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ON T-EXTENSIONS OF ABELIAN GROUPS

ALIAKBAR ALIJANI[1*] AND HOSSEIN SAHLEH[2]

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Abstract. Let $\Re$ be the category of all discrete abelian groups, and let $\pounds$ be the category of all locally compact abelian (LCA) groups. For a group $G \in \pounds$, the maximal torsion subgroup of G is denoted by tG. A short exact sequence $0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 0$ in $\Re$ is said to be a t-extension if $0 \to \mathrm{t}A \overset{\phi}{\to} tB \overset{\psi}{\to} tC \to 0$ is a short exact sequence. We show that the set of all t-extensions of A by C is a subgroup of $Ext(C, A)$, which contains $Pext(C, A)$ for discrete abelian groups A and C. We establish conditions under which the t-extensions split and determine those groups in $\Re$ which are t-injective or t-projective in $\Re$. Finally we determine the compact groups G in $\pounds$ such that every pure extension of G by a compact connected group $C \in \pounds$ splits.

1. Introduction and preliminaries

Throughout, all groups are Hausdorff topological abelian groups and will be written additively. Let $\pounds$ denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms, and let $\Re$ be the category of discrete abelian groups. The Pontrjagin dual and the maximal torsion subgroup of a group $G \in \pounds$ are denoted by $\hat{G}$ and tG, respectively. A morphism is called proper if it is open onto its image, and a short exact sequence $0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 0$ in $\pounds$ is said to be proper exact if ϕ and ψ are proper morphisms. In this case the sequence is called an extension of A by C (in $\pounds$). Following [4], let $Ext(C, A)$ denote the (discrete) group of extensions of A by C. Some of the subgroups of $Ext(C, A)$ such as $Pext(C, A)$,$^*Pext(C, A)$,$Tpext(C, A)$, and $Apext(C, A)$ have been studied in [2, 7, 8, 9, 11]. In this paper, we introduce

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* Corresponding author.
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a new subgroup of $Ext(C, A)$ whenever A and C are discrete abelian groups. In Sections 2 and 3, all groups are discrete abelian groups. An extension $0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 0$ in $\Re$ will be called a t-extension if $0 \to tA \overset{\phi|_{tA}}{\to} tB \overset{\psi|_{tB}}{\to} tC \to 0$ is an extension. Let $Ext_t(C, A)$ denote the set of all elements in $Ext(C, A)$ represented by t-extensions. In Section 2, we show that $Ext_t(C, A)$ is a subgroup of $Ext(C, A)$ which contains $Pext(C, A)$ (see Theorem 2.5 and Lemma 2.6). In Section 3, we establish some results on splitting of t-extensions (see Lemma 3.1, Theorem 3.11, and Theorem 3.13). Assume that $\Im$ is any subcategory of $\pounds$. The Section 4 is a part of an investigation which answers the following question:
Under what conditions on $G \in \pounds$, $Ext(X, G) = 0$ or $Pext(X, G) = 0$ for all $X \in \Im$? In [2, 3, 4, 5, 8, 10] the question is answered in some subcategories of $\pounds$ such as the category of divisible locally compact abelian groups. In [5, Corollary 3.4], Fulp and Griffith proved that a compact group G satisfies $Ext(C, G) = 0$ for all compact connected groups C if and only if $G \cong (\mathbb{R}/\mathbb{Z})^\sigma$ where σ is a cardinal. It may happen that $Ext(X, G) \neq 0$ but $Pext(X, G) = 0$. For example, $Ext(\mathbb{Z}(n), \mathbb{Z}) \neq 0$ but $Pext(\mathbb{Z}(n), \mathbb{Z}) = 0$, where $\mathbb{Z}$ is the group of integers and $\mathbb{Z}(n)$ is the cyclic group of order n. In this paper, we show that a compact group G satisfies $Pext(C, G) = 0$ for all compact connected groups C if and only if $G \cong (\mathbb{R}/\mathbb{Z})^\sigma \bigoplus H$, where H is a compact totally disconnected group (see Theorem 4.2). For the characterization of compact groups G which $Pext(C, G) = 0$ for all compact connected groups C, we need to show that $Pext(X, A) = 0$ for a discrete torsion group X and a discrete torsion-free group A (see Corollary 3.2).

