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ON APPROXIMATION BY (p, q)-MEYER–KÖNIG–ZELLER DURRMEYER OPERATORS

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Abstract. In this paper, we introduce a Durrmeyer type modification of Meyer–König–Zeller operators based on (p, q)-integers. The rate of convergence of these operators is explored with the help of Korovkin type theorems. We establish some direct results for proposed operators. We also obtain statistical approximation properties of operators. In the last section, we show the rate of convergence of (p, q)-Meyer–König–Zeller Durrmeyer operators for some functions by means of MATLAB programming.

1. Introduction and Preliminaries

Recently, Mursaleen et al. [12] introduced (p, q)-analogue of Bernstein type operators. After that many researchers gave the (p, q)-analogue of various well known positive linear operators and studied their approximation properties; for details, we refer the reader to [1, 7, 11, 16, 17]. Now, We begin by recalling certain notation of (p, q) calculus.

Let $0 < q < p \leq 1$. The (p, q)-integer is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad n = 1, 2 \dots,$$

and the (p, q)-factorial is given by

$$[n]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q} \dots [n]_{p,q}, & \text{n} \geq 1, \\ 1, & \text{n=0}. \end{cases}$$

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For integers $0 \leq k \leq n$, the (p,q)-binomial coefficient is defined as

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

Further, (p,q)-binomial function is expressed as

$$(x+y)^n_{p,q} = \prod_{j=0}^{n-1}(p^j x + q^j y).$$

Recently, Sharma [16] introduced the (p,q)-Beta function for $s,t \in \Re^+$ as

$$\beta_{p,q}(t,s) = \int_0^1 x^{t-1}(1-qx)^{s-1}_{p,q} d_{p,q}x$$

and also obtained the relation between (p,q)-Beta function and q-Beta function as

$$\beta_{p,q}(t,s) = p^{(s-1)(s-2)/2-(t-1)}\beta_{\frac{q}{p}}(t,s),$$

where, $\beta_{\frac{q}{p}}(t,s)$ is the $\frac{q}{p}$-analogue of the beta function. Using $\beta_q(t,s) = \frac{[t-1]!_q[s-1]!_q}{[s+t-1]!_q}$ and $[n]!_{\frac{q}{p}} = p^{-n(n-1)/2}[n]!_{p,q}$, we can write

$$\beta_{p,q}(t,s) = p^{((s+t-1)(s+t-2)-(t-1)(t-2))/2-t+1}\frac{[t-1]!_{p,q}[s-1]!_{p,q}}{[s+t-1]!_{p,q}}. \tag{1.1}$$

For $p=1$, all the notations of (p,q)-calculus are reduced to q-calculus. Further details on (p,q)-calculus can be found in [3, 13, 14].

In a recent study, Kadak et al. [10] introduced a (p,q)-analogue of Meyer–König–Zeller operators, for $0 < q < p \leq 1$, on a function defined on $[0,1]$ as

$$M_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n+1)}{2}}}\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k \\ k \end{array}\right]_{p,q} x^k p^{-kn}(1-x)^{n+1}_{p,q} f\left(\frac{p^n[k]_{p,q}}{[n+k]_{p,q}}\right), \quad x \in [0,1)$$

and $M_{n,p,q}(f;1) = f(1)$ for $x=1$.

Further, the moment of the operators are given in the following lemma.

Lemma 1.1 (see[10]). *For all $x \in [0,1]$ and $0 < q < p \leq 1$, we have*

$$\begin{aligned} M_{n,p,q}(1;x) &= 1, \\ M_{n,p,q}(t;x) &= x, \\ x^2 \leq M_{n,p,q}(t^2;x) &\leq \frac{p^n}{[n+1]_{p,q}}x + x^2. \end{aligned}$$

In the past two decades, Studies of Durrmeyer variants of various operators remained the center of attraction for the researchers, for which we refer the reader to [2, 6, 8, 15, 9]. Motivated by these studies, now we introduce the Meyer–König–Zeller Durrmeyer operators based on (p,q)-integers in the following section.

