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ON THE STABILITY OF THE QUASI-LINEAR IMPLICIT EQUATIONS IN HILBERT SPACES

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Abstract. We use the generalized theorem of Liapounov to obtain some necessary and sufficient conditions for the stability of the stationary implicit equation

$$Ax'(t) = Bx(t), \quad t \geq 0,$$

where A and B are bounded operators in Hilbert spaces. The achieved results can be applied to the stability for the quasi-linear implicit equation

$$Ax'(t) = Bx(t) + \theta(t, x(t)), \quad t \geq 0.$$

1. Introduction

Consider the abstract implicit differential equation

$$Ax'(t) = Bx(t) + \theta(t, x(t)), \quad t \geq 0, \tag{1.1}$$

where A and B are two linear bounded operators on a Hilbert space $\mathcal{H}$ and $\theta(\cdot,\cdot)$ is a continuous function from $[0,\infty) \times \mathcal{H}$ to $\mathcal{H}$. The operator A is not necessarily invertible.

The equation (1.1) has been considered in various forms by many authors as Favini and Yagi [5], Rutkas [6], Vlasenko [7] and others.

In the present paper, we study the stationary implicit equation

$$Ax'(t) = Bx(t), \quad t \geq 0, \tag{1.2}$$

and also the quasi-linear implicit equation (1.1), with the initial condition

$$x(0) = x_0.$$

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In [2] the authors obtained results concerning the stability of the degenerate difference systems that is similar to (1.1).

Some practical examples of (1.1) can be found in [5, 6, 7]. The organization of this paper is as follows: in Section 2, we introduce some preliminaries and expand the famous Liapounov general theorem [4], which has an important role in this paper. In section 3, we present our main results concerning the exponential stability of the solution for the quasi-linear implicit equation (1.1).

We use the following definitions.

Definition 1.1. The equation (1.2) is called exponentially stable, if there exist two constants $M > 0$ and $\alpha < 0$ such that, for any solution $x(t)$, we have

$$||x(t)|| \leq Me^{\alpha t}||x_0|| \quad \text{for any} \quad t \geq 0. \tag{1.3}$$

Definition 1.2. The equation (1.2) is said to be well-posed, if it satisfies the following properties:

(i) for any solution $x(.)$ such that $x(0) = x_0 = 0$, then $x(t) = 0$ for all $t \geq 0$;
(ii) it generates an evolution semigroup of bounded operators $S(t) : x_0 \mapsto x(t)$ for all $t \geq 0$.

The operators $S(t)$ are defined on the set $D_0 = \{x_0\}$ of the admissible initial vectors x_0.

Definition 1.3 (see [6])**.** The complex number $\lambda \in \mathbb{C}$ is called a regular value of the pencil $\lambda A - B$, if the resolvent $(\lambda A - B)^{-1}$ exists and is bounded. The set of all regular values is denoted by $\rho(A, B)$ and its complement $\sigma(A, B) = \mathbb{C} \setminus \rho(A, B)$ is called the spectrum of the pencil $\lambda A - B$. The set of all eigenvalues of the pencil $\lambda A - B$ is denoted by

$$\sigma_p(A, B) = \big\{\lambda \in \mathbb{C} : \ \exists v \neq 0; (\lambda A - B)v = 0\big\}.$$

2. Stationary implicit equation

For the stationary implicit equation (1.2), we can obtain the following criterion for the exponential stability .

Theorem 2.1. *The equation* (1.2) *is exponentially stable if and only if it is well-posed.*

Proof. Suppose that (1.2) is exponentially stable. Then, it has a unique solution $x(t)$. In fact, if $x_0 = 0$, then by (1.3), we obtain $||x(t)|| \leq 0$, and consequently $x(t) = 0$ for all $t \geq 0$. On the other hand, we have

$$||x(t)|| = ||S(t)x_0|| \leq Me^{\alpha t}||x_0||.$$

It means that the operator $S(t)$ is bounded and $||S(t)|| \leq Me^{\alpha t}$. So, (1.2) is well-posed. Conversely, if (1.2) is well-posed, then one obtains

$$\omega = \lim_{t\to\infty} \frac{\ln ||S(t)||}{t} < \infty$$

(see [4, p. 26]), where ω is the strict Liapounov exponent of $\Phi(t) = ||S(t)||$. More precisely, ω is the greatest lower bound of the set of real numbers ϱ for which there exists a positive constant N_ϱ (see [3, pp. 8–9]) such that

$$\Phi(t) = ||S(t)|| \leq N_\varrho e^{\varrho t} \quad \text{for all } t \geq 0, \omega \leq \varrho < 0.$$

