Khayyam J. Math. 5 (2019), no. 1, 125–139
DOI: 10.22034/kjm.2019.81226

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ON TWO GENERATION METHODS FOR THE SIMPLE LINEAR GROUP $PSL(3,5)$

AYOUB B.M. BASHEER[1*] AND THEKISO SERETLO[1]

Communicated by J. Moori

Abstract. A finite group G is said to be (l,m,n)-*generated*, if it is a quotient group of the triangle group $T(l,m,n) = \langle x,y,z | x^l = y^m = z^n = xyz = 1 \rangle$. In [Nova J. Algebra and Geometry, **2** (1993), no. 3, 277–285], Moori posed the question of finding all the (p,q,r) triples, where p, q, and r are prime numbers, such that a nonabelian finite simple group G is (p,q,r)-generated. Also for a finite simple group G and a conjugacy class X of G, the *rank* of X in G is defined to be the minimal number of elements of X generating G. In this paper, we investigate these two generational problems for the group $PSL(3,5)$, where we will determine the (p,q,r)-generations and the ranks of the classes of $PSL(3,5)$. We approach these kind of generations using the structure constant method. GAP [The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.9.3;* 2018. (http://www.gap-system.org)] is used in our computations.

1. Introduction

The problem of generation of finite groups has great interest and has many applications to groups and their representations. The classification of finite simple groups is involved heavily and plays a pivotal role in most general results on the generation of finite groups. The study of generating sets in finite groups has a rich history, with numerous applications. We are interested in two kinds of generations of a finite simple group G, namely, the (p,q,r)-generation and the ranks of conjugacy classes of G.

A finite group G is said to be (l,m,n)-*generated*, if $G = \langle x, y \rangle$, with $o(x) = l$, $o(y) = m$ and $o(z) = n$, where $z = (xy)^{-1}$. Here $[x] = lX$ is the conjugacy

Date: Received: 31 July 2018; Revised: 7 October 2018; Accepted: 12 December 2018.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 20C15, 20C40; Secondary 20D08.
Key words and phrases. Conjugacy classes, (p,q,r)-generation, rank, structure constant.

class of x in G, and the elements in this class are of order l; similarly for the classes $[y] = mY$ and $[z] = nZ$. In this case, G is also a quotient group of the triangular group $T(l, m, n)$, and by definition of the triangular group, G is also a $(\sigma(l), \sigma(m), \sigma(n))$-generated group for any $\sigma \in S_3$. Therefore we may assume that $l \leq m \leq n$. In a series of papers [14, 15, 16, 17, 18, 21, 24], Moori and Ganief established all possible (p, q, r)-generations, $p,\ q$, and r are distinct primes, of the sporadic groups $J_1,\ J_2,\ J_3,\ HS,\ McL,\ Co_3,\ Co_2$, and F_{22}. Ashrafi in [1, 2] did the same for the sporadic groups He and HN. Also Darafsheh and Ashrafi established in [11, 10, 13, 12], the (p, q, r)-generations of the sporadic groups $Co_1,\ Ru,\ O'N$ and Ly. The authors in [6] and [7] established the (p, q, r)-generations of the Mathieu sporadic group M_{22} and the alternating group A_{10}, respectively.

From another side, for a finite simple group G and nontrivial class nX of G, the *rank* of nX in G, denoted by $\text{rank}(G : nX)$, is defined to be the minimal number of elements of nX generating G. One of the applications of ranks of conjugacy classes of a finite group is that they are involved in the computations of the covering number of the finite simple group. We recall from Zisser [26] that for a finite simple group G, the *covering number* of G is the smallest integer n such that $C^n = G$, for all nontrivial conjugacy classes C of G and by C^n we mean $\{c_1c_2 \cdots c_n |\ c_1, c_2, \ldots, c_n \in C\}$. In [22, 23, 25], Moori computed the ranks of the involutry classes of the Fischer sporadic simple group Fi_{22}. He found that $\text{rank}(Fi_{22}{:}2B) = \text{rank}(Fi_{22}{:}2C) = 3$, while $\text{rank}(Fi_{22}{:}2A) \in \{5, 6\}$. The work of Hall and Soicher [20] implies that $\text{rank}(Fi_{22}{:}2A) = 6$. Then in a considerable number of publications (see the list of references of [4]) various authors explored the ranks for many of the sporadic simple groups.

The motivation for studying the (p, q, r)-generations and the ranks of classes in a finite simple group G is outlined in the above mentioned papers and the reader is encouraged to consult these papers for background material as well as basic computational techniques.

This paper intends to be a continuation to the above series on simple groups, where we will establish all the (p, q, r)-generations together with the ranks of the conjugacy classes of the projective special linear group $PSL(3, 5)$. Note that, in general, if G is a $(2, 2, n)$-generated group, then G is a dihedral group and therefore G is not simple. Also by [8], if G is a nonabelian (l, m, n)-generated group, then either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Thus for our purpose of establishing the (p, q, r)-generations of $G = PSL(3, 5)$, the only cases we need to consider are when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Therefore excluding the triples $(2, 2, p)$ and those that do not satisfy the condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, we remain with 544 triples (p, q, r), $p \leq q \leq r$ to consider. We found that out of these 544 triples, 490 of them generate $PSL(3, 5)$. The main result on the (p, q, r)-generations of the projective special linear group $PSL(3, 5)$ can be summarized in the following theorem.

Theorem 1.1. *Let* $T := \{A, B, C, D, E, F, G, H, I, J\}$. *The projective special linear group* $PSL(3, 5)$ *is generated by all the triples* (pX, qY, rZ), *where* $p,\ q$, *and* r *are primes dividing* $|PSL(3, 5)|$ *if and only if* (pX, qY, rZ) *is one of the following triples:*

(1) $(2A, 3A, 31X)$; $(2A, 5B, 31X)$; $(2A, 31X, 31Y)$, $X, Y \in T$;

(2) $(3A,3A,5B)$; $(3A,3A,31X)$, $X\in T$; $(3A,5X,31Y)$, $X\in\{A,B\}, Y\in T$; $(3A,31X,31Y)$, $X,Y\in T$;
(3) $(5B,5B,5B)$; $(5X,5B,31Y)$, $X\in\{A,B\}$, $Y\in T$; $(5A,31X,31Y)$, $X,Y\in T$, $X\neq Y$; $(5B,31X,31Y)$, $X,Y\in T$;
(4) $(31X,31Y31Z)$, $X,Y,Z\in T$.