2. Construction of operator and moment estimate

For $0<q<p\leq 1$ and the function f defined on $[0,1]$, the (p,q)-Meyer–König–Zeller Durrmeyer operators are defined as follows:

$$\widetilde{M}_{n,k}^{(p,q)}(f;x)=\frac{[n+1]_{p,q}}{p^n}\sum_{k=0}^{\infty}m_{n,k}^{(p,q)}(x)(pq)^{-}k\int_0^1 b_{n,k}^{(p,q)}(qt)f(t)d_{p,q}t, \quad 0\leq x<1,$$

here,

$$m_{n,k}^{(p,q)}(x)=\frac{1}{p^{kn+n(n+1)/2}}\begin{bmatrix}n+k\\k\end{bmatrix}_{p,q}x^k(1-x)_{p,q}^{n+1},$$

$$b_{n,k}^{(p,q)}(qt)=\frac{1}{p^{k(n-1)+n(n-1)/2}}\begin{bmatrix}n+k+1\\k\end{bmatrix}_{p,q}(qt)^k(1-qt)_{p,q}^n$$

and $\widetilde{M}_{n,k}^{(p,q)}(f;1)=1$. Before computing the moments of (p,q)-Meyer–König–Zeller Durrmeyer operators, we prove some lemmas as follows.

Lemma 2.1. *Let $0<q<p\leq 1$ and let $s=0,1,2,\ldots$. We have*

$$\int_0^1 b_{n,k}^{(p,q)}(qt)t^s d_{p,q}t=\frac{[n+k+1]_{p,q}![k+s]_{p,q}!}{[k]_{p,q}![n+k+s+1]_{p,q}!}\frac{(pq)^k}{[n+1]_{p,q}}p^{n(s+1)}.$$

Proof. This lemma can be proved directly by using the definition of (p,q)-beta operator and Equation (1.1). □

Lemma 2.2. *For $r=1,2\ldots$ and $n>r$, we have*

$$\sum_{k=0}^{\infty}\left[\begin{array}{c}n+k\\k\end{array}\right]_{p,q}\frac{x^k(1-x)_{p,q}^{n+1}}{[n+k]_{p,q}^{\underline{r}}}p^{(r-n)k}=\frac{\prod_{j=0}^{r-1}(p^{n-j}-q^{n-j}x)}{[n]_{p,q}^{\underline{r}}}p^{\frac{(n-r)(n-r+1)}{2}},$$

where $[n]_{p,q}^{\underline{r}}=[n]_{p,q}[n-1]_{p,q}[n-2]_{p,q}\ldots[n-r+1]_{p,q}$.

Lemma 2.3. *The following inequality holds:*

$$\frac{1}{[n+k+r]_{p,q}}\leq\frac{1}{q^r[n+k]_{p,q}}, \quad r\geq 0.$$

Theorem 2.4. *For all $x\in[0,1]$, $n\in\mathbb{N}$ and $0<q<p\leq 1$, we have*

$$\begin{aligned}\widetilde{M}_{n,k}^{(p,q)}(e_0;x) &= 1,\\ \frac{x}{q^2}\left(1-\frac{q+1}{[n]_{p,q}}\right)\leq\widetilde{M}_{n,k}^{(p,q)}(e_1;x) &\leq \frac{x}{q}+\frac{(p^n-q^nx)}{q^2[n]_{p,q}},\\ \widetilde{M}_{n,k}^{(p,q)}(e_2;x) &\leq \frac{x^2}{q^2}+\frac{(p+q)^2}{q^5}\frac{(p^n-q^nx)}{[n]_{p,q}}x\\ &\quad+\frac{p(p+q)}{q^6}\frac{(p^n-q^nx)(p^{n-1}-q^{n-1}x)}{[n]_{p,q}[n-1]_{p,q}},\end{aligned}$$

where $e_i=t^i$ for $i=0,1,2$.

Proof. First moment can be directly computed. We use the moments obtained for (p,q)-Meyer–König–Zeller operators in Lemma 1.1 to estimate moments of proposed Durrmeyer operators. By using Lemma 2.1 for $s=1$ and Lemma 2.2, we get the lower bound of second moment as follows:

