Khayyam J. Math. 5 (2019), no. 1, 140–149
DOI: 10.22034/kjm.2019.81231

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

COEFFICIENTS OF BI-UNIVALENT FUNCTIONS INVOLVING PSEUDO-STARLIKENESS ASSOCIATED WITH CHEBYSHEV POLYNOMIALS

IBRAHIM T. AWOLERE[1*], ABIODUN T. OLADIPO[2]

Communicated by A.K. Mirmostafaee

Abstract. In the present paper, a new subclass of analytic and bi-univalent functions by means of pseudo starlike function, activation function and Chebyshev polynomials are introduced. Coefficient bounds for functions belonging to the said subclass are obtained. Relevance connection of our class to second Hankel determinants is established.

1. Introduction and preliminaries

Let A denote the class of analytic functions in the open unit disk

$$E = \{z : |z| < 1\}$$

and let H denote the subclass of A, which satisfies the normalization condition $f(0) = f'(0) = 1$ and has the series form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

A function f is said to be univalent on the domain E if it is injective on E. The normalized analytic function defined by (1.1) is said to be starlike, convex, and close-to-convex with the provision that the geometric quantities

$$\frac{zf'(z)}{f(z)}, \quad 1 + \frac{zf''(z)}{f'(z)}, \quad f'(z),$$

Date: Received: 11 June 2018; Revised: 31 October 2018; Accepted 16 December 2018.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C650, 30C80.
Key words and phrases. Univalent functions, Chebyshev polynomials, pseudo starlike function, Hankel determinant.

respectively, belong to the class P (*Caratheodory functions)*, the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

where $p(0) = 0$ and $\mathrm{Re}(p(z)) > 0$ (that is functions with positive real part). The well-known Koebe one-quarter theorem established that the image of E under every univalent function $f \in A$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \qquad (z \in E)$$

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4. \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in E if both f and f^{-1} are univalent in E.

Let Σ denote the class of bi-univalent functions in E given by (1.1). In 1967 Lewin [14] introduced the concept of bi-univalent analytic functions and proved that the second coefficient satisfies $|a_2| < 1.51$. Later, Brannan and Clunie [6] asserted $|a_2| < \sqrt{2}$ while Netanyahu [17] showed $\max_f \in \Sigma \, |a_2| = \frac{4}{3}$. The coefficient problem for each of the following Taylor–Maclaurin coefficients

$$|a_n| \leq n, \quad n \in N \setminus \{1,2\}, \quad N = \{1,2,3,\ldots\},$$

is still an open problem (see, [6, 7, 14, 17] for details). Recently, many authors [1, 2, 11, 12, 15, 16, 18, 19] introduced and investigated several interesting subclasses of bi-univalent functions. Moreover, they have obtained nonsharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$.

The Chebyshev polynomials are a sequence of orthogonal polynomials which are related to the De Moires formula and which are defined recursively. Chebyshev polynomials continuously find applications both in practical and theoretical points of view [3, 8, 10]. There are four kinds of Chebyshev polynomials, we shall concern ourselves with $U_k(t)$ polynomials of the second kind which is defined by

$$U_k(t) = \frac{\sin(k+1)\alpha}{\sin\alpha}, \quad t \in [-1,1],$$

where k denotes the degree of the polynomial and $t = \cos\alpha$. Chebyshev polynomials of the second kind have the generating function of the form

$$H(z,t) = \frac{1}{1 - 2tz + z^2} = 1 + \sum_{k=1}^{\infty} \frac{\sin(k+1)\alpha}{\sin\alpha} z^n.$$

Note that if $t = \cos\alpha$, $\alpha \in (\frac{-\pi}{3}, \frac{\pi}{3})$, then

$$H(z,t) = \frac{1}{1 - 2\cos\alpha z + z^2} = 1 + \sum_{k=1}^{\infty} \frac{\sin(k+1)\alpha}{\sin\alpha} z^n.$$

