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ON RANDERS CHANGE OF GENERALIZED mTH ROOT METRIC

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ABSTRACT. In the present paper, we find a condition under which a Finsler space with Randers change of generalized mth root metric is projectively related to an mth root metric. Then we find a condition under which Randers change of generalized mth root Finsler metric is locally projectively flat and also find a condition for locally dually flatness.

1. Introduction

In 1941, Randers [6] introduced a special class of Finsler space defined by $\bar{F} = F + \beta$, where F is a Riemannian metric and $\beta = b_i(x)y^i$ is a one-form. Matsumoto [4] studied Randers spaces and generalized Randers spaces in which F is Finslerian. The theory of mth root metrics has been developed by Matsumoto and Shimada (see [5] and [7]) and applied to ecology by Antonelli [1] and studied by many authors (see [10, 11, 12, 13]). Recent studies show that mth root Finsler metrics plays a very important role in physics, space-time, general relativity as well as in unified gauge field theory (see [2]). In [9], Tayebi and Najafi characterized locally dually flat and Antonelli mth root metrics.

Recently, Tayebi, Peyghan, and Shahbazi Nia [10] has introduced generalized mth root Finsler metrics in which they studied locally dually flatness of generalized mth root Finsler metrics and found a condition under which a generalized mth root metric is projectively related to an mth root metric.

Throughout this paper, we call the Finsler metric $\bar{F}$ as transformed mth root metric and $(M^n, \bar{F}) = \bar{F}^n$ as transformed Finsler space. We restrict ourselves to $m > 2$, throughout the paper and also the quantities corresponding to the

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transformed Finsler space $\bar{F}^n$ will be denoted by putting bar on the top of that quantity.

In this paper, we obtain the condition under which the transformed Finsler space is projectively related with the given Finsler space. Also we find the condition under which the transformed Finsler space is locally projectively flat and locally dually flat.

2. Preliminaries

Let M^n be an n-dimensional C^∞-manifold, and let T_xM denote the tangent space of M^n at x. The tangent bundle TM is the union of tangent spaces, $TM := \bigcup_{x\in M} T_xM$. We denote the elements of TM by (x, y), where $x = (x^i)$ is a point of M^n and $y \in T_xM$, called supporting element. We denote $TM_0 = TM \setminus \{0\}$.

Definition 2.1. A Finsler metric on M^n is a function $F : TM \to [0, \infty)$ with the following properties:
(i) F is C^∞ on TM_0,
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and
(iii) the Hessian of F^2 with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.

The pair $(M^n, F) = F^n$ is called a Finsler space. F is called the fundamental function, and g_{ij} is called the fundamental metric tensor of the Finsler space F^n. The normalized supporting element l_i and angular metric tensor h_{ij} of F^n are defined, respectively, as:

$$l_i = \frac{\partial F}{\partial y^i} \quad \text{and} \quad h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j}. \tag{2.1}$$

Let $F = \sqrt[m]{A}$ be an mth root Finsler metric, where A is given by $A := a_{i_1 i_2 \dots i_m}(x)\, y^{i_1} y^{i_2} \dots y^{i_m}$, with $a_{i_1 \dots i_m}$ is symmetric in all its indices [7].
The generalized mth root metric is defined as $\tilde{F} = \sqrt{F^2 + B}$. Here F is called an associated mth root metric of the generalized mth root metric $\tilde{F}$ [8].

Consider the transformed Finsler metric

$$\bar{F} = \sqrt{F^2 + B} + C, \tag{2.2}$$

where $B = b_{ij}(x)y^i y^j$ and $C = c_i(x)y^i$ is a one-form on the manifold M^n.
This transformed metric $\bar{F}$ is called Randers change of generalized mth root metric. Clearly $\bar{F}$ is also a Finsler metric on M^n.
Let,

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i,$$

$$B_i = \frac{\partial B}{\partial y^i}, \quad B_{x^i} = \frac{\partial B}{\partial x^i}, \quad B_0 = B_{x^i} y^i,$$

$$C_i = \frac{\partial C}{\partial y^i}, \quad C_{x^i} = \frac{\partial C}{\partial x^i}, \quad C_0 = C_{x^i} y^i.$$