Hence

$$||x(t)|| = ||S(t)x_0|| \leq N_\varrho e^{\varrho t}||x_0||,$$

which achieves the proof . □

Theorem 2.2. *If* (1.2) *is exponentially stable, then all eigenvalues of the pencil* $\lambda A - B$ *are in the half-plane* $\text{Re}\lambda \leq \alpha$*, where* α *is the constant defined in* (1.3)*.*

Proof. Suppose that there exists an eigenvalue $\lambda_0 \in \sigma_p(A, B)$ such that $\text{Re}\lambda_0 > \alpha$. Then $(\lambda_0 A - B)v = 0$, where v is the corresponding eigenvector. Consequently, $y(t) = e^{\lambda_0 t}v$ is a solution of (1.2) verifying the condition $y(0) = v$, and we have

$$||y(t)|| = ||e^{\lambda_0 t}v|| = e^{(\text{Re}\lambda_0)t}||v|| > e^{\alpha t}||y(0)||.$$

So, the solution $y(t)$ does not satisfy (1.3) and consequently (1.2) is not exponentially stable. □

Remark 2.3. If (1.2) is exponentially stable, then all the eigenvalues of the pencil $\lambda A - B$ are inside the left half-plane, that is

$$\sigma_p(A, B) \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda < 0\},$$

since $\alpha < 0$, where α is given by (1.3).

We can now extend the generalized Liapounov theorem [4] for the spectrum of the bounded operator T, to the spectrum of the pencil $\lambda A - B$ of the bounded operators A and B on a Hilbert space $\mathcal{H}$, using the spectral theory of the pencil of operators and an appropriate conformal mapping as follows.

Theorem 2.4. *A necessary condition, for the spectrum* $\sigma(A, B)$ *of the pencil* $\lambda A - B$ *to lie in the interior of the half-plane* $\text{Re}\lambda < \alpha$ $(\alpha < 0)$*, is that, for any uniformly positive operator* $U \gg 0$ [1]*, there exists an operator* $W \gg 0$ *such that*

$$A^*WB + B^*WA - 2\alpha A^*WA = -2U, \tag{2.1}$$

and a sufficient condition is that $\alpha + 1$ *is a regular value of the pencil* $\lambda A - B$ *and there exists an operator* $W \gg 0$ *such that*

$$A^*WB + B^*WA - 2\alpha A^*WA \ll 0. \tag{2.2}$$

Proof. Necessary condition. Suppose that $\sigma(A, B) \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda < \alpha\}$. Then, $(\alpha + 1) \in \rho(A, B)$ and the operator $T = [(\alpha - 1)A - B][(\alpha + 1)A - B]^{-1}$ is well defined and bounded. Now, using the conformal mapping $z = \varphi(\lambda) = \frac{\lambda - \alpha + 1}{\lambda - \alpha - 1}$ which transforms the vertical line $\text{Re}\lambda = \alpha$ into the unit circle $|z| = 1$, we obtain

[1]It means that $U^* = U$ and that $\langle Ux, x\rangle > 0$ for all x with $||x|| = 1$.

$$
\begin{aligned}
zI - T &= (\tfrac{\lambda-\alpha+1}{\lambda-\alpha-1})[(\alpha+1)A-B]\big[(\alpha+1)A-B]^{-1} \\
&\qquad\qquad\qquad - [(\alpha-1)A-B][(\alpha+1)A-B\big]^{-1} \\
&= \tfrac{1}{(\lambda-\alpha-1)}\big\{(\lambda-\alpha+1)[(\alpha+1)A-B] \\
&\qquad\qquad - (\lambda-\alpha-1)[(\alpha-1)A-B]\big\}[(\alpha+1)A-B]^{-1} \\
&= \tfrac{2}{(\lambda-\alpha-1)}(\lambda A-B)[(\alpha+1)A-B]^{-1}.
\end{aligned}
$$

So, the operator $zI - T$ is invertible if and only if the pencil $\lambda A - B$ is also invertible. Therefore, $\rho(T) = \rho(I, T) = \varphi(\rho(A, B))$.