The additive topological group of real numbers is denoted by $\mathbb{R}$, and $\mathbb{Q}$ is the group of rationals with the discrete topology. We denote the identity component of a group $G \in \pounds$ by G_0. For more on locally compact abelian groups, see [6].

2. T-EXTENSIONS

In this section, we define the concept of a t-extension of A by C. We show that the set of all t-extensions of A by C forms a subgroup of $Ext(C, A)$ which contains $Pext(C, A)$.

Lemma 2.1. *A pushout or a pullback of a t-extension is a t-extension.*

Proof. Suppose that $0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 0$ is a t-extension and that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \overset{\phi}{\longrightarrow} & B & \overset{\psi}{\longrightarrow} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow & & \downarrow{\scriptstyle 1_C} & & \\
0 & \longrightarrow & A' & \overset{\phi'}{\longrightarrow} & (A' \bigoplus B)/H & \overset{\psi'}{\longrightarrow} & C & \longrightarrow & 0
\end{array}$$

is a standard pushout diagram (see [1]). Then

$$H = \{(\mu(a), -\phi(a)), a \in A\}$$

and

$$\phi' : a' \longmapsto (a', 0) + H, \ \psi' : (a', b) + H \longmapsto \psi(b).$$

We show that $0 \to tA' \xrightarrow{\phi'} t((A' \bigoplus B)/H) \xrightarrow{\psi'} tC \to 0$ is exact. First, we show that $\psi' : t((A' \bigoplus B)/H) \to tC$ is surjective. Let $c \in tC$. Since $0 \to tA \xrightarrow{\phi} tB \xrightarrow{\psi} tC \to 0$ is exact, so there exists $b \in tB$ such that $\psi(b) = c$. Clearly, $(0, b) + H \in t((A' \bigoplus B)/H)$. On the other hand, $\psi'((0, b) + H) = \psi(b) = c$. Hence ψ' is surjective. Now, we show that $\ker \psi' |_X \subseteq Im\phi' |_{tA'}$ where $X = t((A' \bigoplus B)/H)$. Let $(a', b) + H \in X$, and let $\psi'((a', b) + H) = 0$. So, $\psi(b) = 0$. Hence, there exists $a \in A$ such that $\phi(a) = -b$. On the other hand, there exists a positive integer n such that $(na', nb) \in H$. So, there exists $a_1 \in A$ such that $\mu(a_1) = na'$ and $-\phi(a_1) = nb$. Now, we have

$$\phi(a_1 - na) = \phi(a_1) - n\phi(a) = 0.$$

So $a_1 = na$ and $n(a' - \mu(a)) = 0$. It follows that $a' - \mu(a) \in tA'$ and $\phi'(a' - \mu(a)) = (a' - \mu(a), 0) + H = (a', b) + H$ (since $(a' - \mu(a), 0) - (a', b) = (-\mu(a), -b) = (\mu(-a), -\phi(-a)) \in H$). Now, suppose that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\phi'} & B' & \xrightarrow{\psi'} & C' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_A} & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0
\end{array}$$

is a standard pullback diagram. Then

$$B' = \{(b, c'); \psi(b) = \gamma(c')\}$$

and

$$\phi' : a \longmapsto (\phi(a), 0), \ \ \psi' : (b, c') \longmapsto c'.$$

We show that $0 \to tA \xrightarrow{\phi'} tB' \xrightarrow{\psi'} tC' \to 0$ is exact. Let $c' \in tC'$. Then, there exists a positive integer n such that $nc' = 0$. Since ψ is surjective, $\psi(b) = \gamma(c')$ for some $b \in B$. Now, $n\psi(b) = \gamma(nc') = 0$. Hence, $\psi(b) \in tC$. Since $0 \to tA \xrightarrow{\phi} tB \xrightarrow{\psi} tC \to 0$ is exact, so $\psi(b_1) = \psi(b)$ for some $b_1 \in tB$. Hence, $(b_1, c') \in tB'$ and $\psi'(b_1, c') = c'$. Therefore, $\psi' : tB' \to tC'$ is surjective. Now, suppose that $(b, c') \in tB'$ and $\psi'(b, c') = 0$. Then $c' = 0$ and $nb = 0$ for some positive integer n. So $b \in tB$. Since $\psi(b) = \gamma(c') = 0$ and $0 \to tA \xrightarrow{\phi} tB \xrightarrow{\psi} tC \to 0$ is exact, there exists $a \in tA$ such that $\phi(a) = b$. Now, we have