3. Fundamental metric tensor with Randers change of generalized mth root metric

From (2.2), we have

$$\bar{F}^2 = A^{\frac{2}{m}} + B + C^2 + 2C\sqrt{A^{\frac{2}{m}} + B}. \tag{3.1}$$

Differentiating equation (3.1) with respect to y^i yields

$$\frac{\partial \bar{F}^2}{\partial y^i} = \frac{2}{m}A^{\frac{2}{m}-1}A_i + 2b_{ij}y^j + 2CC_i + 2C_i\sqrt{A^{\frac{2}{m}} + B} + \frac{\left(\frac{2}{m}A^{\frac{2}{m}-1}A_i + 2b_{ij}y^j\right)C}{\sqrt{A^{\frac{2}{m}} + B}}. \tag{3.2}$$

Again, differentiating equation (3.2) with respect to y^j yields

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 \bar{F}^2}{\partial y^i \partial y^j} \;=\;& \frac{1}{m}(\frac{2}{m}-1)A^{\frac{2}{m}-2}A_iA_j + \frac{1}{m}A^{\frac{2}{m}-1}A_{ij} + b_{ij} + C_iC_j \\
&+ \frac{C_i\left(\frac{2}{m}A^{\frac{2}{m}-1}A_j + 2b_{jk}y^k\right)}{2\sqrt{A^{\frac{2}{m}} + B}} \\
&+ \frac{\sqrt{A^{\frac{2}{m}} + B}\left\{\left(\frac{1}{m}(\frac{2}{m}-1)A^{\frac{2}{m}-2}A_iA_j + \frac{1}{m}A^{\frac{2}{m}-1}A_{ij} + b_{ij}\right)C\right\}}{A^{\frac{2}{m}} + B} \\
&+ \frac{C_j\sqrt{A^{\frac{2}{m}} + B}\left(\frac{1}{m}A^{\frac{2}{m}-1}A_i + b_{ij}y^j\right)}{A^{\frac{2}{m}} + B} \\
&- C\left(\frac{1}{m}A^{\frac{2}{m}-1}A_i + b_{ij}y^j\right)\frac{\frac{1}{m}A^{\frac{2}{m}-1}A_j + b_{ij}y^i}{(A^{\frac{2}{m}} + B)^{\frac{3}{2}}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\bar{g_{ij}} \;=\;& \frac{1}{m}(\frac{2}{m}-1)A^{\frac{2}{m}-2}A_iA_j + \frac{1}{m}A^{\frac{2}{m}-1}A_{ij} + b_{ij} + C_iC_j \\
&+ \frac{C_i\left(\frac{1}{m}A^{\frac{2}{m}-1}A_j + b_{jk}y^k\right)}{\sqrt{A^{\frac{2}{m}} + B}} \\
&+ (A^{\frac{2}{m}} + B)^{-\frac{1}{2}}\left\{\left(\frac{1}{m}(\frac{2}{m}-1)A^{\frac{2}{m}-2}A_iA_j + \frac{1}{m}A^{\frac{2}{m}-1}A_{ij} + b_{ij}\right)C\right. \\
&\left. + C_j\left(\frac{1}{m}A^{\frac{2}{m}-1}A_i + b_{ij}y^j\right)\right\} \\
&- C(A^{\frac{2}{m}} + B)^{-\frac{3}{2}}\left(\frac{1}{m}A^{\frac{2}{m}-1}A_i + b_{ij}y^j\right)\left(\frac{1}{m}A^{\frac{2}{m}-1}A_j + b_{ij}y^i\right).
\end{aligned}$$

The metric tensor of Finsler space with Randers change of generalized mth root metric is given by

$$\bar{g}_{ij} = \hat{g_{ij}}\left(1 + \frac{C}{\sqrt{A^{\frac{2}{m}} + B}}\right) + C_iC_j + \frac{C_iD_j + C_jD_i}{\sqrt{A^{\frac{2}{m}} + B}} - C\frac{D_iD_j}{(A^{\frac{2}{m}} + B)^{3/2}}, \tag{3.3}$$

where

$$\hat{g_{ij}} = g_{ij} + b_{ij}, \quad D_i = \frac{1}{m} A^{\frac{2}{m}-1} A_i + b_{ij} y^j, \quad D_j = \frac{1}{m} A^{\frac{2}{m}-1} A_j + b_{ij} y^i$$

and

$$g_{ij} = \frac{A^{\frac{2}{m}-2}}{m^2} \left[mAA_{ij} + (2-m) A_i A_j \right].$$