The proof of Theorem 1.1 will be done through sequence of propositions that will be established in Subsections 3.1, 3.2, and 3.3.

Also the main result on the ranks of nontrivial classes of G can be summarized in the following theorem.

Theorem 1.2. *Let G be the projective special linear group $PSL(3,5)$. Then*

(1) $\text{rank}(G{:}2A)=\text{rank}(G{:}4A)=\text{rank}(G{:}4B)=\text{rank}(G{:}5A)=3$,
(2) $\text{rank}(G{:}nX)=2$ *for all* $nX\notin\{1A,2A,4A,4B,5A\}$.

The proof of Theorem 1.2 will be established in Propositions 4.1, 4.3, and 4.5.

In [3], the first author determined the ranks of the classes of the group A_{10}, using the structure constant method. In this paper, we use the same technique to determine the (p,q,r)-generations and ranks of conjugacy classes of $PSL(3,5)$. Therefore for the notation, description of the structure constant method and known results, we follow precisely [3, 4, 5, 6].

2. The projective special linear group $PSL(3,5)$

The projective special linear group $PSL(3,5)$ is a simple group of order $372000 = 2^5\times 3\times 5^3\times 31$. By the Atlas [9], the group $PSL(3,5)$ has exactly 30 conjugacy classes of its elements, of which 14 of these classes have elements of prime orders. These are the classes $2A$, $3A$, $5A$, $5B$, $31A$, $31B$, $31C$, $31D$, $31E$, $31F$, $31G$, $31H$, $31I$, and $31J$. Also $PSL(3,5)$ has 5 conjugacy classes of maximal subgroups, where representatives of these classes of maximal subgroups can be taken as follows:

$H_1=5^2{:}GL(5,2)$ $\quad H_2=5^2{:}GL(5,2)$ $\quad H_3=S_5$
$H_4=4^2{:}S_3$ $\quad H_5=31{:}3$

Throughout this paper and unless otherwise stated, by G we always mean the projective special linear group $PSL(3,5)$. For a subgroup H of G containing a fixed element g such that $\gcd(o(g),[N_G(H){:}H])=1$, we let $h(g,H)$ be the number of conjugates of H in G containing g. This number is given by $\chi_H(g)$, where χ_H is the permutation character of G with action on the conjugates of H. Using [4, Theorem 2.2], we computed the values of $h(g,H_i)$ for all the nonidentity classes of elements and all the maximal subgroups H_i, $1\leq i\leq 5$, of G, and we list these values in Table 1.

3. The (p,q,r)-generations of the $PSL(3,5)$

In this section, we investigate all the generation of $PSL(3,5):=G$ by the triples (pX,qY,rZ), where p, q and r are primes that divide the order of G, that is, $p,\ q,\ r\in\{2,3,5,31\}$.

TABLE 1. The values $h(g, H_i)$, $1 \leq i \leq 5$, for nonidentity classes and maximal subgroups of $PSL(3, 5)$

	H_1	H_2	H_3	H_4	H_5
$2A$	7	7	100	75	0
$3A$	1	1	4	8	16
$4A$	7	7	0	15	0
$4B$	7	7	0	15	0
$4C$	3	3	4	3	0
$5A$	6	6	0	0	0
$5B$	1	1	5	0	0
$6A$	1	1	4	0	0
$8A$	1	1	0	3	0
$8B$	1	1	0	3	0
$10A$	2	2	0	0	0
$12A$	1	1	0	0	0
$12B$	1	1	0	0	0
$20A$	2	2	0	0	0
$20B$	2	2	0	0	0
$24A$	1	1	0	0	0
$24B$	1	1	0	0	0
$24C$	1	1	0	0	0
$24D$	1	1	0	0	0
$31A$	0	0	0	0	1
$31B$	0	0	0	0	1
$31C$	0	0	0	0	1
$31D$	0	0	0	0	1
$31E$	0	0	0	0	1
$31F$	0	0	0	0	1
$31G$	0	0	0	0	1
$31H$	0	0	0	0	1
$31I$	0	0	0	0	1
$31J$	0	0	0	0	1

3.1. **The $(2, q, r)$-generations of G.** The $(2, q, r)$-generations comprise three cases, namely $(2, 3, r)$-, $(2, 5, r)$-, and $(2, 31, r)$-generations. The condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, implies that if G is $(2A, 3A, rZ)$-generated, then we must have $r > 6$, that is, $r = 31$. Throughout the paper, we assume that T is the same as in Theorem 1.1, that is, $T = \{A, B, C, D, E, F, G, H, I, J\}$.

Proposition 3.1. *The group G is $(2A, 3A, 31X)$-generated for $X \in T$.*

Proof. The computations with GAP [19] show that $\Delta_G(2A, 3A, 31X) = 31$, for all $X \in T$. From Table 1, we can see that only $H_5 = 31{:}3$ is the maximal subgroup of G that contains elements of order 31. However we can see that the order of H_5 is odd and thus there is no fusion from this subgroup into the class $2A$ of G. It follows that there is no contribution from any maximal subgroup of G to $\Delta^*_G(2A, 3A, 31X)$ for any $X \in T$. Thus $\Delta^*_G(2A, 3A, 31X) = \Delta_G(2A, 3A, 31X) =$

31 for all $X \in T$. Hence G is generated by all the triples $(2A, 3A, 31X)$ for $X \in T$. □

Proposition 3.2. *The group G is neither $(2A, 5A, 5X)$- nor $(2A, 5A, 31Y)$-generated for all $X \in \{A, B\}$ and $Y \in T$.*

Proof. The GAP computations reveals that $\Delta_G(2A, 5A, 5X) = \Delta_G(2A, 5A, 31Y) = 0$ for all $X \in \{A, B\}$ and $Y \in T$. Hence the result. □

