ON THE STABILITY OF THE QUASI-LINEAR IMPLICIT EQUATIONS IN HILBERT SPACES

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Communicated by J. Brzdęk

ABSTRACT. We use the generalized theorem of Liapounov to obtain some necessary and sufficient conditions for the stability of the stationary implicit equation

\[ Ax'(t) = Bx(t), \quad t \geq 0, \]

where \( A \) and \( B \) are bounded operators in Hilbert spaces. The achieved results can be applied to the stability for the quasi-linear implicit equation

\[ Ax'(t) = Bx(t) + \theta(t, x(t)), \quad t \geq 0. \]

1. Introduction

Consider the abstract implicit differential equation

\[ Ax'(t) = Bx(t) + \theta(t, x(t)), \quad t \geq 0, \quad (1.1) \]

where \( A \) and \( B \) are two linear bounded operators on a Hilbert space \( \mathcal{H} \) and \( \theta(\cdot, \cdot) \) is a continuous function from \([0, \infty) \times \mathcal{H}\) to \( \mathcal{H} \). The operator \( A \) is not necessarily invertible.

The equation (1.1) has been considered in various forms by many authors as Favini and Yagi [5], Rutkas [6], Vlasenko [7] and others.

In the present paper, we study the stationary implicit equation

\[ Ax'(t) = Bx(t), \quad t \geq 0, \quad (1.2) \]

and also the quasi-linear implicit equation (1.1), with the initial condition

\[ x(0) = x_0. \]

\textit{Date:} Received: 01 May 2018; Revised: 10 October 2018; Accepted: 29 November 2018.

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2010 \textit{Mathematics Subject Classification.} Primary 34D20; Secondary 47A10, 34K32.

\textit{Key words and phrases.} Exponential stability, operator theory, implicit equations.
In [2] the authors obtained results concerning the stability of the degenerate difference systems that is similar to (1.1).

Some practical examples of (1.1) can be found in [5, 6, 7]. The organization of this paper is as follows: in Section 2, we introduce some preliminaries and expand the famous Liapounov general theorem [4], which has an important role in this paper. In section 3, we present our main results concerning the exponential stability of the solution for the quasi-linear implicit equation (1.1).

We use the following definitions.

**Definition 1.1.** The equation (1.2) is called exponentially stable, if there exist two constants $M > 0$ and $\alpha < 0$ such that, for any solution $x(t)$, we have

$$||x(t)|| \leq Me^{\alpha t}||x_0|| \text{ for any } t \geq 0.$$  \hfill (1.3)

**Definition 1.2.** The equation (1.2) is said to be well-posed, if it satisfies the following properties:

(i) for any solution $x(.)$ such that $x(0) = x_0 = 0$, then $x(t) = 0$ for all $t \geq 0$;

(ii) it generates an evolution semigroup of bounded operators $S(t) : x_0 \mapsto x(t)$ for all $t \geq 0$.

The operators $S(t)$ are defined on the set $D_0 = \{x_0\}$ of the admissible initial vectors $x_0$.

**Definition 1.3** (see [6]). The complex number $\lambda \in \mathbb{C}$ is called a regular value of the pencil $\lambda A - B$, if the resolvent $(\lambda A - B)^{-1}$ exists and is bounded. The set of all regular values is denoted by $\rho(A, B)$ and its complement $\sigma(A, B) = \mathbb{C} \setminus \rho(A, B)$ is called the spectrum of the pencil $\lambda A - B$. The set of all eigenvalues of the pencil $\lambda A - B$ is denoted by

$$\sigma_p(A, B) = \{\lambda \in \mathbb{C} : \exists v \neq 0; (\lambda A - B)v = 0\}.$$

2. **Stationary implicit equation**

For the stationary implicit equation (1.2), we can obtain the following criterion for the exponential stability.