Proposition 3.1. *The covariant metric tensor $\bar{g}_{ij}$ of Randers change of generalized mth root metric is given as:*

$$\bar{g}_{ij} = \hat{g_{ij}} \left(1 + \frac{C}{\sqrt{A^{\frac{2}{m}} + B}} \right) + C_i C_j + \frac{C_i D_j + C_j D_i}{\sqrt{A^{\frac{2}{m}} + B}} - C \frac{D_i D_j}{(A^{\frac{2}{m}} + B)^{3/2}}.$$

4. Spray coefficients of Randers change of generalized mth root metric

The geodesics of a Finsler space F^n are given by the following system of equations:

$$\frac{d^2 x^i}{dt^2} + G^i \left(x, \frac{dx}{dt} \right) = 0, \quad i = 1, 2, \ldots, n,$$

where

$$G^i = \frac{1}{4} g^{il} \left\{ \left[F^2 \right]_{x^k y^l} y^k - \left[F^2 \right]_{x^l} \right\} \tag{4.1}$$

are called the spray coefficients of F^n.
Two Finsler metrics F and $\bar{F}$ on a manifold M^n are called projectively related if there is a scalar function $P(x, y)$ defined on TM_0 such that $\bar{G}^i = G^i + Py^i$, where $\bar{G}^i$ and G^i are the geodesic spray coefficients of $\bar{F}^n$ and F^n, respectively.

In other words, two metrics F and $\bar{F}$ are called projectively related if any geodesic of the first is also geodesic for the second and vice versa.
The spray coefficients of Randers change of generalized mth root Finsler space $\bar{F}^n$ is given by

$$\bar{G}^i = \frac{1}{4} \bar{g}^{il} \left\{ \left[\bar{F}^2 \right]_{x^k y^l} y^k - \left[\bar{F}^2 \right]_{x^l} \right\}. \tag{4.2}$$

Lemma 4.1. [10]. *Let $A = [A_{ij}]$ be an $n \times n$ invertible and symmetric matrix, and let $D = [D_i]$ and $E = [E_i]$ be two nonzero $n \times 1$ and $1 \times n$ vectors, respectively, such that $D_i E_j = D_j E_i$. Suppose that $1 + A^{pq} D_p E_q \neq 0$. Then, the matrix $B = [B_{ij}]$ defined by $B_{ij} := A_{ij} + D_i E_j$ is invertible and*

$$B^{ij} := (B_{ij})^{-1} = A^{ij} - \frac{1}{1 + A^{pq} D_p E_q} A^{ki} A^{lj} D_k E_l,$$

where matrix (A^{ij}) denotes the inverse of matrix (A_{ij}).

Theorem 4.2. *Let $\bar{F} = \sqrt{A^{\frac{2}{m}} + B} + C$ and $F = A^{\frac{1}{m}}$ be Randers change of generalized mth root and an mth root Finsler metric on an open subset $U \subset \mathbb{R}^n$,* *respectively, where* $A := a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$*,* $B = b_{ij}(x) y^i y^j$*, and* $C = c_i(x) y^i$ *with* $m \neq 1$*. Suppose that the following holds:*

$$\frac{1}{(m-1)} A^{\frac{m-2}{m}} A^{il}(\zeta_l + 2J_l) + \left[-Lb^{il} - KC^iC^l + I^{il}\right](\zeta_l + 2J_l)$$
$$-4\gamma\left(1 + \frac{Lb^{il} - I^{il}}{KC^iC^l}\right)\mathbf{A^i} = 0, \tag{4.3}$$

where

$$\zeta_l = B_{0l} - B_{x^l} + 2C_lC_0 + 2CC_{0l} - 2CC_{x^l},$$
$$J_l = \Big[\frac{\partial^2(C\sqrt{A^{\frac{2}{m}} + B})}{\partial x^k \partial y^l} y^k - \frac{\partial(C\sqrt{A^{\frac{2}{m}} + B})}{\partial x^l}\Big],$$
$$\gamma = \frac{KC^l}{4}\left\{\left[F^2\right]_{x^k y^j} y^k - \left[F^2\right]_{x^j}\right\},$$
$$\mathbf{A^i} = \frac{1}{(m-1)} A^{\frac{m-2}{m}} A^{ip} C_p,$$