Proposition 3.3. *The group G is not $(2A, 5B, 5B)$-generated.*

Proof. Firstly note from Table 1 that there are three maximal subgroups of G with elements that fuse to class $5B$ of G. These are the subgroups $H_1 = H_2 = 5^2{:}GL(2,5)$ and $H_3 = S_5$. Now the intersection of any two of these maximal subgroups are as follows: $H_1 \cap H_2 \cong GL(2,5)$, $H_1 \cap H_3 \cong D_8$, and $H_2 \cap H_3 \cong 5{:}4$. There is no fusion from any class of $H_1 \cap H_2 \cong GL(2,5)$ into the class $5B$ of G, while we can see that $H_1 \cap H_3 \cong D_8$ has no element of order 5. For the group $H_2 \cap H_3 \cong 5{:}4$ there are only one class of invloutions and only one class of elements of order 5, and both these classes fuse to classes $2A$ and $5B$ of G, respectively. Also $H_1 \cap H_2 \cap H_3 \cong \mathbb{Z}_2$, which is clearly has no element of order 5. The computations show that $h(5B, 5{:}4) = 5$ and $\sum_{5:4}(2a, 5a, 5a) = 0$. Finally the computations also show that $\Delta_G(2A, 5B, 5B) = 25$, $\sum_{H_1}(2a, 5a, 5a) + \sum_{H_1}(2b, 5a, 5a) = 0 + 0 = 0$, $\sum_{H_2}(2a, 5a, 5a) + \sum_{H_2}(2b, 5a, 5a) = 0+0 = 0$, and $\sum_{H_3}(2a, 5a, 5a) + \sum_{H_3}(2b, 5a, 5a) = 0+5 = 5$. Now from Table 1, we have $h(5B, H_1) = h(5B, H_2) = 1$ and $h(5B, H_3) = 5$. It follows that

$$\begin{aligned}\Delta^*_G(2A, 5B, 5B) &= \Delta_G(2A, 5B, 5B) - 1 \times \sum_{H_1}(2a, 5a, 5a) - 1 \times \sum_{H_2}(2a, 5a, 5a) \\ &\quad -5 \times \sum_{H_3}(2a, 5a, 5a) + 5 \times \sum_{5:4}(2a, 5a, 5a) \\ &= 25 - 0 - 0 - 25 + 0 = 0,\end{aligned}$$

and hence G is not generated by $(2A, 5B, 5B)$. □

Proposition 3.4. *The group G is $(2A, 5B, 31X)$-generated for all $X \in T$.*

Proof. The computations with GAP show that $\Delta_G(2A, 5B, 31X) = 31$ for all $X \in T$. From Table 1, we can see that only $H_5 = 31{:}3$ is the maximal subgroup of G that contains elements of order 31. However we can see that the order of H_5 is neither divisible by 2 nor by 5, and thus there is no fusion from classes of H_5 into the classes $2A$ and $5B$ of G. It follows that there is no contribution from any maximal subgroup of G to $\Delta^*_G(2A, 5B, 31X)$ for any $X \in T$. Thus $\Delta^*_G(2A, 5B, 31X) = \Delta_G(2A, 5B, 31X) = 31$ for all $X \in T$. Hence G is generated by all the triples $(2A, 5B, 31X)$ for $X \in T$. □

We now look at the last case of the $(2, q, r)$-generations, namely, the $(2, 31, 31)$-generations.

Proposition 3.5. *The group G is $(2A, 31X, 31Y)$-generated for all $X,\ Y \in T$.*

Proof. The computations with GAP show that $\Delta_G(2A, 31X, 31Y) = 31$ for all $X,\ Y \in T$. The treatment is same as in Propositions 3.1 and 3.4, and thus $\Delta^*_G(2A, 31X, 31Y) = \Delta_G(2A, 31X, 31Y) = 31$ for all $X,\ Y \in T$. Hence G is generated by all the triples $(2A, 31X, 31Y)$ for $X,\ Y \in T$. □

3.2. **The $(3, q, r)$-generations of G.** In this subsection, we consider all the $(3, q, r)$-generations, which constitutes the cases $(3, 3, r)$-, $(3, 5, r)$-, and $(3, 31, r)$-generations. The condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, implies that if G is $(3A, 3A, rZ)$-generated, then we must have $r > 3$, that is, $r = 5$ or $r = 31$.

Proposition 3.6. *The group G is not $(3A, 3A, 5A)$-generated.*

Proof. From Table 1, we can see that only $H_1 = H_2 = 5^2{:}GL(2, 5)$ is the maximal subgroup of G that contains elements of orders 3 and 5, and there are fusions into $3A$ and $5A$. We also know that $H_1 \cap H_2 \cong GL(2, 5)$, which has fusions into both $3A$ and $5A$. Now the computations give that $\Delta_G(3A, 3A, 5A) = 1000$, $\sum_{H_1}(3a, 3a, 5a) = \sum_{H_2}(3a, 3a, 5a) = 500$, $\sum_{H_1 \cap H_2}(3a, 3a, 5a) = 4$, and $h(5A, H_1 \cap H_2) = 25$. From Table 1, we also have $h(5A, H_1) = h(5A, H_2) = 6$. It follows that

$$\begin{aligned}\Delta^*_G(3A, 3A, 5A) &= \Delta_G(3A, 3A, 5A) - 6 \times \sum_{H_1}(3a, 3a, 5a) - 6 \times \sum_{H_2}(3a, 3a, 5a) \\ &\quad +25 \times \sum_{H_1 \cap H_2}(3a, 3a, 5a) = 1000 - 3000 - 3000 + 100 < 0,\end{aligned}$$

showing the nongeneration of G by $(3A, 3A, 5A)$. □

Proposition 3.7. *The group G is $(3A, 3A, 5B)$-generated.*

Proof. Here we have three maximal subgroups of G which are involved, namely $H_1 = H_2$ and H_3. We know from the proof of Proposition 3.3 that $H_1 \cap H_2 \cong GL(2, 5)$, $H_1 \cap H_3 \cong D_8$, and $H_2 \cap H_3 \cong 5{:}4$. The unique class of elements of order 5 in $H_1 \cap H_2 \cong GL(2, 5)$ fuses into the class $5A$ of G. We can see that the order of $H_1 \cap H_3 \cong D_8$ is not divisible by 5, while the order of $H_2 \cap H_3 \cong 5{:}4$ is not divisible by 3. Also the order of $H_1 \cap H_2 \cap H_3 \cong \mathbb{Z}_2$ is neither divisible by 3 nor by 5. We conclude that there will be no contribution from the intersections of H_1, H_2, and H_3 (pairwise or the three of them) in the computations of $\Delta^*_G(3A, 3A, 5B)$. Now the computations show that $\Delta_G(3A, 3A, 5B) = 625$, $\sum_{H_1}(3a, 3a, 5b) = \sum_{H_2}(3a, 3a, 5b) = 0$, and $\sum_{H_3}(3a, 3a, 5b) = 5$. From Table 1, we also have $h(5B, H_1) = h(5B, H_2) = 1$, $h(5B, H_3) = 5$, and $h(5B, \mathbb{Z}_2) = 0$. It follows that