Passing to the complement, we conclude that $\sigma(T) = \sigma(I, T) = \varphi(\sigma(A, B))$. Consequently $\sigma(T)$ is in the unit disk. Using [2, Theorem 2], we conclude that there exists an operator $W \gg 0$ such that

$$T^*WT - W = -G \quad \text{for all } G \gg 0, \tag{2.3}$$

which is equivalent to

$$
\begin{aligned}
&\{[(\alpha-1)A-B][(\alpha+1)A-B]^{-1}\}^*W \\
&\qquad\qquad \{[(\alpha-1)A-B][(\alpha+1)A-B]^{-1}\} - W = -G \\
&\Longleftrightarrow [(\alpha+1)A^*-B^*]^{-1}[(\alpha-1)A^*-B^*]W \\
&\qquad\qquad [(\alpha-1)A-B][(\alpha+1)A-B]^{-1} - W = -G \\
&\Longleftrightarrow [(\alpha-1)A^*-B^*]W[(\alpha-1)A-B] \\
&\qquad\qquad - [(\alpha+1)A^*-B^*]W[(\alpha+1)A-B] \\
&\qquad\qquad = -[(\alpha+1)A^*-B^*]G[(\alpha+1)A-B] \\
&\Longleftrightarrow (2A^*WB + 2B^*WA - 4\alpha A^*WA) \\
&\qquad\qquad = -[(\alpha+1)A^*-B^*]G[(\alpha+1)A-B] \\
&\Longleftrightarrow A^*WB + B^*WA - 2\alpha A^*WA \\
&\qquad\qquad = -\tfrac{1}{2}[(\alpha+1)A^*-B^*]G[(\alpha+1)A-B] \\
&\Longleftrightarrow A^*WB + B^*WA - 2\alpha A^*WA = -2U,
\end{aligned}
$$

where

$$U = \tfrac{1}{4}[(\alpha+1)A^* - B^*]G[(\alpha+1)A - B] \gg 0.$$

In fact,

$$U^* = \tfrac{1}{4}[(\alpha+1)A^* - B^*]G[(\alpha+1)A - B] = U,$$

and for each $x \in \mathcal{H}$, we have

$$\begin{aligned}\langle Ux, x\rangle &= \tfrac{1}{4}\langle[(\alpha+1)A^* - B^*]G[(\alpha+1)A - B]x, x\rangle \\ &= \tfrac{1}{4}\langle G[(\alpha+1)A - B]x, [(\alpha+1)A - B]x\rangle \\ &= \tfrac{1}{4}\langle Gy, y\rangle \geq \tfrac{k}{4}||y||^2, \;\; y = [(\alpha+1)A - B]x,\end{aligned}$$

where k is a positive constant. But,

$$\begin{aligned}||x||^2 &= ||[(\alpha+1)A - B]^{-1}y||^2 \\ &\leq ||[(\alpha+1)A - B]^{-1}||^2 \;\; ||y||^2.\end{aligned}$$

Therefore

$$||y||^2 \geq \frac{||x||^2}{||[(\alpha+1)A-B]^{-1}||^2} \; .$$

Thus

$$\langle Ux, x\rangle \;\geq\; \frac{1}{4}\; \frac{k}{||[(\alpha+1)A-B]^{-1}||^2}\; ||x||^2 \;\; > 0.$$

Consequently $U \gg 0$, and (2.2) holds.