$C^l = g^{lk}C_k$ *and* K*,* L *are constants. Then* $\bar{F}$ *is projectively related to* F*.*

Proof: By equation (2.2), we have

$$\bar{F}^2 = F^2 + B + C^2 + 2C\sqrt{A^{\frac{2}{m}} + B}. \tag{4.4}$$

Then using Proposition (3.1), we obtain

$$\bar{g}_{ij} = \hat{g_{ij}}\left(1 + \frac{C}{H}\right) + C_iC_j + \frac{C_iD_j + C_jD_i}{H} - C\frac{D_iD_j}{H^3}, \tag{4.5}$$

where

$$\hat{g_{ij}} = g_{ij} + b_{ij},$$
$$g_{ij} = A^{(\frac{2}{m}-2)}\left[(m-1)AA_{ij} + (2-m)A_iA_j\right], \tag{4.6}$$

and $H = \sqrt{A^{\frac{2}{m}} + B}$.
From Lemma 4.1, we get

$$\bar{g}^{ij} = \hat{g^{ij}} - KC^iC^j + I^{ij}, \tag{4.7}$$

where K is a constant, I^{ij} will be the inverse of the last two term in equation (4.5), which easily may not be calculate explicitly,
and

$$\hat{g^{ij}} = g^{ij} - Lb^{ij} = A^{-\frac{2}{m}}\left[\frac{1}{(m-1)}AA^{ij} + \frac{(m-2)}{(m-1)}y^iy^j\right] - Lb^{ij}, \tag{4.8}$$

where L is a constant and g^{ij} is the inverse of g_{ij}.
Then by equations (4.2), (4.4), and (4.7), we have

$$\bar{G}^i = \frac{1}{4}\left[\hat{g^{il}} - KC^iC^l + I^{il}\right]$$
$$\times\Big[\frac{\partial^2(F^2 + B + C^2 + 2C\sqrt{A^{\frac{2}{m}} + B})}{\partial x^k \partial y^l} y^k - \frac{\partial(F^2 + B + C^2 + 2C\sqrt{A^{\frac{2}{m}} + B})}{\partial x^l}\Big].$$

Then from equation (4.8), we can write

$$\begin{aligned}\bar{G^i} &= \frac{1}{4}\left[g^{il}-Lb^{il}-KC^iC^l+I^{il}\right]\Big[\frac{\partial^2(F^2)}{\partial x^k\partial y^l}y^k-\frac{\partial(F^2)}{\partial x^l}\Big]\\ &+\frac{1}{4}\left[g^{il}-Lb^{il}-KC^iC^l+I^{il}\right]\Big[\frac{\partial^2(B)}{\partial x^k\partial y^l}y^k-\frac{\partial(B)}{\partial x^l}\Big]\\ &+\frac{1}{4}\left[g^{il}-Lb^{il}-KC^iC^l+I^{il}\right]\Big[\frac{\partial^2(C^2)}{\partial x^k\partial y^l}y^k-\frac{\partial(C^2)}{\partial x^l}\Big]\\ &+\frac{1}{2}\left[g^{il}-Lb^{il}-KC^iC^l+I^{il}\right]\Big[\frac{\partial^2(C\sqrt{A^{\frac{2}{m}}+B})}{\partial x^k\partial y^l}y^k-\frac{\partial(C\sqrt{A^{\frac{2}{m}}+B})}{\partial x^l}\Big].\end{aligned}$$