$$\begin{aligned}\Delta^*_G(3A, 3A, 5B) &= \Delta_G(3A, 3A, 5B) - 1 \times \sum_{H_1}(3a, 3a, 5b) - 1 \times \sum_{H_2}(3a, 3a, 5b) \\ &\quad -5 \times \sum_{H_3}(3a, 3a, 5b) + 0 \times \sum_{\mathbb{Z}_2}(3a, 3a, 5b) \\ &= 625 - 0 - 0 - 25 + 0 = 600 > 0,\end{aligned}$$

establishing the generation of G by $(3A, 5A, 5A)$. □

Proposition 3.8. *The group G is $(3A, 3A, 31X)$-generated for all $X \in T$.*

Proof. From Table 1, we can see that $H_5 = 31{:}3$ is the only maximal subgroup of G containing elements of order 31 and also has fusions into the class $3A$ of G. In addition to the identity class $1a$, the group H_5 has two classes of elements of order 3, namely $3a$ and $3b$; and has 10 conjugacy classes of elements of order 31, namely $31a$, $31b$, $31c$, $31d$, $31e$, $31f$, $31g$, $31h$, $31i$ and $31j$. Let $M := \{a, b, c, d, e, f, g, h, i, j\}$. For class $3a$ of H_5 and with the aid of GAP, we find that $\sum_{H_5}(3a, 3a, 31x) = 0$ for all $x \in M \setminus \{a\}$, where $\sum_{H_5}(3a, 3a, 31a) = 31$, while for class $3b$ of H_5 we found that $\sum_{H_5}(3b, 3b, 31x) = 0$ for all $x \in M$. We also found that $\Delta_G(3A, 3A, 31X) = 651$ for all $X \in T$. We can also see from Table 1 that $h(31X, H_5) = 1$ for all $X \in T$. It follows that

$$\begin{aligned}\Delta^*_G(3A, 3A, 31X) &= \Delta_G(3A, 3A, 31X) - 1 \times \sum_{H_5}(3a, 3a, 31x) \\ &= 651 - 31 = 620 > 0.\end{aligned}$$

Hence G is generated by all the triples $(3A, 3A, 31X)$ for $X \in T$. □

Next we turn to look at the $(3, 5, r)$-generations.

Proposition 3.9. *The group G is not $(3A, 5A, 5X)$-generated for $X \in \{A, B\}$.*

Proof. The computations with GAP show that $\Delta_G(3A, 5A, 5A) = 125$, and from the Atlas [9], we can see that $|C_G(5A)| = 500$, where by $C_G(nX)$ we mean the centralizer of a representative of class nX of G. Now the nongeneration of G by $(3A, 5A, 5A)$ follows by [4, Lemma 2.7].

For the other case $(3A, 5A, 5B)$, we can see from Table 1 that only $H_1 = H_2 = 5^2{:}GL(2, 5)$ are the maximal subgroups of G that have fusions into the classes $3A$, $5A$, and $5B$ of G. In fact each of H_1 and H_2 has two classes of elements of order 5 that fuse to class $5A$ of G, one class of elements of order 5 that fuse to class $5B$ of G and one class of elements of order 3 that fuse to class $3A$ of G. The intersection of H_1 and H_2 has no element of order 5 that fuses to class $5B$ of G, and thus $\sum_{H_1 \cap H_2}(3a, 5a, 5b) = 0$. Now the computations with GAP reveal $\Delta_G(3A, 5A, 5B) = 50$, $\sum_{H_1}(3a, 5a, 5b) + \sum_{H_1}(3a, 5a, 5b) = 0 + 25 = 25$, $\sum_{H_2}(3a, 5a, 5b) + \sum_{H_2}(3a, 5a, 5b) = 0 + 25 = 25$. Also from Table 1, we have $h(5B, H_1) = h(5B, H_2) = 1$. Therefore, we get

$$\begin{aligned}\Delta^*_G(3A, 5A, 5B) &= \Delta_G(3A, 5A, 5B) - 1 \times \Big(\sum_{H_1}(3a, 5a, 5b) + \sum_{H_1}(3a, 5a, 5b)\Big) \\ &\quad -1 \times \Big(\sum_{H_2}(3a, 5a, 5b) + \sum_{H_2}(3a, 5a, 5b)\Big) = 50 - 25 - 25 = 0,\end{aligned}$$

showing the nongeneration of G by $(3A, 5A, 5B)$ and completing the proof. □

Proposition 3.10. *The group G is not $(3A, 5B, 5B)$-generated.*

Proof. In this case three maximal subgroups are involved, namely, H_1, H_2, and H_3. Neither the intersection $H_i \cap H_j$ for $i, j \in \{1, 2, 3\}$ nor $H_1 \cap H_2 \cap H_3$ has classes of elements of orders 3 and 5 such that these classes simultaneously fuse to classes $3A$ and $5B$ of G, respectively. It follows that there will be no contribution from the intersections of any set of maximal subgroups of G in the computations of $\Delta^*_G(3A, 5B, 5B)$. By GAP, we have $\Delta_G(3A, 5B, 5B) = 700$, $\sum_{H_1}(3a, 5b, 5b) = 100$, $\sum_{H_2}(3a, 5b, 5b) = 100$, and $\sum_{H_3}(3a, 5b, 5b) = 10$. We also have $h(5B, H_1) = h(5B, H_2) = 1$ and $h(5B, H_3) = 5$. It renders that

$$\begin{aligned}\Delta^*_G(3A, 5B, 5B) &= \Delta_G(3A, 5B, 5B) - 1 \times \sum_{H_1}(3a, 5b, 5b) - 1 \times \sum_{H_2}(3a, 5b, 5b) \\ &\quad -5 \times \sum_{H_2}(3a, 5b, 5b) = 700 - 100 - 100 - 50 = 450 > 0.\end{aligned}$$