$$\phi'(a) = (\phi(a), 0) = (b, 0) = (b, c').$$

It follows that $\ker \psi' |_{tB'} \subseteq Im\phi' |_{tA}$. □

Remark 2.2. Let $\beta : B \to X$ be an isomorphism, and let $x \in tX$. Then $nx = 0$ for some positive integer n. Since β is surjective, so there exist $b \in B$ such that $\beta(b) = x$. Hence, $\beta(nb) = 0$. Since β is injective, so $nb = 0$. Therefore, $\beta |_{tB}: tB \to tX$ is an isomorphism.

Recall that two extensions $0 \to A \xrightarrow{\phi_1} B \xrightarrow{\psi_1} C \to 0$ and $0 \to A \xrightarrow{\phi_2} X \xrightarrow{\psi_2} C \to 0$ are said to be equivalent if there is an isomorphism $\beta : B \to X$ such that the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi_1} & B & \xrightarrow{\psi_1} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_C} & & \\ 0 & \longrightarrow & A & \xrightarrow{\phi_2} & X & \xrightarrow{\psi_2} & C & \longrightarrow & 0 \end{array}$$

is commutative.

Lemma 2.3. *An extension, being equivalent to a t-extension, is a t-extension.*

Proof. Let

$$E_1 : 0 \to A \overset{\phi_1}{\to} B \overset{\psi_1}{\to} C \to 0$$

and

$$E_2 : 0 \to A \overset{\phi_2}{\to} X \overset{\psi_2}{\to} C \to 0$$

be two equivalent extensions such that E_1 is a t-extension. Then, there is an isomorphism $\beta : B \to X$ such that $\beta\phi_1 = \phi_2$ and $\psi_2\beta = \psi_1$. Let $x \in tC$. Since E_1 is a t-extension, so $\psi_1(b) = x$ for some $b \in tB$. Hence, $\psi_2(\beta(b)) = \psi_1(b) = x$. So, $\psi_2 : tX \to tC$ is surjective. Now, let $\psi_2(x) = 0$ for some $x \in tX$. By Remark 2.2, there exists $b \in tB$ such that $\beta(b) = x$. Hence, $\psi_1(b) = \psi_2(\beta(b)) = 0$. Since E_1 is t-extension, so $\phi_1(a) = b$ for some $a \in tA$. Consequently, $\phi_2(a) = \beta(\phi_1(a)) = x$. □

Remark 2.4. Let C and A be two groups, and let $0 \to A \overset{\phi_1}{\to} B_1 \overset{\psi_1}{\to} C \to 0$ and $0 \to A \overset{\phi_2}{\to} B_2 \overset{\psi_2}{\to} C \to 0$ be two t-extensions of A by C. An easy calculation shows that $0 \to A \bigoplus A \overset{(\phi_1 \bigoplus \phi_2)}{\to} B_1 \bigoplus B_2 \overset{(\psi_1 \bigoplus \psi_2)}{\to} C \bigoplus C \to 0$ is a t-extension where $(\phi_1 \bigoplus \phi_2)(a_1, a_2) = (\phi_1(a_1), \phi_2(a_2))$ and $(\psi_1 \bigoplus \psi_2)(b_1, b_2) = (\psi_1(b_1), \psi_2(b_2))$.

Theorem 2.5. *Let A and C be two groups. Then, the class $Ext_t(C, A)$ of all equivalence classes of t-extensions of A by C is an subgroup of $Ext(C, A)$ with respect to the operation defined by*

$$[E_1] + [E_2] = [\nabla_A(E_1 \bigoplus E_2)\triangle_C],$$

where E_1 and E_2 are t-extensions of A by C and ∇_A and $\triangle_C$ are the diagonal and codiagonal homomorphisms.