$$\begin{aligned}
\widetilde{M}_{n,k}^{(p,q)}(e_1,x) &= \sum_{k=0}^{\infty} m_{n,k}^{(p,q)}(x)\frac{[n+k+1]_{p,q}![k+1]_{p,q}!}{[k]_{p,q}![n+k+2]_{p,q}!}p^n \\
&= p^n\sum_{k=0}^{\infty}\frac{1}{p^{kn+n(n+1)/2}}\begin{bmatrix} n+k \\ k\end{bmatrix}_{p,q} x^k(1-x)_{p,q}^{n+1}\frac{[k+1]_{p,q}}{[n+k+2]_{p,q}} \\
&= p^n\sum_{k=1}^{\infty}\frac{1}{p^{kn+n(n+1)/2}}\begin{bmatrix} n+k-1 \\ k-1\end{bmatrix}_{p,q} x^k(1-x)_{p,q}^{n+1}\frac{[k+1]_{p,q}}{[k]_{p,q}}\frac{[n+k]_{p,q}}{[n+k+2]_{p,q}} \\
&\geq p^n\sum_{k=0}^{\infty}\frac{1}{p^{(k+1)n+n(n+1)/2}}\begin{bmatrix} n+k \\ k\end{bmatrix}_{p,q} x^{k+1}(1-x)_{p,q}^{n+1}\frac{[n+k+1]_{p,q}}{[n+k+3]_{p,q}} \\
&= \frac{x}{q}\sum_{k=0}^{\infty} m_{n,k}^{(p,q)}(x)\left(\frac{[n+k+2]_{p,q}-p^{n+k+1}}{[n+k+3]_{p,q}}\right) \\
&\geq \frac{x}{q}\sum_{k=0}^{\infty} m_{n,k}^{(p,q)}(x)\left(\frac{[n+k+2]_{p,q}}{[n+k+3]_{p,q}}-\frac{1}{[n]_{p,q}}\right) \\
&= \frac{x}{q}\sum_{k=0}^{\infty} m_{n,k}^{(p,q)}(x)\frac{[n+k+2]_{p,q}}{[n+k+3]_{p,q}}-\frac{x}{[n]_{p,q}q} \\
&= \frac{x}{q^2}\sum_{k=0}^{\infty} m_{n,k}^{(p,q)}(x)\left(\frac{[n+k+3]_{p,q}-p^{n+k+2}}{[n+k+3]_{p,q}}\right)-\frac{x}{[n]_{p,q}q} \\
&\geq \frac{x}{q^2}\sum_{k=0}^{\infty} m_{n,k}^{(p,q)}(x)\left(1-\frac{1}{[n]_{p,q}}\right)-\frac{x}{[n]_{p,q}q} \\
&= \frac{x}{q^2}\left(1-\frac{q+1}{[n]_{p,q}}\right).
\end{aligned}$$

By using the inequality of Lemma 2.3, the upper bound can be obtained as below:

$$\begin{aligned}
\widetilde{M}_{n,k}^{(p,q)}(e_1,x) &= p^n\sum_{k=0}^{\infty} m_{n,k}^{(p,q)}(x)\frac{[k+1]_{p,q}}{[n+k+2]_{p,q}} \\
&\leq p^n\sum_{k=0}^{\infty} m_{n,k}^{(p,q)}(x)\frac{p^k+q[k]_{p,q}}{q^2[n+k]_{p,q}} \\
&= \frac{p^n}{q^2}\sum_{k=0}^{\infty} m_{n,k}^{(p,q)}(x)\frac{p^k}{[n+k]_{p,q}}+\frac{1}{q}\sum_{k=0}^{\infty} m_{n,k}^{(p,q)}(x)\frac{p^n[k]_{p,q}}{[n+k]_{p,q}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{p^n}{q^2}\sum_{k=0}^{\infty}\frac{1}{p^{kn}+n(n+1)/2}\begin{bmatrix} n+k \\ k \end{bmatrix}_{p,q} x^k(1-x)_{p,q}^{n+1}\frac{p^k}{[n+k]_{p,q}}+\frac{x}{q}\\
&= \frac{(p^n-q^nx)}{q^2[n]_{p,q}}+\frac{x}{q}.
\end{aligned}$$

To estimate the third moment, we use Lemma 2.1 for $s=2$ and Lemma 2.2 as follows:

$$\begin{aligned}
\widetilde{M}_{n,k}^{(p,q)}(e_2,x) &= \sum_{k=0}^{\infty} m_{n,k}^{(p,q)}(x)\frac{[n+k+1]_{p,q}![k+2]_{p,q}!}{[k]_{p,q}![n+k+3]_{p,q}!}p^{2n}\\
&= p^{2n}\sum_{k=0}^{\infty} m_{n,k}^{(p,q)}(x)\frac{[k+2]_{p,q}[k+1]_{p,q}}{[n+k+3]_{p,q}[n+k+2]_{p,q}}\\
&\leq \frac{p^{2n}}{q^6}\sum_{k=0}^{\infty} m_{n,k}^{(p,q)}(x)\frac{(p+q)p^{2k}+(p+2q)qp^k[k]_{p,q}+q^3[k]_{p,q}^2}{[n+k]_{p,q}[n+k-1]_{p,q}}\\
&= \frac{p^{\frac{-n(n-3)}{2}}}{q^6}(p+q)\sum_{k=0}^{\infty}p^{-kn}\begin{bmatrix} n+k \\ k \end{bmatrix}_{p,q}\frac{x^k(1-x)_{p,q}^{n+1}}{[n+k]_{p,q}[n+k-1]_{p,q}}p^{2k}\\
&\quad+\frac{p^{\frac{-n(n-3)}{2}}}{q^5}(p+2q)\sum_{k=0}^{\infty}p^{-kn}\begin{bmatrix} n+k \\ k \end{bmatrix}_{p,q}\frac{x^k(1-x)_{p,q}^{n+1}}{[n+k]_{p,q}[n+k-1]_{p,q}}p^k[k]_{p,q}\\
&\quad+\frac{p^{\frac{-n(n-3)}{2}}}{q^3}\sum_{k=0}^{\infty}p^{-kn}\begin{bmatrix} n+k \\ k \end{bmatrix}_{p,q}\frac{x^k(1-x)_{p,q}^{n+1}}{[n+k]_{p,q}[n+k-1]_{p,q}}[k]_{p,q}^2\\
&= I_1+I_2+I_3.
\end{aligned}$$