Using [10], we state that

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots, \quad (z \in E, \ \ t \in [-1,1]),$$

where

$$U_{k-1} = \frac{\sin(k \ \text{arc}\cos t)}{\sqrt{1-t^2}}, \quad k \in N$$

is the Chebyshev polynomial of the second kind. It is well known that

$$U_k(t) = 2tU_{k-1}(t) - U_{k-2}(t),$$

so that

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t.$$

Recently, Fadipe et al. in [10] considered the function

$$f_\gamma(z) = z + \sum_{k=2}^{\infty} \gamma(s) a_k z^k, \tag{1.3}$$

where $\gamma(s) = \frac{2}{1+e^{-s}}$ is the sigmoid activation function and $s \geq 0$. Functions of the form (1.3) belong to the class A_γ, where $A_1 = A$.

Motivated by the recent work of Rajya et al. [13] and previous works of Babalola [5] and Awolere et al. [4], in this work, we introduce a new subclasses of Σ associated with Chebyshev Polynomials and obtain the initial Taylor coefficient $|a_2|$, $|a_3|$ and $|a_4|$ and second Hankel determinants by means of subordination.

Lemma 1.1 (See [9])**.** *If $w(z) = b_1 z + b_2 z^2 + \cdots,\ b_1 \neq 0$, is analytic and satisfies $|w(z)| < 1$ on the unit disk E, then for each $0 < r < 1, |w'(z)| < 1$ and $\left|w(re^{i\theta})\right| < 1$ unless $w(z) = e^{i\theta} z$ for some real number θ.*

2. Main results

Definition 2.1. A function $f \in \Sigma$ is said to be in the class $HT_\Sigma^\gamma(\lambda, \phi(z,t), \gamma(s)),\ \lambda \geq 1,\ \ \gamma(s) = \frac{2}{1+e^{-s}},\ \ s \geq 0,\ t \in [-1,1]$, if it satisfies the following conditions:

$$\frac{z\left[f_\gamma'(z)\right]^\lambda}{f_\gamma(z)} \prec \phi(z,t), \quad z \in E,$$

$$\frac{w\left[g_\gamma'(w)\right]^\lambda}{g_\gamma(w)} \prec \phi(w,t), \quad w \in E,$$

where g is an extension of $f^{-1} \in E$.

Remark 2.2. If $\phi(z,t) = \left(\frac{1}{1-2tz+z^2}\right)^\beta$, then the class $HT_\Sigma^\gamma(\lambda, \phi(z,t), \gamma(s))$ reduces to the class $HT_\Sigma^\gamma(\lambda, \beta, \gamma(s)),\ \ 0 < \beta \leq 1$ and satisfies the following conditions:

$$\left|\arg \frac{z\left[f_\gamma'(z)\right]^\lambda}{f_\gamma(z)}\right| < \frac{\beta\pi}{2}, \quad z \in E,$$

$$\left|\arg \frac{w\left[g_\gamma'(w)\right]^\lambda}{g_\gamma(w)}\right| < \frac{\beta\pi}{2}, \quad w \in E,$$

where g is an extension of $f^{-1} \in E$.

Remark 2.3. If $s = 0$, then the class $HT^{\gamma}_{\Sigma}(\lambda, \phi(z,t), \gamma(s))$ reduces to the class $ST^{\gamma}_{\Sigma}(\lambda, \phi(z,t))$ and satisfies the following conditions:

$$\frac{z\left[f'_{\gamma}(z)\right]}{f_{\gamma}(z)} \prec \phi(z,t), \quad z \in E,$$

$$\frac{w\left[g'_{\gamma}(w)\right]}{g_{\gamma}(w)} \prec \phi(w,t), \quad w \in E,$$

where g is an extension of $f^{-1} \in E$.