Proof. Clearly, $0 \to A \to A \bigoplus C \to C \to 0$ is a t-extension. By Remark 2.4 and Lemma 2.1 , $[E_1] + [E_2] \in Ext_t(C, A)$ for two t-extensions E_1 and E_2 of A by C. So, $Ext_t(C, A)$ is a subgroup of $Ext(C, A)$. □

Lemma 2.6. *Let A and C be two groups. Then, $Pext(C, A) \subseteq Ext_t(C, A)$.*

Proof. Let $0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 0$ be an element of $Pext(C, A)$. It is sufficient to show that $tB/t\phi(A) \cong t(B/\phi(A))$. Consider the map $\varphi : tB \to t(B/\phi(A))$ given by $b \longmapsto b + \phi(A)$. Clearly, φ is a homomorphism. We show that φ is surjective. Let $b + \phi(A) \in t(B/\phi(A))$. Then, there exists a positive integer n such that $nb \in \phi(A)$. Since $\phi(A)$ is pure in B, so $nb = n\phi(a)$ for some $a \in A$. Hence,

$n(b-\phi(a))=0$. This shows that $b-\phi(a)\in tB$ and $\varphi(b-\phi(a))=b+\phi(A)$. So φ is surjective. We have

$$\ker\varphi=\{b\in tB; b\in\phi(A)\}=\phi(A)\cap tB=t\phi(A).$$

Hence $tB/t\phi(A)\cong t(B/\phi(A))$. □

Corollary 2.7. *If A is a divisible group or C is a torsion-free group, then $Pext(C,A)=Ext_t(C,A)=Ext(C,A)$.*

Proof. It is clear. □

3. Splitting of t-extensions

In this section, we establish some conditions on A and C such that $Ext_t(C,A)=0$. We also determine the t-injective and t-projective groups in $\Re$.

Lemma 3.1. *Let A be a torsion-free group, and let C be a torsion group. Then, $Ext_t(C,A)=0$.*

Proof. Let $E:0\to A\xrightarrow{\phi}B\xrightarrow{\psi}C\to 0$ be a t-extension. Then $\psi_{|tB}:tB\to C$ is an isomorphism. Let $b\in B$. Then, $\psi(b)\in C$. So $\psi(b)=\psi(b')$ for some $b'\in tB$. Hence, $b-b'=\phi(a)$ for some $a\in A$. This follows that $B=\phi(A)+tB$. Since $\phi(A)$ is torsion-free and tB torsion, so $\phi(A)\bigcap tB=0$. Hence, $B=\phi(A)\bigoplus tB$ and E splits. □

Corollary 3.2. *Let A be a torsion-free group, and let C be a torsion group. Then, $Pext(C,A)=0$.*

Proof. It is clear by Lemma 2.6 and Lemma 3.1. □

Lemma 3.3. *Let A and C be two torsion groups. Then $Ext(C,A)=Ext_t(C,A)$.*

Proof. Let A and C be two torsion groups. It is clear that $Ext_t(C,A)\subseteq Ext(C,A)$. Suppose that $E:0\to A\to B\to C\to 0$ is an extension. Then, B is a torsion group. Hence, E is a t-extension. □

Lemma 3.4. *Let C be a torsion group. Then, $Ext(C,tA)\cong Ext_t(C,A)$ for every group A.*

Proof. The exact sequence $0\to tA\xrightarrow{i}A\xrightarrow{\pi}A/tA\to 0$ induces the following exact sequence

$$Hom(C,A/tA)\to Ext(C,tA)\xrightarrow{i_*}Ext(C,A)\xrightarrow{\pi_*}Ext(C,A/tA)\to 0.$$

Note that $Hom(C,A/tA)=0$. By Lemma 3.1, $Ext_t(C,A/tA)=0$. Since $\pi_*(Ext_t(C,A))\subseteq Ext_t(C,A/tA)$, so $\pi_*(Ext_t(C,A))=0$. Hence, $Ext_t(C,A)\subseteq\ker\pi_*=i_*(Ext(C,tA))$. By Lemma 3.3, $Ext(C,tA)=Ext_t(C,tA)$. Therefore, $i_*(Ext(C,tA))\subseteq Ext_t(C,A)$. Hence, $Ext(C,tA)\cong Ext_t(C,A)$. □

Corollary 3.5. *Let A be a group. Then, $Ext_t(\mathbb{Z}(m),A)\cong tA/m(tA)$ for every positive integer m.*

Proof. It is clear by Lemma 3.4 and [1, p. 222]. □

Corollary 3.6. *Let C be a torsion group, and let $\{A_i : i \in I\}$ be a collection of groups. If I is finite, then $Ext_t(C, \Pi_{i\in I} A_i) \cong \Pi_{i\in I} Ext_t(C, A_i)$.*