That is,

$$\begin{aligned}\bar{G^i} &= \frac{1}{4}\left[g^{il}\right]\Big[\frac{\partial^2(F^2)}{\partial x^k\partial y^l}y^k-\frac{\partial(F^2)}{\partial x^l}\Big]-\frac{(KC^iC^l+Lb^{il}-I^{il})}{4}\Big[\frac{\partial^2(F^2)}{\partial x^k\partial y^l}y^k-\frac{\partial(F^2)}{\partial x^l}\Big]\\ &+\frac{1}{4}\left[g^{il}-Lb^{il}-KC^iC^l+I^{il}\right][B_{0l}-B_{x^l}+2C_lC_0+2CC_{0l}-2CC_{x^l}]\\ &+\frac{1}{2}\left[g^{il}-Lb^{il}-KC^iC^l+I^{il}\right]J_l,\end{aligned}$$

where

$$J_l=\Big[\frac{\partial^2(C\sqrt{A^{\frac{2}{m}}+B})}{\partial x^k\partial y^l}y^k-\frac{\partial(C\sqrt{A^{\frac{2}{m}}+B})}{\partial x^l}\Big].$$

Therefore,

$$\begin{aligned}\bar{G^i} &= G^i-\gamma\left[1+\frac{(Lb^{il}-I^{il})}{KC^iC^l}\right]C^i+\frac{1}{4}\left[g^{il}-Lb^{il}-KC^iC^l+I^{il}\right]\zeta_l\\ &+\frac{1}{2}\left[g^{il}-Lb^{il}-KC^iC^l+I^{il}\right]J_l,\end{aligned}\tag{4.9}$$

where

$$\begin{aligned}&\gamma=\frac{KC^l}{4}\left\{\left[F^2\right]_{x^ky^l}y^k-\left[F^2\right]_{x^l}\right\},\\ &\zeta_l=B_{0l}-B_{x^l}+2C_lC_0+2CC_{0l}-2CC_{x^l}.\end{aligned}\tag{4.10}$$

Putting

$$\Phi:=\frac{(m-2)}{(m-1)}A^{-\frac{2}{m}}y^pC_p,\quad \mathbf{A^i}:=\frac{1}{(m-1)}A^{\frac{m-2}{m}}A^{ip}C_p,\tag{4.11}$$

we have

$$C^i=g^{ip}C_p=A^{-\frac{2}{m}}\left[\frac{1}{(m-1)}AA^{ip}+\frac{(m-2)}{(m-1)}y^iy^p\right]C_p=\mathbf{A^i}+\Phi y^i.\tag{4.12}$$

By (4.8), (4.9), and (4.12), we get

$$\begin{aligned}\bar{G^i} &= G^i+\Big[A^{-\frac{2}{m}}\frac{(m-2)}{4(m-1)}y^l(\zeta_l+2J_l)-\gamma\left(1+\frac{Lb^{il}-I^{il}}{KC^iC^l}\right)\Phi\Big]y^i\\ &-\gamma\left(1+\frac{Lb^{il}-I^{il}}{KC^iC^l}\right)\mathbf{A^i}+\frac{1}{4(m-1)}A^{\frac{m-2}{m}}A^{il}(\zeta_l+2J_l)\\ &+\frac{1}{4}\left[-Lb^{il}-KC^iC^l+I^{il}\right](\zeta_l+2J_l).\end{aligned}\tag{4.13}$$

If the relation (4.3) holds, then by (4.13) the Finsler metric $\bar{F}$ is projectively related to F.

5. Locally projectively flatness of Randers change of generalized mth root metric

A Finsler metric is called locally projectively flat if, for any point, there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric $F(x,y)$ on an open domain $U \subset \mathbb{R}^{\ltimes}$ is locally projectively flat if and only if its geodesic coefficients G^i take the form $G^i(x,y) = P(x,y)y^i$, where $P : TU = U \times \mathbb{R}^{\ltimes} \to \mathbb{R}$ is positively homogeneous with degree one and $P(x, \lambda y) = \lambda P(x,y), \lambda > 0$ [3]. We say that $P(x,y)$ is the projective factor of F.