Hence G is a $(3A, 5B, 5B)$-generated group. □

Proposition 3.11. *The group G is $(3A, 5X, 31Y)$-generated for $X \in \{A, B\}$ and $Y \in T$.*

Proof. From Table 1, we can see that $H_5 = 31{:}3$ is the only maximal subgroup of G that contains elements of order 31. Clearly order of H_5 is not divisible by 5, and thus there is no contribution by any maximal subgroup of G in the computations of $\Delta^*_G(3A, 5X, 31Y)$ for $X \in \{A, B\}$ and $Y \in T$, that is, $\Delta^*_G(3A, 5X, 31Y) = \Delta_G(3A, 5X, 31Y)$ for $X \in \{A, B\}$ and $Y \in T$. Now the computations show that $\Delta_G(3A, 5A, 31Y) = 31$ and $\Delta_G(3A, 5B, 31Y) = 620$ for all $Y \in T$. Hence G is a $(3A, 5X, 31Y)$-generated group for $X \in \{A, B\}$ and $Y \in T$. □

The last part of our investigation on the $(3, q, r)$-generations of G is to look at the $(3, 31, 31)$-generations, which is the context of the next proposition.

Proposition 3.12. *The group G is $(3A, 31X, 31Y)$-generated for all X, $Y \in T$.*

Proof. The computations with GAP show that $\Delta_G(3A, 31X, 31Y) = 496$ for all X, $Y \in T$. Again H_5 is the only maximal subgroup of G with classes that fuse to classes $3A$ and $31X$ of G for $X \in T$. By GAP, we obtained that $\sum_{H_5}(3x, 31y, 31z) = 0$, for $x \in \{a, b\}$ and $y, z \in M$, where M is the same as in the proof of Proposition 3.8. Thus $\Delta^*_G(3A, 31X, 31Y) = \Delta_G(3A, 31X, 31Y) = 496$ for all X, $Y \in T$. Hence G is generated by all the triples $(3A, 31X, 31Y)$ for X, $Y \in T$. □

3.3. **The $(5, q, r)$- and $(31, q, r)$-generations of G.** In this subsection, we look at the $(5, q, r)$-generations of G, which comprise of the cases $(5, 5, r)$- and $(5, 31, r)$-generations.

Proposition 3.13. *The group G is neither $(5A, 5A, 5X)$- nor $(5A, 5A, 31Y)$-generated for $X \in \{A, B\}$ and $Y \in T$.*

Proof. The GAP computations give $\Delta_G(5A,5A,5A)=43$ and $\Delta_G(5A,5A,5B)=10$. From the Atlas, we can see that $|C_G(5A)|=500$ and $|C_G(5B)|=25$. Now the nongeneration of G by $(3A,5A,5X)$ for $X\in\{A,B\}$ follows by [4, Lemma 2.7]. For the other case of $(5A,5A,31Y)$, for $Y\in T$, the direct computations show that $\Delta_G(5A,5A,31Y)=0$ for all $Y\in T$. Hence the result holds. □

Proposition 3.14. *The group G is not $(5A,5B,5B)$-generated while it is a $(5B,5B,5B)$-generated group.*

Proof. The maximal subgroups of G with elements, that fuse to class $5B$ of G, are H_1, H_2, and H_3, while those maximal subgroups with elements that fuse to both classes $5A$ and $5B$ of G are H_1 and H_2 only. Now we consider the case $(5A,5B,5B)$ firstly. The intersection of H_1 and H_2 has no element of order 5 that fuses to the class $5B$ of G, and thus $\sum_{H_1\cap H_2}(5a,5b,5b)=0$. Using GAP, we get $\Delta_G(5A,5B,5B)=34$, $\sum_{H_1}(5a,5b,5b)+\sum_{H_1}(5c,5b,5b)=19+15=34$, $\sum_{H_2}(5a,5b,5b)+\sum_{H_2}(5c,5b,5b)=19+15=34$. We also have $h(5B,H_1)=h(5B,H_2)=1$. It follows that

$$\begin{aligned}\Delta^*_G(5A,5B,5B) &= \Delta_G(5A,5B,5B)-1\times(\sum_{H_1}(5a,5b,5b)+\sum_{H_1}(5c,5b,5b))\\ &\quad -1\times(\sum_{H_2}(5a,5b,5b)+\sum_{H_2}(5c,5b,5b))\\ &= 34-34-34=-34<0,\end{aligned}$$

showing the nongeneration of G by $(5A,5B,5B)$.

For the other case $(5B,5B,5B)$, the intersection of H_1 and H_2 has no element of order 5 that fuses to class $5B$ of G. The intersection of H_1 and H_3 has no element of order 5 at all, and the intersection of H_2 and H_3 has no element of order 5 that fuses to class $5A$ of G. The intersection of H_1, H_2, and H_3 is $\mathbb{Z}_2$. Therefore there is no any contribution from the intersection of any subgroups of G in the computations of $\Delta^*_G(5B,5B,5B)$. Using GAP, we obtained that $\Delta_G(5B,5B,5B)=670$, $\sum_{H_1}(5b,5b,5b)=45$, $\sum_{H_2}(5b,5b,5b)=45$, and $\sum_{H_3}(5b,5b,5b)=8$. We also have $h(5B,H_3)=5$. Therefore we get

$$\begin{aligned}\Delta^*_G(5B,5B,5B) &= \Delta_G(5B,5B,5B)-1\times\sum_{H_1}(5b,5b,5b)-1\times\sum_{H_2}(5b,5b,5b)\\ &\quad -5\times\sum_{H_2}(5b,5b,5b)=670-45-45-40=540>0.\end{aligned}$$

Hence G is a $(5B,5B,5B)$-generated group. □

Proposition 3.15. *The group G is $(5X,5B,31Y)$-generated for $X\in\{A,B\}$ and $Y\in T$.*

Proof. For the triple $(5A, 5B, 31Y)$, we can see from Table 1 that $H_5 = 31{:}3$ is the only maximal subgroup of G containing elements of order 31. However H_5 does not contain elements of order 5. Thus there will be no contribution by any maximal subgroup of G in the computations of $\Delta^*_G(5A, 5B, 31Y)$ for $Y \in T$, that is, $\Delta^*_G(5A, 5B, 31Y) = \Delta_G(5A, 5B, 31Y)$ for $Y \in T$. Now the computations show that $\Delta_G(5A, 5B, 31Y) = 31$ for all $Y \in T$. Hence G is a $(5A, 5B, 31Y)$-generated group for $Y \in T$.