**Theorem 2.1.** The equation (1.2) is exponentially stable if and only if it is well-posed.

**Proof.** Suppose that (1.2) is exponentially stable. Then, it has a unique solution $x(t)$. In fact, if $x_0 = 0$, then by (1.3), we obtain $||x(t)|| \leq 0$, and consequently $x(t) = 0$ for all $t \geq 0$. On the other hand, we have

$$||x(t)|| = ||S(t)x_0|| \leq Me^{\alpha t}||x_0||.$$

It means that the operator $S(t)$ is bounded and $||S(t)|| \leq Me^{\alpha t}$. So, (1.2) is well-posed. Conversely, if (1.2) is well-posed, then one obtains

$$\omega = \lim_{t \to \infty} \frac{\ln ||S(t)||}{t} < \infty$$

(see [4, p. 26]), where $\omega$ is the strict Liapounov exponent of $\Phi(t) = ||S(t)||$. More precisely, $\omega$ is the greatest lower bound of the set of real numbers $\rho$ for which
there exists a positive constant $N_\varrho$ (see [3, pp. 8–9]) such that
\[
\Phi(t) = \|S(t)\| \leq N_\varrho e^{\varrho t} \quad \text{for all } t \geq 0, \omega \leq \varrho < 0.
\]
Hence
\[
\|x(t)\| = \|S(t)x_0\| \leq N_\varrho e^{\varrho t}\|x_0\|,
\]
which achieves the proof.

**Theorem 2.2.** If (1.2) is exponentially stable, then all eigenvalues of the pencil $\lambda A - B$ are in the half-plane $\text{Re}\lambda \leq \alpha$, where $\alpha$ is the constant defined in (1.3).

**Proof.** Suppose that there exists an eigenvalue $\lambda_0 \in \sigma_p(A, B)$ such that $\text{Re}\lambda_0 > \alpha$. Then $(\lambda_0 A - B)v = 0$, where $v$ is the corresponding eigenvector. Consequently, $y(t) = e^{\lambda_0 t}v$ is a solution of (1.2) verifying the condition $y(0) = v$, and we have
\[
\|y(t)\| = \|e^{\lambda_0 t}v\| = e^{(\text{Re}\lambda_0)t}\|v\| > e^{\alpha t}\|y(0)\|.
\]
So, the solution $y(t)$ does not satisfy (1.3) and consequently (1.2) is not exponentially stable.

**Remark 2.3.** If (1.2) is exponentially stable, then all the eigenvalues of the pencil $\lambda A - B$ are inside the left half-plane, that is
\[
\sigma_p(A, B) \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda < 0\},
\]
since $\alpha < 0$, where $\alpha$ is given by (1.3).

We can now extend the generalized Liapounov theorem [4] for the spectrum of the bounded operator $T$, to the spectrum of the pencil $\lambda A - B$ of the bounded operators $A$ and $B$ on a Hilbert space $\mathcal{H}$, using the spectral theory of the pencil of operators and an appropriate conformal mapping as follows.

**Theorem 2.4.** A necessary condition, for the spectrum $\sigma(A, B)$ of the pencil $\lambda A - B$ to lie in the interior of the half-plane $\text{Re}\lambda < \alpha$ ($\alpha < 0$), is that, for any uniformly positive operator $U \gg 0$ \footnote{It means that $U^* = U$ and that $\langle Ux, x \rangle > 0$ for all $x$ with $\|x\| = 1$.}, there exists an operator $W \gg 0$ such that
\[
A^*WB + B^*WA - 2\alpha A^*WA = -2U, \quad (2.1)
\]
and a sufficient condition is that $\alpha + 1$ is a regular value of the pencil $\lambda A - B$ and there exists an operator $W \gg 0$ such that
\[
A^*WB + B^*WA - 2\alpha A^*WA \ll 0. \quad (2.2)
\]