Sufficient condition. If $\alpha+1 \in \rho(A,B)$ is a regular value for the pencil $\lambda A - B$, then the operator $T = [(\alpha-1)A - B][(\alpha+1)A - B]^{-1}$ is bounded and (2.2) becomes

$$A^*WB + B^*WA - 2\alpha A^*WA = -\frac{1}{2}[(\alpha+1)A^* - B^*]G[(\alpha+1)A - B] \ll 0.$$

Therefore, $G = W - T^*WT \gg 0$ (see (2.3)). Using again [2, Theorem 2], the spectrum $\sigma(T)$ will be inside the unit disk. We conclude that $\sigma(A,B) = \varphi^{-1}(\sigma(T)) \subset \{\lambda : \mathrm{Re}\lambda < \alpha\}$, where $\lambda = \varphi^{-1}(z) = \alpha + \frac{z+1}{z-1}$ is a conformal mapping and Theorem 2.4 is proved. □

Theorem 2.5. *If* (2.1) *is satisfied for the pair of the positive uniform operators* (W, U)*, then* $\lambda = \alpha + 1$ *is not an eigenvalue for the pencil* $\lambda A - B$*.*

Proof. Suppose that $\lambda = \alpha + 1$ is an eigenvalue. We denote by $v \neq 0$ the corresponding eigenvector. Then, $[(\alpha+1)A - B]v = 0$ or $(\alpha+1)Av = Bv$, and in the two cases the scalar product becomes

$$\begin{aligned}\langle Uv, v\rangle &= -\tfrac{1}{2}\langle(A^*WB + B^*WA - 2\alpha A^*WA)v, v\rangle \\ &= -\tfrac{1}{2}\langle A^*WBv, v\rangle - \tfrac{1}{2}\langle B^*WAv, v\rangle + \langle \alpha A^*WAv, v\rangle \\ &= -\tfrac{1}{2}\langle WBv, Av\rangle - \tfrac{1}{2}\langle WAv, Bv\rangle + \alpha\langle WAv, Av\rangle \\ &= -\langle WAv, Av\rangle < 0.\end{aligned}$$

We obtain a contradiction, with the hypothesis $U \gg 0$, since $W \gg 0$. Consequently Theorem 2.5 is proved. □

Corollary 2.6. *In the case of a finite-dimensional space $\mathcal{H}$, the following statements are equivalent:*

(a) *The equation (1.2) is exponentially stable;*
(b) $\sigma(A, B) = \sigma_p(A, B) \subset \{\lambda : \mathrm{Re}\lambda < \alpha\}$*;*
(c) *There exists a positive definite matrix $W \gg 0$ such that*

$$A^*WB + B^*WA - 2\alpha A^*WA \ll 0.$$

3. Quasi-linear implicit equation

In this section, we give some stability conditions of the quasi-linear implicit equation of the form (1.1), using the following variation of constants method (Lemma 3.1) and the Gronwall–Bellman inequality (Lemma 3.2).

Remember that $D_0 = \{x(0)\}$ denotes the initial manifold subspace of $\mathcal{H}$ for the stationary equation (1.2).

Lemma 3.1. *Suppose that*

(i) *the restriction operator $A_0 = A|_{D_0}$ on D_0 is invertible;*[2]
(ii) *for any $\tau \geq 0$, the space $\theta(\tau, x(\tau))$ is in the domain of A_0 and the function $S(t-\tau)A_0^{-1}\theta(\tau, x(\tau))$ is integrable (with respect to τ), where $\{S(t)\}_{t\geq 0}$ is the semigroup of the operators for (1.2).*

Then the quasi-linear equation (1.1) is equivalent to the integral equation

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)A_0^{-1}\theta(\tau, x(\tau))d\tau. \tag{3.1}$$

Lemma 3.2 (Gronwall–Bellman)**.** (*see* [1]). *If*

$$g(t) \leq c + \int_0^t g(\tau)h(\tau)d\tau \quad \textit{for all } t \geq 0,$$

where h is a continuous positive real function and $c > 0$ is an arbitrary constant, then

$$g(t) \leq c \;\; exp\left[\int_0^t h(\tau)d\tau\right].$$

For the quasi-linear equation (1.1), we have the next result.

Theorem 3.3. *Suppose that*

(i) *the equation (1.2) is well-posed;*
(ii) *the quasi-linear operator $\theta(t, x(t))$, for all $t \geq 0$, transforms D_0 into AD_0 such that*

$$\int_0^\infty ||A_0^{-1}\theta(t, x(t))||dt < \infty.$$

Then the quasi-linear equation (1.1) is exponentially stable .