For the other case $(5B, 5B, 31Z)$, $Z \in T$, we recall from Proposition 3.4 that G is a $(2A, 5B, 31Z)$-generated group for all $Z \in T$. It follows by [4, Lemma 2.5] that G is also a $(5B, 5B, (31Z)^2)$-generated group for all $Z \in T$, that is, G is a $(5B, 5B, 31Y)$-generated group for all $Y \in T$, where $(31Z)^2 = 31Y$. Hence the result holds. □

The last part of this subsection is to study the $(5, 31, r)$-generations of G.

Proposition 3.16. *For $X \in T$, the group G is not $(5A, 31X, 31X)$-generated, while for $X, Y \in T$ and $X \neq Y$, the group G is $(5A, 31X, 31Y)$-generated.*

Proof. The direct computations show that $\Delta_G(5A, 31X, 31X) = 0$ for all $X \in T$. Thus G is not generated by $(5A, 31X, 31X)$ for $X \in T$.
For the case $(5A, 31X, 31Y)$, where $X, Y \in T$ and $X \neq Y$, the computations show that $\Delta_G(5A, 31X, 31Y) = 31$. We know that H_5 is the only maximal subgroup of G that has elements of order 31. However it does not contains elements of orders 5. Thus there will be no contribution from any maximal subgroup of G in the computations of $\Delta^*_G(5A, 31X, 31Y)$, that is, $\Delta^*_G(5A, 31X, 31Y) = \Delta_G(5A, 31X, 31Y) = 31$, establishing the generation of G by $(5A, 31X, 31Y)$ for $X, Y \in T$ and $X \neq Y$. □

Proposition 3.17. *The group G is $(5B, 31X, 31Y)$-generated for all $X, Y \in T$.*

Proof. As in the proof of Proposition 3.15, there will be no contribution from any maximal subgroup of G in the computations of $\Delta^*_G(5B, 31X, 31Y)$, that is, $\Delta^*_G(5B, 31X, 31Y) = \Delta_G(5B, 31X, 31Y)$. The computations with GAP show that $\Delta_G(5B, 31X, 31X) = 620$ while $\Delta_G(5B, 31X, 31Y) = 465$, for $X \neq Y$, and both X and Y are in T. This establishes the generation of G by $(5B, 31X, 31Y)$ for $X, Y \in T$. □

Finally we handle the case $(31, q, r)$-generation of G. This comprises of only the case $(31, 31, 31)$.

Proposition 3.18. *The group G is $(31X, 31Y, 31Z)$-generated for all $X, Y, Z \in T$.*

Proof. Using GAP, we obtained that $\Delta_G(31X, 31Y, 31Z) \in \{341, 466, 591\}$. Now H_5 is the only maximal subgroup of G that has elements of order 31. It has 10 conjugacy classes of elements of order 31, where each class fuses into a class of elements of order 31 in G. Again the computations with GAP show that $\sum_{H_5}(31a, 31b, 31c) \in \{0, 1, 2\}$. Since $h(31X, H_5) = 1$ for all $X \in T$, it follows that

$$\begin{aligned}\Delta_G^*(31X,31Y,31Z) &= \Delta_G(31X,31Y,31Z) - 1\times\sum_{H_5}(31x,31y,31z)\\ &\in \{339,340,341,464,465,466,589,590,591\}.\end{aligned}$$

Therefore $\Delta_G^*(31X,31Y,31Z) > 0$, and hence G is generated by all the triples $(31X,31Y,31Z)$ for all $X,Y,Z\in T$. □

4. The ranks of the classes of $PSL(3,5)$

In this section, we determine the ranks for all the nontrivial conjugacy classes of elements of the group $PSL(3,5)$.

We start our investigation on the ranks of the nontrivial classes of $PSL(3,5) := G$ by looking at the unique class of involutions $2A$. It is well-known that two involutions generate a dihedral group. Thus the lower bound of the rank of an involutry class in a finite group $G \neq D_{2n}$ (the dihedral group of order $2n$) is 3. The following proposition gives the rank of class $2A$ in G.

Proposition 4.1. $\text{rank}(G{:}2A) = 3$.

Proof. By Proposition 3.1, we have G is a $(2A,3A,31X)$-generated group, for all $X\in T$, where T as in the previous section. It follows by [4, Lemma 2.3], that G is a $(2A,2A,2A,(31X)^3)$-generated group, that is, G is a $(2A,2A,2A,31Y)$-generated group for some $Y\in T$. Therefore $\text{rank}(G:2A)\leq 3$. Since $\text{rank}(G:2A)\neq 2$, it follows that $\text{rank}(G:2A)=3$. □

Lemma 4.2. $\text{rank}(G{:}nX)\neq 2$, *for* $nX\in\{4A,4B,5A\}$.

Proof. For the classes $4A$ and $4B$ of G, let $M := \{3A,4A,4B,4C,5A,5B,6A,8A,8B,24A,24B,24C,24D,31A,31B,31C,31D,31E,31F,31G,31H,31I,31J\}$. The direct computations show that $\Delta_G(4A,4A,mY)=0$ and $\Delta_G(4B,4B,kZ)=0$ for all $mY\in M\setminus\{4B\}$ and $kZ\in M\setminus\{4A\}$. Thus G is neither a $(4A,4A,mY)$- nor $(4B,4B,kZ)$-generated group for all $mY\in M\setminus\{4B\}$ and $kZ\in M\setminus\{4A\}$. Also we have

$$\begin{aligned}
\Delta_G(4A,4A,2A) &= \Delta_G(4B,4B,2A) = 49 < 480 = |C_G(2A)|,\\
\Delta_G(4A,4A,4B) &= \Delta_G(4B,4B,4A) = 30 < 480 = |C_G(4B)| = |C_G(4A)|,\\
\Delta_G(4A,4A,10A) &= \Delta_G(4B,4B,10A) = 4 < 20 = |C_G(10A)|,\\
\Delta_G(4A,4A,12A) &= \Delta_G(4B,4B,12A) = 6 < 24 = |C_G(12A)|,\\
\Delta_G(4A,4A,12B) &= \Delta_G(4B,4B,12B) = 6 < 24 = |C_G(12B)|,\\
\Delta_G(4A,4A,20B) &= \Delta_G(4B,4B,20A) = 10 < 20 = |C_G(20B)| = |C_G(20A)|,\\
\Delta_G(4A,4A,20A) &= \Delta_G(4B,4B,20B) = 5 < 20 = |C_G(20A)| = |C_G(20B)|.
\end{aligned}$$

Thus by [4, Lemma 2.7], G is neither a $(4A,4A,mY)$- nor $(4B,4B,kZ)$-generated group for $mY\in\{2A,4B,10A,12A,12B,20A,20B\}$ and $kZ\in\{2A,4A,10A,12A,12B,20A,20B\}$. It follows that G cannot be generated by only two elements from class $4A$ or $4B$.