By using Lemma 2.3, I_1 can be obtained as

$$I_1=\frac{p(p+q)}{q^6}\frac{(p^n-q^nx)\left(p^{n-1}-q^{n-1}x\right)}{[n]_{p,q}[n-1]_{p,q}}.$$

Computations for I_2 are as follows:

$$\begin{aligned}
I_2 &= \frac{p^{\frac{-n(n-3)}{2}}}{q^5}(p+2q)\sum_{k=1}^{\infty}p^{-kn}\begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_{p,q}\frac{x^k(1-x)_{p,q}^{n+1}}{[n+k-1]_{p,q}}p^k\\
&= \frac{p^{\frac{-n(n-1)}{2}+1}}{q^5}(p+2q)x\sum_{k=0}^{\infty}p^{-kn}\begin{bmatrix} n+k \\ k \end{bmatrix}_{p,q}\frac{x^k(1-x)_{p,q}^{n+1}}{[n+k]_{p,q}}p^k\\
&= \frac{p(p+2q)}{q^5[n]_{p,q}}\left(p^n-q^nx\right)x.
\end{aligned}$$

I_3 can be obtained as follows:

$$\begin{aligned} I_3 &= \frac{p^{\frac{-n(n-3)}{2}}}{q^3}\sum_{k=1}^{\infty} p^{-kn}\begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_{p,q} \frac{x^k(1-x)_{p,q}^{n+1}}{[n+k-1]_{p,q}}[k]_{p,q} \\ &= \frac{p^{\frac{-n(n-1)}{2}}}{q^3}x\sum_{k=0}^{\infty} p^{-kn}\begin{bmatrix} n+k \\ k \end{bmatrix}_{p,q} \frac{x^k(1-x)_{p,q}^{n+1}}{[n+k]_{p,q}}[k+1]_{p,q} \\ &= \frac{p^{\frac{-n(n-1)}{2}}}{q^3}x\sum_{k=0}^{\infty} p^{-kn}\begin{bmatrix} n+k \\ k \end{bmatrix}_{p,q} \frac{x^k(1-x)_{p,q}^{n+1}}{[n+k]_{p,q}}p^k \\ &\quad +\frac{p^{\frac{-n(n-1)}{2}}}{q^3}x\sum_{k=0}^{\infty} p^{-kn}\begin{bmatrix} n+k \\ k \end{bmatrix}_{p,q} \frac{x^k(1-x)_{p,q}^{n+1}}{[n+k]_{p,q}}q[k]_{p,q} \\ &= \frac{x^2}{q^2}+\left(\frac{(p^n-q^nx)}{q^3[n]_{p,q}}\right)x. \end{aligned}$$

By using I_1, I_2, and I_3, we get the upper bound of second moment. □

Corollary 2.5. *Central moments of operators are*

$$\begin{aligned} \widetilde{M}_{n,k}^{(p,q)}(\psi_1;x) &\leq \frac{p^n-q^nx}{q^2[n]_{p,q}}+\left(\frac{1}{q}-1\right)x, \\ \widetilde{M}_{n,k}^{(p,q)}(\psi_2;x) &\leq x^2\left(1-\frac{1}{q^2}+\frac{2(q+1)}{q^2[n]_{p,q}}\right)+\frac{(p+q)^2}{q^5}\frac{(p^n-q^nx)}{[n]_{p,q}}x \\ &\quad +\frac{p(p+q)}{q^6}\frac{(p^n-q^nx)(p^{n-1}-q^{n-1}x)}{[n]_{p,q}[n-1]_{p,q}}, \end{aligned}$$

where $\psi_i(x)=(t-x)^i$ *for* $i=1,2$.