**Proof.** Necessary condition. Suppose that $\sigma(A, B) \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda < \alpha\}$. Then, $(\alpha + 1) \in \rho(A, B)$ and the operator $T = [(\alpha - 1)A - B][(\alpha + 1)A - B]^{-1}$ is well defined and bounded. Now, using the conformal mapping $z = \varphi(\lambda) = \frac{\lambda - \alpha + 1}{\lambda - \alpha - 1}$ which transforms the vertical line $\text{Re}\lambda = \alpha$ into the unit circle $|z| = 1$, we obtain
\[ zI - T = \left( \frac{\lambda - \alpha + 1}{\lambda - \alpha - 1} \right) [(\alpha + 1)A - B][((\alpha + 1)A - B)^{-1} - [(\alpha - 1)A - B][(\alpha + 1)A - B]^{-1} \]

\[ = \frac{1}{(\lambda - \alpha - 1)} \left\{ (\lambda - \alpha + 1)((\alpha + 1)A - B) \right. \]

\[ - (\lambda - \alpha - 1)[(\alpha - 1)A - B][(\alpha + 1)A - B]^{-1} \]

\[ = \frac{2}{(\lambda - \alpha - 1)}(\lambda A - B)[(\alpha + 1)A - B]^{-1}. \]

So, the operator \( zI - T \) is invertible if and only if the pencil \( \lambda A - B \) is also invertible. Therefore, \( \rho(T) = \rho(I, T) = \varphi(\rho(A, B)) \).

Passing to the complement, we conclude that \( \sigma(T) = \sigma(I, T) = \varphi(\sigma(A, B)) \). Consequently \( \sigma(T) \) is in the unit disk. Using [2, Theorem 2], we conclude that there exists an operator \( W \gg 0 \) such that

\[ T^*WT - W = -G \quad \text{for all} \ G \gg 0, \quad (2.3) \]

which is equivalent to

\[ \{[(\alpha - 1)A - B][(\alpha + 1)A - B]^{-1}\}^*W \]

\[ \{[(\alpha - 1)A - B][(\alpha + 1)A - B]^{-1}\} - W = -G \]

\[ \iff [(\alpha + 1)A^* - B^*]^{-1}[(\alpha - 1)A^* - B^*]W \]

\[ [(\alpha - 1)A - B][(\alpha + 1)A - B]^{-1} - W = -G \]


\[ - [(\alpha + 1)A^* - B^*]W[(\alpha + 1)A - B] \]

\[ = -[(\alpha + 1)A^* - B^*]G[(\alpha + 1)A - B] \]

\[ \iff (2A^*WB + 2B^*WA - 4\alpha A^*WA) \]

\[ = -[(\alpha + 1)A^* - B^*]G[(\alpha + 1)A - B] \]

\[ \iff A^*WB + B^*WA - 2\alpha A^*WA \]

\[ = -\frac{1}{2}[(\alpha + 1)A^* - B^*]G[(\alpha + 1)A - B] \]

\[ \iff A^*WB + B^*WA - 2\alpha A^*WA = -2U, \]
where
\[ U = \frac{1}{4}[(\alpha + 1)A^* - B^*]G[(\alpha + 1)A - B] \gg 0. \]
In fact,
\[ U^* = \frac{1}{4}[(\alpha + 1)A^* - B^*]G[(\alpha + 1)A - B] = U, \]
and for each \( x \in H \), we have
\[ \langle Ux, x \rangle = \frac{1}{4} \langle [(\alpha + 1)A - B]^{-1}y, y \rangle \]
where \( k \) is a positive constant. But,
\[ ||x||^2 = ||[(\alpha + 1)A - B]^{-1}y||^2 \]
\[ \leq ||[(\alpha + 1)A - B]^{-1}||^2 ||y||^2. \]
Therefore
\[ ||y||^2 \geq \frac{||x||^2}{||[(\alpha + 1)A - B]^{-1}||^2} \cdot \]
Thus
\[ \langle Ux, x \rangle \geq \frac{1}{4} \frac{k}{||[(\alpha + 1)A - B]^{-1}||^2} ||x||^2 > 0. \]
Consequently \( U \gg 0 \), and (2.2) holds.

Sufficient condition. If \( \alpha + 1 \in \rho(A,B) \) is a regular value for the pencil \( \lambda A - B \), then the operator \( T = [(\alpha - 1)A - B][(\alpha + 1)A - B]^{-1} \) is bounded and (2.2) becomes
\[ A^*WB + B^*WA - 2\alpha A^*WA = -\frac{1}{2}[(\alpha + 1)A^* - B^*]G[(\alpha + 1)A - B] \ll 0. \]
Therefore, \( G = W - T^*WT \gg 0 \) (see (2.3)). Using again [2, Theorem 2], the spectrum \( \sigma(T) \) will be inside the unit disk. We conclude that \( \sigma(A,B) = \varphi^{-1}(\sigma(T)) \subset \{ \lambda : \text{Re}\lambda < \alpha \} \), where \( \lambda = \varphi^{-1}(z) = \alpha + \frac{z + \frac{1}{z}}{\alpha - 1} \) is a conformal mapping and Theorem 2.4 is proved.