For the class $nX = 5A$, the direct computations show that $\Delta_G(5A, 5A, mY) = 0$, for all nontrivial classes of G except for $mY \in \{3A, 4C, 5A, 5B, 6A, 10A\} := K$. For $mY \in K$, we have

$$\begin{aligned}
\Delta_G(5A, 5A, 3A) &= 6 < 24 = |C_G(3A)|,\\
\Delta_G(5A, 5A, 4C) &= 4 < 16 = |C_G(4C)|,\\
\Delta_G(5A, 5A, 5A) &= 43 < 500 = |C_G(5A)|,\\
\Delta_G(5A, 5A, 5B) &= 10 < 25 = |C_G(5B)|,\\
\Delta_G(5A, 5A, 6A) &= 6 < 24 = |C_G(6A)|,\\
\Delta_G(5A, 5A, 10A) &= 5 < 20 = |C_G(10A)|.
\end{aligned}$$

It follows that G is not a $(5A, 5A, mY)$-generated group for any nontrivial class mY of G, and hence $\text{rank}(G{:}5A) \neq 2$. This completes the result for all $nX \in \{4A, 4B, 5A\}$. □

Proposition 4.3. $\text{rank}(G{:}4A) = \text{rank}(G{:}4B) = \text{rank}(G{:}5A) = 3$.

Proof. For $nX \in \{4A, 4B, 5A\}$, the computations show that $\Delta_G(nX, nX, nX, 31A) = 961$. From Table 1, we can see that $h(31A, H_i) = 0$ for all $i \in \{1, 2, 3, 4\}$, while $h(31A, H_5) = 1$. Also the direct computations show that $\sum_{H_5}(nX, nX, nX, 31A) = 0$ for $nX \in \{4A, 4B, 5A\}$. It follows that

$$\begin{aligned}
\Delta^*_G(nX, nX, nX, 31A) &= \Delta_G(nX, nX, nX, 31A)\\
&\quad - \sum_{i=1}^{5} h(g_i, H_i) \sum_{H_i}(nX, nX, nX, 31A)\\
&= 961 - 0 = 961,
\end{aligned}$$

showing the generation of G by $(nX, nX, nX, 31A)$. Hence the result holds. □

Remark 4.4. The result of Proposition 4.3 can also be established by the results of [4, Section 2] as follows. Let

$$\begin{aligned}
a &:= (2,4,8)(3,7,31)(5,12,17)(6,16,23)(9,11,22)(10,13,21)(14,18,30)\\
&\quad\ (15,25,28)(19,29,27)(20,26,24) \in 3A,\\
b &:= (1,3,2,13)(4,22,18,30)(5,21,25,23)(7,17,14,27)(8,19,24,29)\\
&\quad\ (10,16,26,28) \in 4A,\\
c &:= (1,3,13,9)(4,15,22,18)(5,25,20,23)(7,27,14,17)(8,19,29,11)\\
&\quad\ (10,16,28,12) \in 4B,\\
d &:= (1,2,3,6,13)(4,26,27,23,19)(5,22,7,24,28)(8,25,14,16,30)\\
&\quad\ (11,31,15,20,12) \in 5A.
\end{aligned}$$

Then $\langle a,b\rangle = \langle a,c\rangle = \langle a,d\rangle = G$ with

$$\begin{aligned} ab &= (1,3,17,21,16,5,12,14,30,27,24,20,28,15,23,6,26,29,7,31,2,\\ &\quad 22,9,11,18,4,19,8,13,25,10) \in 31A,\\ ac &= (1,3,27,29,14,4,19,11,18,30,17,25,12,7,31,13,21,16,5,10,9,\\ &\quad 8,2,15,20,26,24,23,6,28,22) \in 31A \text{ and}\\ ad &= (1,2,26,28,20,27,4,25,5,11,7,15,14,18,8,3,24,12,17,22,9,31,\\ &\quad 6,30,16,19,29,23,13,21,10) \in 31A. \end{aligned}$$

Thus G is a $(3A, nX, 31A)$-generated group for $nX \in \{4A, 4B, 5A\}$. This implies that it is also a $(nX, 3A, 31A)$-generated group. By [4, Lemma 2.3], it follows that G is a $(nX, nX, nX, (31A)^3)$-generated group for $nX \in \{4A, 4B, 5A\}$. Thus $\text{rank}(G{:}nX) \leq 3$, but by Lemma 4.2, we know that $\text{rank}(G{:}nX) \neq 2$, thus $\text{rank}(G{:}nX) = 3$ for $nX \in \{4A, 4B, 5A\}$.

TABLE 2. Some information on the classes $nX \in S$

	Δ_G	$h(31A, H_5)$	$\sum_{H_5}$	$h(31A, H_5)\sum_{H_5}$	Δ^*_G
$3A$	651	1	62	62	589
$4C$	1271	1	0	0	1271
$5B$	589	1	0	0	589
$6A$	651	1	0	0	651
$8A$	651	1	0	0	651
$8B$	651	1	0	0	651
$10A$	899	1	0	0	899
$12A$	651	1	0	0	651
$12B$	651	1	0	0	651
$20A$	899	1	0	0	899
$20B$	899	1	0	0	899
$24B$	651	1	0	0	651
$24A$	651	1	0	0	651
$24C$	651	1	0	0	651
$24D$	651	1	0	0	651
$31A$	341	1	0	0	341
$31B$	591	1	2	2	589
$31C$	466	1	1	1	465
$31D$	341	1	0	0	341
$31E$	341	1	0	0	341
$31F$	341	1	0	0	341
$31G$	341	1	0	0	341
$31H$	341	1	0	0	341
$31I$	341	1	0	0	341
$31J$	341	1	0	0	341

Proposition 4.5. *Let S be the set* $\{3A, 4C, 5B, 6A, 8A, 8B, 10A, 12A, 12B, 20A, 20B, 24A, 24B, 24C, 24D, 31A, 31B, 31C, 31D, 31E, 31F, 31G, 31H, 31I, 31J\}$. *Then* $\text{rank}(G{:}nX) = 2$ *for all* $nX \in S$.