Proof. By the linearity of $\widetilde{M}_{n,k}^{(p,q)}$ and Theorem 2.4, central moments can be obtained directly. □

Remark 2.6. For $0<q<p\leq 1$, by simple computations $\lim_{n\to\infty}[n]_{p,q}=1/(p-q)$. In order to obtain results for order of convergence of the operator, we take $q_n\in(0,1)$ and $p_n\in(q_n,1]$ such that $\lim_{n\to\infty}p_n=\lim_{n\to\infty}q_n=1$, $\lim_{n\to\infty}p_n^n=a$ and $\lim_{n\to\infty}q_n^n=b$, so that $\lim_{n\to\infty}\frac{1}{[n]_{p_n,q_n}}=0$. Such a sequence can always be constructed, for example, we can take $q_n=1-1/2n$ and $p_n=1-1/3n$, clearly $\lim_{n\to\infty}p_n^n=e^{-1/3}$, $\lim_{n\to\infty}q_n^n=e^{-1/2}$ and $\lim_{n\to\infty}\frac{1}{[n]_{p_n,q_n}}=0$.

3. Rate of convergence

We denote $W^2=\{g\in C[0,1]:g',g''\in C[0,1]\}$. For $\delta>0$, then K-functional is defined as

$$K_2(f,\delta)=\inf_{g\in W^2}\{\|f-g\|+\delta\|g''\|\},$$

where the norm $||.||$ denotes the supremum norm on $C[0,1]$. Following the well-known inequality given by DeVore and Lorentz in [4], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\leq C\omega_2(f,\sqrt{\delta}),$$

where $\omega_2(f,\sqrt{\delta})$ is the second order modulus of continuity for $f \in C[0,1]$, defined as

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x,x+h\in[0,1]} |f(x+2h) - 2f(x+h) + f(x)|.$$

We denote by $\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{x,x+h\in[0,1]} |f(x+h)-f(x)|$, the usual modulus of continuity for $f \in C[0,1]$.

Theorem 3.1. *Let $(p_n)_n$ and $(q_n)_n$ be the sequences as defined in Remark 2.6. Then for each $f \in C[0,1]$, we have $\widetilde{M}_{n,k}^{(p_n,q_n)}(f;x)$ converges uniformly to f.*

Proof. By the Korovkin theorem, it is sufficient to show that

$$\lim_{n\to\infty} \|\widetilde{M}_{n,k}^{(p_n,q_n)}(t^m;x) - x^m\| = 0$$

for $m = 0,1,2$. For $m = 0$, the results hold trivially. Using Theorem 2.4, we obtain the results for $m = 1,2$ as follows:

$$\begin{aligned}
\lim_{n\to\infty} \|\widetilde{M}_{n,k}^{(p_n,q_n)}(t;x) - x\| &\leq \lim_{n\to\infty} \left| \frac{x}{q_n} + \frac{(p_n^n - q_n^n x)}{q_n^2 [n]_{p_n,q_n}} - x \right| \\
&\leq \lim_{n\to\infty} \left| \frac{p_n^n}{q_n^2 [n]_{p_n,q_n}} \right| + \lim_{n\to\infty} \left| \frac{1}{q_n} - \frac{q_n^{(n-2)}}{[n]_{p_n,q_n}} - 1 \right| x \\
&= 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
\lim_{n\to\infty} \|\widetilde{M}_{n,k}^{(p_n,q_n)}(t^2;x) - x^2\| &\leq \lim_{n\to\infty} \left| \frac{1}{q_n^2} - 1 \right| x^2 + \lim_{n\to\infty} \left| \frac{(p_n+q_n)^2}{q_n^5} \frac{(p_n^n - q_n^n x)}{[n]_{p_n,q_n}} \right| x \\
&\quad + \lim_{n\to\infty} \left| \frac{p_n(p_n+q_n)}{q_n^6} \frac{(p_n^n - q_n^n x)(p_n^{n-1} - q_n^{n-1}x)}{[n]_{p_n,q_n}[n-1]_{p_n,q_n}} \right| \\
&= 0.
\end{aligned}$$