Proof. Let $I = \{1, \ldots, n\}$ for some positive integer n. Lemma 3.4 implies that $Ext_t(C, \Pi_{i=1}^n A_i) \cong Ext(C, t(\Pi_{i=1}^n A_i))$. Since $t(\Pi_{i=1}^n A_i) = \Pi_{i=1}^n t(A_i)$, therefore $Ext_t(C, \Pi_{i=1}^n A_i) \cong \Pi_{i\in I} Ext(C, tA_i) \cong \Pi_{i\in I} Ext_t(C, A_i)$. □

Remark 3.7. In general, $Ext_t(C, \Pi_{i\in I} A_i) \ncong \Pi_{i\in I} Ext_t(C, A_i)$.

Example 3.8. Let p be a prime, and let $H = \prod_{n=1}^{\infty} \mathbb{Z}(p^n)$. By Lemma 3.4, $Ext_t(\mathbb{Q}/\mathbb{Z}, H) \cong Ext(\mathbb{Q}/\mathbb{Z}, tH)$. Consider the following exact sequence

$$0 \to Ext(\mathbb{Q}/\mathbb{Z}, tH) \to Ext(\mathbb{Q}/\mathbb{Z}, H) \to Ext(\mathbb{Q}/\mathbb{Z}, H/tH) \to 0. \tag{3.1}$$

By [1, Theorem 52.2] and Lemma 3.3,

$$Ext(\mathbb{Q}/\mathbb{Z}, H) \cong \prod_{n=1}^{\infty} Ext(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}(p^n)) \cong \prod_{n=1}^{\infty} Ext_t(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}(p^n)).$$

If $Ext_t(\mathbb{Q}/\mathbb{Z}, H) \cong \prod_{n=1}^{\infty} Ext_t(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}(p^n))$,then $Ext(\mathbb{Q}/\mathbb{Z}, H) \cong Ext(\mathbb{Q}/\mathbb{Z}, tH)$. It follows from (3.1) that, $Ext(\mathbb{Q}/\mathbb{Z}, H/tH) = 0$ which is a contradiction, since H/tH is not divisible.

Lemma 3.9. *Let A be a torsion-free group. Then, $Ext(C/tC, A) \cong Ext_t(C, A)$ for every group C.*

Proof. The exact sequence $0 \to tC \xrightarrow{i} C \xrightarrow{\pi} C/tC \to 0$ induces the following exact sequence

$$Hom(tC, A) \to Ext(C/tC, A) \xrightarrow{\pi_*} Ext(C, A) \xrightarrow{i_*} Ext(tC, A) \to 0.$$

Note that $Hom(tC, A) = 0$. By Lemma 3.1, $Ext_t(tC, A) = 0$. Therefore, $i_*(Ext_t(C, A)) \subseteq Ext_t(tC, A) = 0$. So, $Ext_t(C, A) \subseteq \ker i_* = \pi_*(Ext(C/tC, A))$. By Corollary 2.7, $Ext(C/tC, A) = Ext_t(C/tC, A)$. So, $\pi_*(Ext(C/tC, A)) \subseteq Ext_t(C, A)$. Hence, $Ext(C/tC, tA) \cong Ext_t(C, A)$. □

Definition 3.10. Let G be a group. We call G a t-injective group in $\Re$ if for every t-extension

$$0 \to A \xrightarrow{\phi} B \to C \to 0$$

and a homomorphism $f : A \to G$, there is a homomorphism $\bar{f} : B \longrightarrow G$ such that $\bar{f}\phi = f$.

We call G a t-projective group in $\Re$ if for every t-extension

$$0 \to A \to B \xrightarrow{\psi} C \to 0$$

and a homomorphism $f : G \to C$, there is a homomorphism $\bar{f} : G \to B$ such that $\psi\bar{f} = f$.

Recall that a group A is said to be cotorsion if $Ext(\mathbb{Q}, A) = 0$ (see [1]).

Theorem 3.11. *Let A be a group. The following statements are equivalent:*

(1) *A is t-injective in $\Re$.*
(2) *$Ext_t(C, A) = 0$ for all $C \in \Re$.*

(3) *$A \cong B \bigoplus D$ where B is a torsion divisible group and D a torsion-free cotorsion group.*

Proof. (1) $\Rightarrow$ (2): Let A be a t-injective in $\Re$, and let $E : 0 \to A \overset{\phi}{\to} B \to C \to 0$ be a t-extension of A by C. Then, there is a homomorphism $\bar{\phi} : B \to G$ such that $\bar{\phi}\phi = 1_G$. Consequently, E splits.