In other words, a Finsler metric $\bar{F} = \bar{F}(x,y)$ on a manifold M^n is said to be locally projectively flat, if and only if

$$(\bar{F})_{x^k y^l} y^k = (\bar{F})_{x^l}.$$

Using equation (2.2), we have

$$[\bar{F}]_{x^l} = \frac{\frac{2}{m} A^{\frac{2-m}{m}} A_{x^l} + B_{x^l}}{2\sqrt{A^{\frac{2}{m}} + B}} + C_{x^l}. \tag{5.1}$$

From (5.1), we get

$$[\bar{F}]_{x^k} = \frac{\frac{2}{m} A^{\frac{2-m}{m}} A_{x^k} + B_{x^k}}{2\sqrt{A^{\frac{2}{m}} + B}} + C_{x^k}$$

and

$$\begin{aligned} [\bar{F}]_{x^k y^l} \quad = \quad & \frac{(2\sqrt{A^{\frac{2}{m}} + B})\left(\frac{2}{m}(\frac{2-m}{m}) A^{\frac{2-2m}{m}} A_{y^l} A_{x^k} + \frac{2}{m} A^{\frac{2-m}{m}} A_{x^k y^l} + B_{x^k y^l}\right)}{4(A^{\frac{2}{m}} + B)} \\ & - \frac{1}{4(A^{\frac{2}{m}} + B)^{\frac{3}{2}}} \left(\frac{2}{m} A^{\frac{2-m}{m}} A_{x^k} + B_{x^k}\right)\left(\frac{2}{m} A^{\frac{2-m}{m}} A_{y^l} + B_{y^l}\right) + C_{x^k y^l}. \end{aligned} \tag{5.2}$$

Let the Finsler metric $\bar{F}$ be locally projectively flat. Then we have

$$[\bar{F}]_{x^k y^l} y^k - [\bar{F}]_{x^l} = 0. \tag{5.3}$$

Therefore, from (5.1), (5.2), and (5.3), we obtain

$$\begin{aligned} [\bar{F}]_{x^k y^l} y^k - [\bar{F}]_{x^l} \quad = \quad & \frac{(2\sqrt{A^{\frac{2}{m}} + B})\left(\frac{2}{m}(\frac{2-m}{m}) A^{\frac{2-2m}{m}} A_l A_0 + \frac{2}{m} A^{\frac{2-m}{m}} A_{0l} + B_{0l}\right)}{4(A^{\frac{2}{m}} + B)} \\ & - \frac{1}{4(A^{\frac{2}{m}} + B)^{\frac{3}{2}}} \left(\frac{2}{m} A^{\frac{2-m}{m}} A_0 + B_0\right)\left(\frac{2}{m} A^{\frac{2-m}{m}} A_l + B_l\right) + C_{0l} \\ & - \frac{\frac{2}{m} A^{\frac{2-m}{m}} A_{x^l} + B_{x^l}}{2\sqrt{A^{\frac{2}{m}} + B}} - C_{x^l} = 0. \end{aligned}$$

Hence, $\bar{F}$ is locally projectively flat metric if and only if

$$A_{x^l} = \frac{(2-m)}{mA} A_l A_0 + A_{0l} + \frac{m}{2} A^{\frac{m-2}{m}} B_{0l} - \frac{1}{2(A^{\frac{2}{m}} + B)} \left(A_0 + \frac{m}{2} A^{\frac{m-2}{m}} B_0\right) \times \left(\frac{2}{m} A^{\frac{2-m}{m}} A_l + B_l\right) + (C_{0l} - C_{x^l})\, mA^{\frac{m-2}{m}} \sqrt{A^{\frac{2}{m}} + B} - \frac{m}{2} B_{x^l} A^{\frac{2-m}{m}}. \tag{5.4}$$

Thus, we have the following result.

Theorem 5.1. *Let $\bar{F} = \sqrt{A^{\frac{2}{m}} + B} + C$ be a Randers change of generalized mth root Finsler metric on a manifold M^n. Then, $\bar{F}$ is a locally projectively flat metric if and only if equation (5.4) satisfied.*