Hence the proof is completed. □

Theorem 3.2. *Let $(p_n)_n$ and $(q_n)_n$ be sequences as defined in Remark 2.6. Let $f \in C[0,1]$. Then for all $n \in N$, there exists an absolute constant $C > 0$ such that*

$$|\widetilde{M}_{n,k}^{(p_n,q_n)}(f;x) - f(x)| \leq C\omega_2(f,\delta_n(x)) + \omega(f,\alpha_n(x)),$$

where,

$$\delta_n(x) = \left\{ \widetilde{M}_{n,k}^{(p_n,q_n)}((t-x)^2;x) + \widetilde{M}_{n,k}^{(p_n,q_n)}(t-x;x) \right\}^{\frac{1}{2}}$$

and

$$\alpha_n(x) = \widetilde{M}_{n,k}^{(p_n,q_n)}(t-x;x).$$

Proof. For $x \in [0,1]$, we consider the operators $M_n^*(f;x)$ as

$$M_n^*(f;x) = \widetilde{M}_{n,k}^{(p_n,q_n)}(f;x) + f(x) - f\left(\frac{x}{q_n} + \frac{p_n^n - q_n^n x}{q_n^2 [n]_{p_n,q_n}} \right).$$

Using the first central moment of $\widetilde{M}_{n,k}^{(p_n,q_n)}$ and the positivity of operator, we immediately get $M_n^*(t-x;x) = 0$.

For $g \in W^2$ and $x \in [0,1]$, using Taylor's formula,

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

Therefore,

$$\begin{aligned} M_n^*(g;x) - g(x) &= g'(x)M_n^*((t-x);x) + M_n^*\left(\int_x^t (t-u)g''(u)du;x\right) \\ &= \widetilde{M}_{n,k}^{(p_n,q_n)}\left(\int_x^t (t-u)g''(u)du;x\right) \\ &\quad -\int_x^{\frac{x}{q_n}+\frac{p_n^n-q_n^n x}{q_n^2[n]_{p_n,q_n}}} \left(\frac{x}{q_n}+\frac{p_n^n-q_n^n x}{q_n^2[n]_{p_n,q_n}}-u\right)g''(u)du. \end{aligned}$$

Finally, we have

$$\begin{aligned} |M_n^*(g;x) - g(x)| &\leq \left|\widetilde{M}_{n,k}^{(p_n,q_n)}\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\ &\quad + \left|\int_x^{\frac{x}{q_n}+\frac{p_n^n-q_n^n x}{q_n^2[n]_{p_n,q_n}}} \left(\frac{x}{q_n}+\frac{p_n^n-q_n^n x}{q_n^2[n]_{p_n,q_n}}-u\right)g''(u)du\right| \\ &\leq \|g''\|\widetilde{M}_{n,k}^{(p_n,q_n)}((t-x)^2;x) + \left(\frac{x}{q_n}+\frac{p_n^n-q_n^n x}{q_n^2[n]_{p_n,q_n}}-x\right)^2\|g''\| \\ &= \delta_n^2(x)\|g''\|. \end{aligned}$$

Also, we have

$$|M_n^*(f;x)| \leq |\widetilde{M}_{n,k}^{(p_n,q_n)}(f;x)| + 2\|f\| \leq 3\|f\|.$$

Therefore,

$$\begin{aligned} |\widetilde{M}_{n,k}^{(p_n,q_n)}(f;x) - f(x)| &\leq |M_n^*(f-g;x) - (f-g)(x)| \\ &\quad + \left|f\left(\frac{x}{q_n}+\frac{p_n^n-q_n^n x}{q_n^2[n]_{p_n,q_n}}\right) - f(x)\right| + |M_n^*(g;x) - g(x)| \\ &\leq |M_n^*(f-g;x)| + |(f-g)(x)| \\ &\quad + \left|f\left(\frac{x}{q_n}+\frac{p_n^n-q_n^n x}{q_n^2[n]_{p_n,q_n}}\right) - f(x)\right| + |M_n^*(g;x) - g(x)| \\ &\leq 4\|f-g\| + \omega\left(f, \alpha_n(x)\right) + \delta_n^2(x)\|g''\|. \end{aligned}$$

Taking the infimum on the right hand side over all $g \in W^2$ and using the definition of K-functional, we get

$$|\widetilde{M}_{n,k}^{(p_n,q_n)}(f;x) - f(x)| \leq 4K_2(f, \delta_n^2(x)) + \omega(f, \alpha_n(x)).$$

$\square$

4. Statistical approximation

In this section, by using a Bohman–Korovkin type theorem proved in [5], we present the statistical approximation properties of purposed operator.