(2) $\Rightarrow$ (3): Let $Ext_t(C, A) = 0$ for every group C. So $Ext_t(\mathbb{Z}(m), A) = 0$ for every positive integer m. By Corollary 3.5, $m(tA) = tA$ for every positive integer m. So, tA is divisible. Hence, $A \cong tA \bigoplus A/tA$. Therefore, $Ext(\mathbb{Q}, A) \cong Ext(\mathbb{Q}, tA) \bigoplus Ext(\mathbb{Q}, A/tA)$. Since $Ext(\mathbb{Q}, A) = Ext_t(\mathbb{Q}, A) = 0$, then $Ext(\mathbb{Q}, A/tA) = 0$. Hence, A/tA is cotorsion. Now, we set $tA = B$ and $A/tA = D$.

(3) $\Rightarrow$ (2): Suppose that $A \cong B \bigoplus D$ where B is a torsion divisible group and D a torsion-free cotorsion group. Let C be a group. Since $Ext(C, B) = 0$, so $p_2^* : Ext(C, B \bigoplus D) \to Ext(C, D)$ is an isomorphism, where $p_2 : B \bigoplus D \to D$ is the projection map. By Lemma 3.9, $Ext_t(C, D) \cong Ext(C/tC, D)$. Since D is a cotorsion group, so $Ext(C/tC, D) = 0$. Hence, $Ext_t(C, D) = 0$. So, $p_2^*(Ext_t(C, B \bigoplus D)) \subseteq Ext_t(C, D) = 0$. Since p_2^* is an isomorphism, therefore $Ext_t(C, B \bigoplus D) = 0$ or $Ext_t(C, A) = 0$.

(2) $\Longrightarrow$ (1): Let $E : 0 \to G \overset{\phi}{\to} B \to C \to 0$ be a t-extension and let $f : G \to A$ be a homomorphism. Then f induces a pushout diagram

$$
\begin{array}{ccccccccc}
E : 0 & \longrightarrow & G & \overset{\phi}{\longrightarrow} & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow & & \\
fE : 0 & \longrightarrow & A & \overset{\mu}{\longrightarrow} & (A \bigoplus B)/H & \longrightarrow & C & \longrightarrow & 0
\end{array}
$$

where $H = \{(-f(a), \phi(a)); a \in A\}$ and $\mu : a \longmapsto (a, 0) + H$. By Lemma 2.1, fE is a t-extension and by assumption it splits. Hence A is t-injective. □

Recall that a group A is called algebraically compact if and only if $Pext(X, A) = 0$ for every group X (see [1]).

Corollary 3.12. *A torsion-free, cotorsion group is algebraically compact.*

Proof. Let A be a torsion-free, cotorsion group. By Theorem 3.11, $Ext_t(C, A) = 0$ for every group C. Hence, $Pext(C, A) = 0$ for every group C. □

Theorem 3.13. *Let C be a group. Consider the following conditions for C:*

(1) *C is t-projective in $\Re$.*
(2) *$Ext_t(C, A) = 0$ for all $A \in \Re$.*
(3) *C is a direct sum of cyclic groups.*

Then: (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) *and* (3) $\not\Rightarrow$ (2).

Proof. (1) $\Rightarrow$ (2): Let C be t-projective in $\Re$, and let $E : 0 \to A \to B \overset{\psi}{\to} C \to 0$ be a t-extension of A by C. Then there is a homomorphism $\bar{\psi} : C \to B$ such that $\psi\bar{\psi} = 1_C$. Consequently, E splits.

(2) $\Rightarrow$ (3): Let $Ext_t(C, A) = 0$ for every group A. By Lemma 2.6, $Pext(C, A) = 0$ for every group A. Hence, C is a direct sum of cyclic groups (see [1, Theorem 30.2]).