**Theorem 2.5.** If (2.1) is satisfied for the pair of the positive uniform operators \( (W,U) \), then \( \lambda = \alpha + 1 \) is not an eigenvalue for the pencil \( \lambda A - B \).

**Proof.** Suppose that \( \lambda = \alpha + 1 \) is an eigenvalue. We denote by \( v \neq 0 \) the corresponding eigenvector. Then, \([(\alpha + 1)A - B]v = 0 \) or \( (\alpha + 1)Av = Bv \), and in the two cases the scalar product becomes
\[ \langle Uv, v \rangle = -\frac{1}{2}\langle (A^*WB + B^*WA - 2\alpha A^*WA)v, v \rangle \]
\[ = -\frac{1}{2}\langle A^*WBv, v \rangle - \frac{1}{2}\langle B^*WAv, v \rangle + \alpha \langle A^*WAv, v \rangle \]
\[ = -\frac{1}{2}\langle WBv, Av \rangle - \frac{1}{2}\langle WAv, Bv \rangle + \alpha \langle WAv, Av \rangle \]
\[ = -\langle WAv, Av \rangle < 0. \]
We obtain a contradiction, with the hypothesis $U \gg 0$, since $W \gg 0$. Consequently Theorem 2.5 is proved. □

**Corollary 2.6.** In the case of a finite-dimensional space $\mathcal{H}$, the following statements are equivalent:

(a) The equation (1.2) is exponentially stable;
(b) $\sigma(A,B) = \sigma_p(A,B) \subset \{\lambda : \text{Re}\lambda < \alpha\}$;
(c) There exists a positive definite matrix $W \gg 0$ such that

$$A^*WB + B^*WA - 2\alpha A^*WA \ll 0.$$  

3. **Quasi-linear implicit equation**

In this section, we give some stability conditions of the quasi-linear implicit equation of the form (1.1), using the following variation of constants method (Lemma 3.1) and the Gronwall–Bellman inequality (Lemma 3.2).

Remember that $D_0 = \{x(0)\}$ denotes the initial manifold subspace of $\mathcal{H}$ for the stationary equation (1.2).

**Lemma 3.1.** Suppose that

(i) the restriction operator $A_0 = A|_{D_0}$ on $D_0$ is invertible;
(ii) for any $\tau \geq 0$, the space $\theta(t,x(\tau))$ is in the domain of $A_0$ and the function $S(t-\tau)A_0^{-1}\theta(t,x(\tau))$ is integrable (with respect to $\tau$), where $\{S(t)\}_{t \geq 0}$ is the semigroup of the operators for (1.2).

Then the quasi-linear equation (1.1) is equivalent to the integral equation

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)A_0^{-1}\theta(\tau,x(\tau))d\tau.$$  

(3.1)

**Lemma 3.2** (Gronwall–Bellman). (see [1]). If

$$g(t) \leq c + \int_0^t g(\tau)h(\tau)d\tau \quad \text{for all } t \geq 0,$$

where $h$ is a continuous positive real function and $c > 0$ is an arbitrary constant, then

$$g(t) \leq c \exp\left[\int_0^t h(\tau)d\tau\right].$$

For the quasi-linear equation (1.1), we have the next result.

**Theorem 3.3.** Suppose that

(i) the equation (1.2) is well-posed;
(ii) the quasi-linear operator $\theta(t,x(t))$, for all $t \geq 0$, transforms $D_0$ into $AD_0$ such that

$$\int_0^\infty ||A_0^{-1}\theta(t,x(t))||dt < \infty.$$  

Then the quasi-linear equation (1.1) is exponentially stable.