6. Locally dually flatness of Randers change of generalized mth root metric

In Finsler geometry, Shen extended the notion of locally dually flatness for Finsler metrics. A Finsler metric $\bar{F} = \bar{F}(x, y)$ on a manifold M^n is said to be locally dually flat, if at any point there is a standard coordinate system (x^i, y^i) in TM such that $\left[\bar{F}^2\right]_{x^k y^l} y^k = 2\left[\bar{F}^2\right]_{x^l}$.
Every locally Minkowskian metric is locally dually flat.
Using equation (2.2), we have

$$\left[\bar{F}^2\right]_{x^l} = \frac{2}{m} A^{\frac{2-m}{m}} A_{x^l} + B_{x^l} + 2CC_{x^l} + \frac{\frac{2}{m} CA^{\frac{2-m}{m}} A_{x^l} + CB_{x^l}}{\sqrt{A^{\frac{2}{m}} + B}} + 2C_{x^l}\sqrt{A^{\frac{2}{m}} + B}. \tag{6.1}$$

From (6.1), we get

$$\left[\bar{F}^2\right]_{x^k} = \frac{2}{m} A^{\frac{2-m}{m}} A_{x^k} + B_{x^k} + 2CC_{x^k} + \frac{\frac{2}{m} CA^{\frac{2-m}{m}} A_{x^k} + CB_{x^k}}{\sqrt{A^{\frac{2}{m}} + B}} + 2C_{x^k}\sqrt{A^{\frac{2}{m}} + B}$$

and

$$\begin{aligned}\left[\bar{F}^2\right]_{x^k y^l} =& \frac{2}{m} A^{\frac{2-m}{m}} A_{x^k y^l} + x\frac{2}{m}\frac{(2-m)}{m} A^{\frac{2-2m}{m}} A_l A_{x^k} \\ &+ B_{x^k y^l} + 2CC_{k y^l} + 2C_l C_{x^k} \\ &+ \frac{1}{(A^{\frac{2}{m}} + B)} \Bigg[\sqrt{A^{\frac{2}{m}} + B}\,\Bigg\{\frac{2}{m}\frac{(2-m)}{m} CA^{\frac{2-2m}{m}} A_l A_{x^k} + \frac{2}{m} CA^{\frac{2-m}{m}} A_{x^k y^l} \\ &\qquad + \frac{2}{m} A^{\frac{2-m}{m}} A_{x^k} C_l + C_l B_{x^k} + CB_{x^k y^l}\Bigg\} \\ &\qquad - \left(\frac{2}{m} CA^{\frac{2-m}{m}} A_{x^k} + CB_{x^k}\right) \frac{1}{2\sqrt{A^{\frac{2}{m}} + B}} \left(\frac{2}{m} A^{\frac{2-m}{m}} A_l + B_l\right)\Bigg] \\ &+ 2C_{x^k y^l}\sqrt{A^{\frac{2}{m}} + B} + C_{x^k} \frac{\frac{2}{m} A^{\frac{2-m}{m}} A_l + B_l}{\sqrt{A^{\frac{2}{m}} + B}}.\end{aligned} \tag{6.2}$$

Let the Finsler metric $\bar{F}$ be locally dually flat. Then we have

$$\left[\bar{F}^2\right]_{x^k y^l} y^k - 2\left[\bar{F}^2\right]_{x^l} = 0. \tag{6.3}$$

Therefore, from (6.1), (6.2), and (6.3), we obtain

$$\begin{aligned}
\left[\bar{F}^2\right]_{x^k y^l} y^k - 2\left[\bar{F}^2\right]_{x^l} =& \frac{2}{m} A^{\frac{2-m}{m}} A_{0l} + \frac{2}{m}\frac{(2-m)}{m} A^{\frac{2-2m}{m}} A_l A_0 \\
&+ B_{0l} + 2CC_{0l} + 2C_l C_0 \\
&+ (A^{\frac{2}{m}} + B)^{-\frac{1}{2}} \Big\{ \frac{2}{m}\frac{(2-m)}{m} C A^{\frac{2-2m}{m}} A_l A_0 + \frac{2}{m} C A^{\frac{2-m}{m}} A_{0l} \\
&\qquad + \frac{2}{m} A^{\frac{2-m}{m}} A_0 C_l + C_l B_0 + C B_{0l} \Big\} \\
&- \left(\frac{2}{m} C A^{\frac{2-m}{m}} A_0 + C B_0\right) \frac{(A^{\frac{2}{m}} + B)^{-\frac{3}{2}}}{2} \left(\frac{2}{m} A^{\frac{2-m}{m}} A_l + B_l\right) \\
&+ 2C_{0l}\sqrt{A^{\frac{2}{m}} + B} + C_0 \frac{\frac{2}{m} A^{\frac{2-m}{m}} A_l + B_l}{\sqrt{A^{\frac{2}{m}} + B}} \\
&- \frac{4}{m} A^{\frac{2-m}{m}} A_{x^l} - 2B_{x^l} - 4CC_{x^l} \\
&- 2\left(\frac{\frac{2}{m} C A^{\frac{2-m}{m}} A_{x^l} + C B_{x^l}}{\sqrt{A^{\frac{2}{m}} + B}}\right) - 4C_{x^l}\sqrt{A^{\frac{2}{m}} + B} \\
=& 0.
\end{aligned}$$

