Y,Z) - \eta(Y)\eta(Z)).
\end{aligned} \tag{5.2}$$

Considering the ϕ-basis and putting $Y = Z = e_i$ in (5.2), we get

$$\sum_{i=1}^{2n+1} S(e_i, e_i) = r$$

$$r = \sum_{i=1}^{2n+1} \left\{ \begin{array}{l} (2n-1)\left(\begin{array}{c} -\alpha^2 g(e_i, e_i) + \alpha^2\eta(e_i)\eta(e_i) + 2\alpha g(\phi he_i, e_i) \\ -g(h^2 e_i, e_i) + g(\phi(\nabla_\xi h)e_i, e_i) \end{array} \right) \\ -g(e_i, e_i)(-2n\alpha^2 - \mathrm{tr}(h^2) + \eta(e_i)(-2n\alpha^2\eta(e_i) - g(div(\phi h), e_i)) \\ +\eta(e_i)(-2n\alpha^2\eta(e_i) - (div(\phi h))e_i) + \frac{r}{2n}(g(e_i, e_i) - \eta(e_i)\eta(e_i)). \end{array} \right\}.$$

Then by (2.8), (2.14), and (2.15), we obtain

$$0 = \mathrm{tr}(\phi(\nabla_\xi h)).$$

$\square$

6. CONCIRCULARLY FLAT ALMOST α-COSYMPLECTIC MANIFOLDS (WITH THE η-PARALLEL TENSOR FIELD ϕh)

Theorem 6.1. *A concircularly flat almost α-cosymplectic manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ has a scalar curvature*

$$r = (2n+1)[S(\xi, \xi) + \mathrm{tr}(\phi(\nabla_\xi h))]. \tag{6.1}$$

Proof. Let us suppose that almost α-cosymplectic manifold is concircularly flat. If we take the inner product of (1.4) with W, we get

$$g(R(X,Y)Z,W) = \frac{r}{2n(2n+1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

By setting $W = X = \xi$ in the last equation and using (2.11), we obtain

$$-\alpha^2 g(\phi Y, \phi Z) + 2\alpha g(\phi h Y, Z) - g(h^2 Y, Z) + g(\phi(\nabla_\xi h)Y, Z) \qquad (6.2)$$
$$= \frac{r}{2n(2n+1)}(g(Y, Z) - \eta(Y)\eta(Z)).$$

Considering the ϕ-basis and and putting $Y = Z = e_i$ in (6.2), we get

$$\sum_{i=1}^{2n+1} \left(-\alpha^2 g(\phi e_i, \phi e_i) + 2\alpha g(\phi h e_i, e_i) - g(h^2 e_i, e_i) + g(\phi(\nabla_\xi h)e_i, e_i)\right)$$

$$= \sum_{i=1}^{2n+1} \left(\frac{r}{2n(2n+1)}(g(e_i, e_i) - \eta(e_i)\eta(e_i))\right).$$

Then by (2.8), (2.14), and (2.15), we obtain (6.1). □

Theorem 6.2. *A concircularly flat α-cosymplectic manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ has a scalar curvature*

$$r = -2\alpha^2 n(2n+1). \qquad (6.3)$$

Proof. If we take $h = 0$ in the proof of Theorem 6.1, we obtain the requested equation. □

Theorem 6.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a concircularly flat almost α-cosymplectic manifold with the η-parallel tensor field ϕh. Then $r = \mathrm{tr}(l)(2n+1)$.*

Proof. Let us suppose that $(M^{2n+1}, \phi, \xi, \eta, g)$ is a concircularly flat almost α-cosymplectic manifold with the η-parallel tensor field ϕh. If we take the inner product of (1.4) with W, we get

$$g(R(X, Y)Z, W) = \frac{r}{2n(2n+1)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].$$

By setting $W = X = \xi$ in the last equation and using Proposition 3.2 and Theorem 3.3, we obtain

$$g(R(Y, \xi)\xi, Z) = g(lY, Z) = \frac{r}{2n(2n+1)}[g(Y, Z) - \eta(Y)\eta(Z)].$$

If we set $Y = Z = e_i$ in the last equation, we complete the proof. □

7. Example

The following α-Kenmotsu manifold example [11] satisfies the conditions, which we proved the previous sections.

Example 7.1. We consider the three-dimensional manifold $M^3 = \{(x, y, z) \in \mathbb{R}^3\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields are

$$e_1 = c_2 e^{-\alpha z}\frac{\partial}{\partial x} + c_1 e^{-\alpha z}\frac{\partial}{\partial y}, \quad e_2 = -c_1 e^{-\alpha z}\frac{\partial}{\partial x} + c_2 e^{-\alpha z}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},$$

where $c_1^2 + c_2^2 \neq 0$ for constants c_1, c_2 and $\alpha \neq 0$. It is obvious that $\{e_1, e_2, e_3\}$ are linearly independent at each point of M^3. Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_1, e_2) = g(e_1, e_3) = 0, \quad g(e_2, e_3) = 0,$$

and given by the tensor product $g = (f_1^2 + f_2^2)^{-1}(dx \otimes dx + dy \otimes dy) + dz \otimes dz$. Let η be the 1-form defined by $\eta(X) = g(X, e_3)$ for any vector field X on M^3, and let ϕ be the $(1,1)$ tensor field defined by $\phi e_1 = e_2$, $\phi e_2 = -e_1$, $\phi e_3 = 0$. Then using linearity of g and ϕ, we have

$$\phi^2 X = -X + \eta(X)e_3, \quad \eta(e_3) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields on M^3.

Let ∇ be the Levi–Civita connection with respect to the metric g. Then we get

$$[e_1, e_3] = \alpha e_1, \quad [e_2, e_3] = \alpha e_2, \quad [e_1, e_2] = 0.$$

Using Koszul's formula, the Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) - g(Z, [X, Y]). \end{aligned}$$

Koszul's formula yields

$$\begin{array}{lll} \nabla_{e_1} e_1 = -\alpha e_3, & \nabla_{e_1} e_2 = -e_3, & \nabla_{e_1} e_3 = \alpha e_1, \\ \nabla_{e_2} e_1 = -e_3, & \nabla_{e_2} e_2 = -\alpha e_3, & \nabla_{e_2} e_3 = \alpha e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = 0. \end{array}$$

Thus it can be easily seen that $(M^3, \phi, \xi, \eta, g)$ is an α-Kenmotsu manifold. Hence, one can obtain by simple calculations that the curvature tensor components are as follows:

$$\begin{array}{ll} R(e_1, e_2)e_1 = \alpha(\alpha e_2 - e_1), & R(e_1, e_2)e_2 = \alpha(e_2 - \alpha e_1), \\ R(e_1, e_2)e_3 = 0, & R(e_1, e_2)e_1 = \alpha^2 e_3, \\ R(e_1, e_3)e_2 = \alpha e_3, & R(e_1, e_2)e_1 = -\alpha^2 e_1, \\ R(e_2, e_3)e_1 = \alpha e_3, & R(e_1, e_2)e_1 = \alpha^2 e_3, \\ R(e_2, e_3)e_3 = -\alpha^2 e_2. & \end{array}$$

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