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2In particular, if (1.2) is well-posed, then $A_0$ is invertible.
Proof. Thanks to Lemma 3.1, equation (1.1) is equivalent to (3.1). According to the hypothesis (i), we have

\[ ||S(t)x_0|| \leq Me^{\alpha t}||x_0||, \]

and

\[ ||S(t-\tau)A_0^{-1}\theta(\tau, x(\tau))|| \leq Me^{\alpha(t-\tau)}||A_0^{-1}\theta(\tau, x(\tau))||. \]

Considering (i) and (ii), we have \( A^{-1}_0\theta(\tau, x(\tau)) \in D_0 \). Using (3.1), we obtain

\[ ||x(t)|| \leq Me^{\alpha t}||x_0|| + M \int_0^t e^{\alpha(t-\tau)}||A_0^{-1}\theta(\tau, x(\tau))|| ||x(\tau)||d\tau \]

or

\[ e^{-\alpha t}||x(t)|| \leq M||x_0|| + M \int_0^t e^{-\alpha \tau}||A_0^{-1}\theta(\tau, x(\tau))|| ||x(\tau)||d\tau. \]

Applying Lemma 3.2 with \( g(t) = e^{-\alpha t}||x(t)||, \ h(\tau) = M||A_0^{-1}\theta(\tau, x(\tau))||, \) and \( c = M||x_0|| \), we obtain

\[ e^{-\alpha t}||x(t)|| \leq M||x_0|| \exp \left[ M \int_0^t ||A_0^{-1}\theta(\tau, x(\tau))||d\tau \right] \]

\[ \leq M||x_0|| \exp \left[ M \int_0^\infty ||A_0^{-1}\theta(\tau, x(\tau))||d\tau \right]. \]

Thus,

\[ ||x(t)|| \leq M_1 e^{\alpha t}||x_0||, \]

where

\[ M_1 = M \exp \left[ M \int_0^\infty ||A_0^{-1}\theta(\tau, x(\tau))||d\tau \right] < \infty. \]

Corollary 3.4. If the conditions (i) and (ii) of Theorem 3.3 are fulfilled and (1.2) is exponentially stable, then the quasi-linear equation (1.1) is also exponentially stable.

Remark 3.5. Theorem 3.3 represents the generalization of the Dini–Hukuhara theorem [1], where \( A \equiv I, \ B \equiv T, \ \theta(t, x(t)) \equiv T(t)\{x(t)\}, \) and \( \alpha = 0. \)

Finally we provide the following example to illustrate our main result.

Example 3.6. Consider (1.1) in the finite-dimensional spaces:

\[ A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \theta(t, x(t)) \equiv e^{-t} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad t \geq 0. \]

In our case

\[ D_0 = \{(a, b) \in \mathbb{R}^2 : b = 0\}, \quad AD_0 = \{(a, b) \in \mathbb{R}^2 : a = b\}, \]

\[ \lambda A - B = \begin{pmatrix} \lambda + 1 & 0 \\ \lambda + 1 & 1 \end{pmatrix}, \quad (\lambda A - B)^{-1} = \frac{1}{\lambda + 1} \begin{pmatrix} 1 & 0 \\ -\lambda - 1 & \lambda + 1 \end{pmatrix}. \]

It is clear that \( \theta(t, x(t)) : D_0 \to AD_0, \) \( t \geq 0, \) and \( A_0 \) is invertible.
Since $\sigma(A, B) = \sigma_p(A, B) = \{-1\}$, then (1.2) is exponentially stable (see Corollary 2.6). From Corollary 3.4, we conclude that the corresponding quasi-linear equation (1.1) is also exponentially stable as far as,

$$\int_0^\infty ||A_0^{-1}\theta(t, x(t))|| dt \leq ||A_0^{-1}|| \int_0^\infty ||\theta(t, x(t))|| dt$$

$$= ||A_0^{-1}|| \int_0^\infty e^{-t} dt$$

$$= ||A_0^{-1}|| < \infty.$$  

Acknowledgement. The authors would like to express their gratitude to the anonymous referees for their comments and suggestions that improve the last version of the manuscript.

References


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