Therefore, $\bar{F}$ is locally dually flat metric if and only if

$$\begin{aligned}
A_{x^l} =& \frac{1}{2\bar{F}} A_{0l}\sqrt{A^{\frac{2}{m}} + B} + \frac{1}{2m\bar{F}A}(2-m) A_l A_0 \sqrt{A^{\frac{2}{m}} + B} \\
&+ \frac{m}{4\bar{F}}\left(B_{0l} + 2CC_{0l} + 2C_l C_0\right) A^{\frac{m-2}{m}} \sqrt{A^{\frac{2}{m}} + B} + \frac{1}{2mA\bar{F}}(2-m) C A_l A_0 \\
&+ \frac{1}{2\bar{F}} C A_{0l} + \frac{1}{2\bar{F}} A_0 C_l + \frac{m}{4\bar{F}}\left(C_l B_0 + C B_{0l}\right) A^{\frac{m-2}{m}} \\
&- \left(\frac{2}{m} C A_0 + C B_0 A^{\frac{m-2}{m}}\right) \frac{m}{8\bar{F}(A^{\frac{2}{m}} + B)} \left(\frac{2}{m} A^{\frac{2-m}{m}} A_l + B_l\right) \\
&+ \frac{m}{2\bar{F}} C_{0l} A^{\frac{m-2}{m}} \left(A^{\frac{2}{m}} + B\right) + \frac{mC_0}{4\bar{F}}\left(\frac{2}{m} A_l + B_l A^{\frac{m-2}{m}}\right) \\
&- \frac{m}{2\bar{F}}\left(B_{x^l} + 2CC_{x^l}\right) A^{\frac{m-2}{m}} \sqrt{A^{\frac{2}{m}} + B} - \frac{mC}{2\bar{F}} B_{x^l} A^{\frac{m-2}{m}} - \frac{m}{\bar{F}} C_{x^l} A^{\frac{m-2}{m}} \left(A^{\frac{2}{m}} + B\right).
\end{aligned} \tag{6.4}$$

Thus, we have the following results

Theorem 6.1. *Let $\bar{F} = \sqrt{A^{\frac{2}{m}} + B} + C$ be a Randers change of generalized mth root Finsler metric on a manifold M^n. Then, $\bar{F}$ is a locally dually flat metric if and only if equation (6.4) satisfied.*

Corollary 6.2. [9]. *Let $F = A^{\frac{1}{m}}$ be an mth root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Then F is a locally dually flat metric if and only if the following holds*

$$A_{x^l} = \frac{1}{2A}\left[\frac{(2-m)}{m} A_0 A_{y^l} + A A_{0l}\right]. \tag{6.5}$$

Putting $C = 0$ in equation (2.2) and equation (6.4), we get the following corollary.

Corollary 6.3. [8]. *Let $F = A^{\frac{1}{m}}$ be an mth root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Then $\tilde{F} = \sqrt{F^2 + B}$ is a locally dually flat metric if and only if the following holds*

$$A_{x^l} = \left(A_{0l} + \frac{1}{mA}(2-m)A_l A_0 + \frac{m}{2} B_{0l} A^{\frac{m-2}{m}} - m B_{x^l} A^{\frac{m-2}{m}} \right) \frac{\sqrt{A^{\frac{2}{m}} + B}}{2\bar{F}}.$$

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