At this moment, we recall the concept of statistical convergence.
A sequence $(x_n)_n$ is said to be statistically convergent to a number L, denoted by $st-\lim\limits_n x_n = L$ if, for every $\varepsilon > 0$,

$$\delta\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0,$$

where

$$\delta(S) := \frac{1}{N}\sum_{k=1}^{N}\chi_S(j)$$

is the natural density of set $S \subseteq \mathbb{N}$ and χ_S is the characteristic function of S.

Let $C_B(D)$ represent the space of all continuous functions on D and bounded on entire real line, where D is any interval on real line. It can be easily shown that $C_B(D)$ is a Banach space with the supreme norm. Also $\widetilde{M}_{n,k}^{(p,q)}(f;x)$ are well defined for any $f \in C_B([0,1])$.

Theorem 4.1 (see [5])**.** *Let $(L_n)_n$ be a sequence of positive linear operators from $C_B([a,b])$ into $B([a,b])$ which satisfies the following condition*

$$st-\lim_n \|L_n e_i - e_i\| = 0 \quad \textit{for all } i = 0,1,2.$$

Then,

$$st-\lim_n \|L_n f - f\| = 0 \quad \textit{for all } f \in C_B([a,b]).$$

Theorem 4.2. *Let $\{p_n\}_n$ and $\{q_n\}_n$ be sequences such that*

$$st-\lim_{n\to\infty} q_n = 1, \qquad st-\lim_{n\to\infty} {q_n}^n = a,$$

$$st-\lim_{n\to\infty} p_n = 1, \qquad st-\lim_{n\to\infty} {p_n}^n = b.$$

Then, $\widetilde{M}_{n,k}^{(p_n,q_n)}(f,x)$ converges statistically to f. Therefore,

$$st-\lim_n \left\|\widetilde{M}_{n,k}^{(p_n,q_n)}(f(t);x) - f\right\|_{C[0,1]} = 0 \quad forall f \in C[0,1].$$

Proof. By Theorem 4.1, it is sufficient to prove that

$$st-\lim_n \left\|\widetilde{M}_{n,k}^{(p_n,q_n)}(f_i(t);x) - f_i(x)\right\|_{C[0,1]} = 0 \quad \textit{for all } i = 0,1,2.$$

Based on Theorem 2.4, we have

$$st-\lim_n \left\|\widetilde{M}_{n,k}^{(p_n,q_n)}(1;x) - 1\right\|_{C[0,1]} = 0,$$

$$|\widetilde{M}_{n,k}^{(p_n,q_n)}(t;x) - x| \leq \left|\frac{p_n^n}{q_n^2[n]_{p_n,q_n}} + \frac{x}{q_n} - \frac{xq_n^{(n-2)}}{[n]_{p_n,q_n}} - x\right|$$

and

$$|\widetilde{M}_{n,k}^{(p_n,q_n)}(t^2;x)-x^2| \leq \left|\frac{x^2}{q_n^2}-x^2+\frac{x(p_n+q_n)^2}{q_n^5}\frac{(p_n^n-q_n^n x)}{[n]_{p_n,q_n}} + \frac{p_n(p_n+q_n)}{q_n^6}\frac{(p_n^n-q_n^n x)(p_n^{n-1}-q_n^{n-1}x)}{[n]_{p_n,q_n}[n-1]_{p_n,q_n}}\right|.$$

By taking supremum over $x \in [0,1]$ in the above inequalities, we get

$$|\widetilde{M}_{n,k}^{(p_n,q_n)}(t;x)-x| \leq \left|\frac{p_n^n}{q_n^2[n]_{p_n,q_n}}+\frac{1}{q_n}-\frac{q_n^{(n-2)}}{[n]_{p_n,q_n}}-1\right|$$

and

$$|\widetilde{M}_{n,k}^{(p_n,q_n)}(t^2;x)-x^2| \leq \left|\frac{1}{q_n^2}-1+\frac{(p_n+q_n)^2}{q_n^5}\frac{(p_n^n-q_n^n x)}{[n]_{p_n,q_n}} + \frac{p_n(p_n+q_n)}{q_n^6}\frac{(p_n^n-q_n^n)(p_n^{n-1}-q_n^{n-1})}{[n]_{p_n,q_n}[n-1]_{p_n,q_n}}\right|.$$

By using facts that $st-\lim_n q_n = 1$ and $st-\lim_n p_n = 1$, we get

$$\begin{aligned} st-\lim_n \left\|\widetilde{M}_{n,k}^{(p_n,q_n)}(t;x)-x\right\|_{C[0,1]} &= 0,\\ st-\lim_n \left\|\widetilde{M}_{n,k}^{(p_n,q_n)}(t^2;x)-x^2\right\|_{C[0,1]} &= 0. \end{aligned}$$

Hence the proof is complete. □

In the next theorem, we estimate the rate of convergence by using the concepts of modulus of continuity.