(2) $\Rightarrow$ (1): Let $E : 0 \to A \to B \xrightarrow{\psi} G \to 0$ be a t-extension, and let $f : C \to G$ be a homomorphism. Then f induces a pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B' & \xrightarrow{\psi'} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\psi} & G & \longrightarrow & 0 \end{array}$$

Where $B' = \{(b, c); \psi(b) = f(c)\}$ and $\psi' : (b, c) \longmapsto c$. By Lemma 2.1, Ef is a t-extension and by assumption, it splits. Hence C is t-projective.

(3) $\not\Rightarrow$ (2): By Corollary 3.5, $Ext_t(\mathbb{Z}_2, \mathbb{Z}_4) \cong \mathbb{Z}_2 \neq 0$. $\square$

4. Splitting pure extensions by compact connected abelian groups

In this section, we determine the structure of a compact group G such that $Pext(C, G) = 0$ for all compact connected groups C.

Lemma 4.1. *If A is a compact totally disconnected group and C a compact connected group, then $Pext(C, A) = 0$.*

Proof. By [8, Lemma 2.3], $Pext(C, A) \cong Pext(\hat{A}, \hat{C})$. On the other hand, by [6, Theorems 24.25 and 24.26], $\hat{A}$ and $\hat{C}$ are a discrete, torsion group and a discrete, torsion-free group, respectively. Hence, by Corollary 3.2, $Pext(\hat{A}, \hat{C}) = 0$. So $Pext(C, A) = 0$. $\square$

Let G be a compact group. Then, $Ext(C, G) = 0$ for every compact connected group C if and only if $G \cong (\mathbb{R}/\mathbb{Z})^\sigma$ (see [5, Corollary 3.4]). In the next Theorem, we show that $Pext(C, G) = 0$ for every compact connected group C if and only if $G \cong (\mathbb{R}/\mathbb{Z})^\sigma \bigoplus H$, where H is a compact totally disconnected group.

Theorem 4.2. *Let G be a compact group. Then, $Pext(C, G) = 0$ for all compact connected groups C if and only if $G \cong (\mathbb{R}/\mathbb{Z})^\sigma \bigoplus H$, where H is a compact totally disconnected group.*

Proof. Let G be a compact group, and let C be a compact connected group. Consider the exact sequence $0 \to G_0 \to G \to G/G_0$. By [2, Proposition 4], we have the exact sequence

$$Hom(C, G/G_0) \to Pext(C, G_0) \to Pext(C, G) \to Pext(C, G/G_0).$$

By Lemma 4.1, $Pext(C, G/G_0) = 0$. Since G/G_0 is totally disconnected, so $Hom(C, G/G_0) = 0$. It follows that $Pext(C, G) \cong Pext(C, G_0)$. But, G_0 is a divisible group. So, $Pext(C, G) \cong Ext(C, G_0)$. Now, Let $Pext(C, G) = 0$ for all compact connected groups C. Then $Ext(C, G_0) = 0$ for all compact connected groups C. By [5, Corollary 3.4], $G_0 \cong (\mathbb{R}/\mathbb{Z})^\sigma$. So, the extension $0 \to G_0 \to G \to G/G_0 \to 0$ splits. This shows that $G \cong (\mathbb{R}/\mathbb{Z})^\sigma \bigoplus G/G_0$. The converse is clear. $\square$

Remark 4.3. Let G be a compact group such that $Ext(C, G) = 0$ for every compact connected group C. Then $Pext(C, G) = 0$ for every compact connected group C. So by Theorem 4.2, $G \cong (\mathbb{R}/\mathbb{Z})^{\sigma} \bigoplus H$, where H is a compact totally disconnected group. Since $\mathbb{R}/\mathbb{Z}$ is a compact connected group, so $Ext(\mathbb{R}/\mathbb{Z}, H) = 0$. Consider the following exact sequence

$$Hom(\mathbb{R}, H) \to Hom(\mathbb{Z}, H) \to Ext(\mathbb{R}/\mathbb{Z}, H) = 0.$$

Since $\mathbb{R}$ is a connected group and H is a totally disconnected group, therefore $Hom(\mathbb{R}, H) = 0$. Hence, $H = 0$ and $G \cong (\mathbb{R}/\mathbb{Z})^{\sigma}$.

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[1] Mollasadra Technical and Vocational College, Technical and Vocational University, Ramsar, Iran.
E-mail address: alijanialiakbar@gmail.com

[2] Department of Mathematics, University of Guilan, P. O. Box 1914, Rasht, Iran.
E-mail address: sahleh@guilan.ac.ir