Proof. The aim here is to show that G is an $(nX, nX, 31A)$-generated group for all $nX \in S$. We recall from Table 1, that $H_5 = 31{:}3$ is the only maximal subgroup of G containing elements of order 31. Now for $nX \in S$, we give in Table 2 some information about $\Delta_G(nX, nX, 31A) := \Delta_G$, $h(31A, H_5)$, $\sum_{H_5}(nX, nX, 31A) := \sum_{H_5}$, and $\Delta^*_G(nX, nX, 31A) := \Delta^*_G$. The last column of Table 2 establishes the generation of G by the triple $(nX, nX, 31A)$ for all $nX \in S$. It follows that $\text{rank}(G{:}nX) = 2$ for all $nX \in S$. □

Remark 4.6. For all $nX \in S$ of Proposition 4.5, it is possible show that G is a $(2A, nX, 31A)$-generated group. Now the result follows by [4, Corollary 2.6].

Acknowledgments

The authors would like to thank the University of Limpopo and the National Research Foundation (NRF) of South Africa for the financial support received.

References

1. A.R. Ashrafi, *Generating pairs for the Held group He*, J. Appl. Math. Comput. **10** (2002), no. 1-2, 167–174.
2. A.R. Ashrafi, *(p,q,r)-generations of the sporadic group HN*, Taiwanese J. Math. **10** (2006), no. 3, 613–629.
3. A.B.M. Basheer, *The ranks of the classes of* A_{10}, Bull. Iranian Math. Soc. **43** (2017), no. 7, 2125–2135.
4. A.B.M. Basheer, J. Moori, *On the ranks of finite simple groups*, Khayyam J. Math. **2** (2016), no. 1, 18–24.
5. A.B.M. Basheer, J. Moori, *On the ranks of the alternating group* A_n, Bull. Malays. Math. Sci. Soc. (2017) DOI 10.1007/s40840-017-0586-5.
6. A.B.M. Basheer, T.T. Seretlo, *The* (p, q, r)*-generations of the Mathieu group* M_{22}, Southeast Asian Bull. Math. to appear.
7. A.B.M. Basheer, T.T. Seretlo, *The* (p, q, r)*-generations of the alternating group* A_{10}, Quaest. Math. accepted.
8. M.D.E. Conder, *Some results on quotients of triangle groups*, Bull. Austral. Math. Soc. **30** (1984) 73–90.
9. J.H. Conway, R.T. Curtis, S.P.Norton, R.A. Parker, R.A. Wilson, *Atlas of Finite Groups, Maximal subgroups and ordinary characters for simple groups*, With computational assistance from J.G. Thackray, Oxford University Press, Eynsham, 1985.
10. M.R. Darafsheh, A.R. Ashrafi, *Generating pairs for the sporadic group Ru*, J. Appl. Math. Comput. **12** (2003), no. 1-2, 143–154.
11. M.R. Darafsheh, A.R. Ashrafi, G.A. Moghani, (p, q, r)*-generations of the Conway group* Co_1*, for odd* p, Kumamoto J. Math. **14** (2001) 1–20.
12. M.R. Darafsheh, A.R. Ashrafi, G.A. Moghani, *(p,q,r)-generations of the sporadic group O'N*, London Math. Soc. Lecture Note Ser. Groups St. Andrews in Oxford, Cambridge Univ. Press, **304** (2001), 101–109.
13. M.R. Darafsheh, A.R. Ashrafi, G.A. Moghani, *(p,q,r)-generations and nX-complementary generations of the sporadic group Ly*, Kumamoto J. Math. **16** (2003) 13–25.
14. S. Ganief, J. Moori, *2-generations of the smallest Fischer group* Fi_{22}, Nova J. Math. Game Theory Algebra **6** (1997), no. 2-3, 127–145.
15. S. Ganeif, J. Moori, *(p,q,r)-generations and nX-complementary generations of the sporadic groups HS and McL.*, J. Algebra **188** (1997), no. 2, 531–546.
16. S. Ganief, J. Moori, *(p,q,r)-generations of the smallest Conway group* Co_3, J. Algebra **188** (1997), no. 2, 516–530.
17. S. Ganief, J. Moori, *Generating pairs for the Conway groups* Co_2 *and* Co_3, J. Group Theory **1** (1998), no. 3, 237–256.

18. S. Ganief, J. Moori, *2-generations of the fourth Janko group J_4*, J. Algebra **212** (1999), no. 3, 305–322.
19. The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.9.3;* 2018. (http://www.gap-system.org)
20. J.I. Hall, L.H. Soicher, *Presentations of some 3-transposition groups,* Comm. Algebra **23** (1995), 2517–2559.
21. J. Moori, *(p, q, r)-generations for the Janko groups J_1 and J_2*, Nova J. Algebra Geom. **2** (1993), no. 3, 277–285.
22. J. Moori, *Generating sets for F_{22} and its automorphism group*, J. Algebra **159** (1993) 488–499.
23. J. Moori, *Subgroups of 3-transposition groups generated by four 3-transpositions,* Quaest. Math. **17** (1994) 483–494.
24. J. Moori, *$(2, 3, p)$-generations for the Fischer group F_{22}*, Comm. Algebra **2** (1994), no. 11, 4597–4610.
25. J. Moori, *On the ranks of the Fischer group F_{22}*, Math. Japonica **43** (1996) 365–367.
26. I. Zisser, *The covering numbers of the sporadic simple groups*, Israel J. Math. **67** (1989) 217–224.

[1] School of Mathematical and Computer Sciences, University of Limpopo (Turfloop), P Bag X1106, Sovenga 0727, South Africa

E-mail address: ayoubbasheer@gmail.com; ayoub.basheer@ul.ac.za

E-mail address: thekisoseretlo@gmail.com; thekiso.seretlo@ul.ac.za