Theorem 4.3. *Let $\{p_n\}_n$ and $\{q_n\}_n$ be sequences such that*

$$st-\lim_{n\to\infty} q_n = 1, \qquad st-\lim_{n\to\infty} {q_n}^n = a,$$
$$st-\lim_{n\to\infty} p_n = 1, \qquad st-\lim_{n\to\infty} {p_n}^n = b.$$

Then,

$$|\widetilde{M}_{n,k}^{(p_n,q_n)}(f;x)-f| \leq 2\omega(f,\sqrt{\delta_n}) \tag{4.1}$$

for all $f \in C[0,1]$, where $\delta_n = \widetilde{M}_{n,k}^{(p_n,q_n)}((t-x)^2;x)$.

Proof. By the linearity and monotonicity of the operator, we get

$$|\widetilde{M}_{n,k}^{(p_n,q_n)}(f;x)-f| \leq \widetilde{M}_{n,k}^{(p_n,q_n)}(|f(t)-f(x)|;x),$$

also, by the continuity property of modulus, we have

$$|f(t)-f(x)| \leq \left(1+\frac{(t-x)^2}{\delta^2}\right)\omega(f,\delta).$$

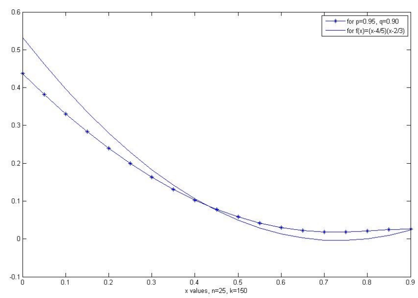
By using the above facts, we get

$$|\widetilde{M}_{n,k}^{(p_n,q_n)}(f;x)-f| \leq \left(\widetilde{M}_{n,k}^{(p_n,q_n)}(1;x)+\frac{1}{\delta^2}\widetilde{M}_{n,k}^{(p_n,q_n)}\left((t-x)^2;x\right)\right)\omega(f,\delta).$$

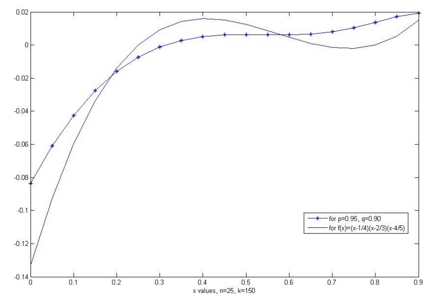
So, letting $\delta_n = \widetilde{M}_{n,k}^{(p_n,q_n)}((t-x)^2;x)$ and taking $\delta = \sqrt{\delta_n}$, we finally get the result. □

5. Graphical illustrations

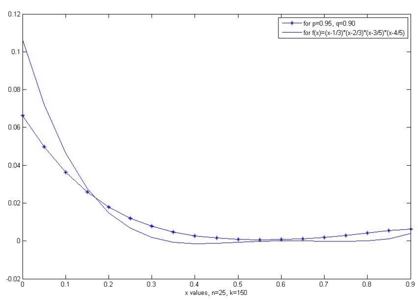
In this section, we show the approximation by (p, q)-Meyer-König-ZellerKantrovich operators using MATLAB programming for functions $f(x) = (x-2/3)(x-4/5)$, $(x-1/4)(x-2/3)(x-4/5)$, $(x-1/3)(x-2/3)(x-3/5)(x-4/5)$, and $(x-1/3)(x-2/3)(x-3/5)(x-4/5)(x-5/7)$ taking $n = 25$ and $k = 150$.



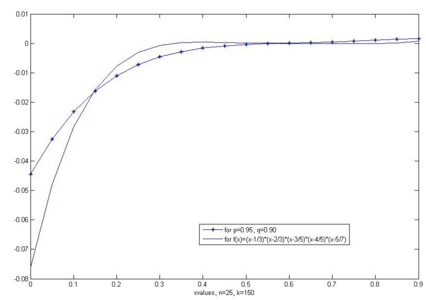
(A) $f(x) = (x-2/3)(x-4/5)$



(B) f(x)=(x-1/4)(x-2/3)(x-4/5)



(C) $f(x) = (x-1/3)(x-2/3)(x-3/5)$ $(x-4/5)$



(D) $f(x) = (x-1/3)(x-2/3)(x-3/5)$ $(x-4/5)(x-5/7)$

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