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[2]In particular, if (1.2) is well-posed, then A_0 is invertible.

Proof. Thanks to Lemma 3.1, equation (1.1) is equivalent to (3.1). According to the hypothesis (i), we have

$$||S(t)x_0|| \leq Me^{\alpha t}||x_0||,$$

and

$$||S(t-\tau)A_0^{-1}\theta(\tau, x(\tau))|| \leq Me^{\alpha(t-\tau)}||A_0^{-1}\theta(\tau, x(\tau))||.$$

Considering (i) and (ii), we have $A_0^{-1}\theta(\tau, x(\tau)) \in D_0$. Using (3.1), we obtain

$$||x(t)|| \leq Me^{\alpha t}||x_0|| + M\int_0^t e^{\alpha(t-\tau)}||A_0^{-1}\theta(\tau, x(\tau))|| \ \ ||x(\tau)||d\tau$$

or

$$e^{-\alpha t}||x(t)|| \leq M||x_0|| + M\int_0^t e^{-\alpha\tau}||A_0^{-1}\theta(\tau, x(\tau))|| \ \ ||x(\tau)||d\tau.$$

Applying Lemma 3.2 with $g(t) = e^{-\alpha t}||x(t)||$, $h(\tau) = M||A_0^{-1}\theta(\tau, x(\tau))||$, and $c = M||x_0||$, we obtain

$$\begin{aligned} e^{-\alpha t}||x(t)|| &\leq M||x_0|| \ \ exp\left[M\int_0^t ||A_0^{-1}\theta(\tau, x(\tau))||d\tau\right] \\ &\leq M||x_0|| \ \ exp\left[M\int_0^\infty ||A_0^{-1}\theta(\tau, x(\tau))||d\tau\right]. \end{aligned}$$

Thus,

$$||x(t)|| \leq M_1e^{\alpha t}||x_0||,$$

where

$$M_1 = M \ \ exp\left[M\int_0^\infty ||A_0^{-1}\theta(\tau, x(\tau))||d\tau\right] < \infty.$$

□

Corollary 3.4. *If the conditions* (i) *and* (ii) *of Theorem* 3.3 *are fulfilled and* (1.2) *is exponentially stable, then the quasi-linear equation* (1.1) *is also exponentially stable.*

Remark 3.5. Theorem 3.3 represents the generalization of the Dini–Hukuhara theorem [1], where $A \equiv I$, $B \equiv T$, $\theta(t, x(t)) \equiv T(t)\{x(t)\}$, and $\alpha = 0$.

Finally we provide the following example to illustrate our main result.

Example 3.6. Consider (1.1) in the finite-dimensional spaces:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \theta(t, x(t)) \equiv e^{-t}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad t \geq 0.$$

In our case

$$D_0 = \{(a, b) \in \mathbb{R}^2 : b = 0\}, \qquad AD_0 = \{(a, b) \in \mathbb{R}^2 : a = b\},$$

$$\lambda A - B = \begin{pmatrix} \lambda+1 & 0 \\ \lambda+1 & 1 \end{pmatrix}, \qquad (\lambda A - B)^{-1} = \frac{1}{\lambda+1}\begin{pmatrix} 1 & 0 \\ -\lambda-1 & \lambda+1 \end{pmatrix}.$$

It is clear that $\theta(t, x(t)) : D_0 \to AD_0$, $t \geq 0$, and A_0 is invertible.

Since $\sigma(A,B) = \sigma_p(A,B) = \{-1\}$, then (1.2) is exponentially stable (see Corollary 2.6). From Corollary 3.4, we conclude that the corresponding quasi-linear equation (1.1) is also exponentially stable as far as,

$$\begin{aligned}\int_0^\infty ||A_0^{-1}\theta(t,x(t))||dt &\leq ||A_0^{-1}|| \int_0^\infty ||\theta(t,x(t))||dt \\ &= ||A_0^{-1}|| \int_0^\infty e^{-t} dt \\ &= ||A_0^{-1}|| < \infty.\end{aligned}$$

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