Theorem 2.4. *Let $f \in HT^{\gamma}_{\Sigma}(\lambda, \phi(z,t), \gamma(s))$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|4t^2(2\lambda^2-\lambda)-(2\lambda-1)^2(4t^2-1)\right|\gamma^2(s)}},$$

$$|a_3| \leq \frac{4t^2}{(2\lambda-1)^2\gamma(s)} + \frac{2t}{(3\lambda-1)\gamma(s)},$$

$$|a_4| \leq \frac{4t^3(9-34\lambda-8\lambda^3)}{3(2\lambda-1)^3(4\lambda-1)\gamma(s)} + \frac{10t^3}{(2\lambda-1)^3\gamma(s)} + \frac{10t^2}{(2\lambda-1)(3\lambda-1)\gamma(s)}$$

$$+\frac{2t}{(4\lambda-1)\gamma(s)} + \frac{8t^2-2}{(4\lambda-1)\gamma(s)} + \frac{8t^3-4t}{(4\lambda-1)\gamma(s)}.$$

Proof. Since $f \in HT^{\gamma}_{\Sigma}(\lambda, \phi(z,t), \gamma(s))$, there exist two Chebyshev polynomials $\phi(z,t)$ and $\phi(w,t)$ such that

$$\frac{z\left[f'_{\gamma}(z)\right]^{\lambda}}{f_{\gamma}(z)} = \phi(z,t), \tag{2.1}$$

$$\frac{w\left[g'_{\gamma}(w)\right]^{\lambda}}{g_{\gamma}(w)} = \phi(w,t). \tag{2.2}$$

Define the functions $u(z)$ and $v(w)$ by

$$u(z) = c_1 z + c_2 z^2 + \cdots, \tag{2.3}$$

$$v(w) = d_1 w + d_2 w^2 + \cdots, \tag{2.4}$$

which are analytic in D with $u(0) = 0 = v(0)$ and $|u(z)| < 1$, $|v(w)| < 1$ for all $z \in E$. It is well known that

$$|u(z)| = \left|c_1 z + c_2 z^2 + \cdots\right| < 1,$$

$$|v(w)| = \left|d_1 w + d_2 w^2 + \cdots\right| < 1,$$

and

$$|c_i| \leq 1,$$

$$|d_i| \leq 1.$$

Using (2.3) and (2.4) in (2.1) and (2.2), respectively, we have

$$\frac{z\left[f'_\gamma(z)\right]^\lambda}{f_\gamma(z)} = 1 + U_1(t)u(z) + U_2(t)u^2(z) + \cdots \tag{2.5}$$

and

$$\frac{w\left[g'_\gamma(w)\right]^\lambda}{g_\gamma(w)} = 1 + U_1(t)v(w) + U_2(t)v^2(w) + \cdots . \tag{2.6}$$

In the light of (1.1), (1.2), (2.1), and (2.2) and from (2.5) and (2.6), we have

$$\begin{aligned}&1 + (2\lambda-1)\gamma(s)a_2 z + [(3\lambda-1)\gamma(s)a_3 + (2\lambda^2-4\lambda+1)\gamma^2(s)a_2^2]z^2\\ &+\left\{(4\lambda-1)\gamma(s)a_4 + (6\lambda^2-11\lambda+2)\gamma^2(s)a_2a_3 + \left[\frac{4\lambda(\lambda-1)(\lambda-2)}{3} + (4\lambda-2\lambda^2-1)\right]\gamma^3 a_2^3\right\}z^3\\ &= 1 + U_1(t)c_1 z + (c_2U_1(t) + c_1^2U_2(t))z^2 + (c_3U_1(t) + 2c_1c_2U_2(t) + c_1^3U_3(t))z^3 + \cdots\end{aligned}$$

and

$$\begin{aligned}&1 - (2\lambda-1)\gamma(s)a_2 w + [(2\lambda^2+2\lambda-1)\gamma^2(s)a_2^2 - (3\lambda-1)\gamma(s)a_3]w^2\\ &+\left\{-(4\lambda-1)\gamma(s)a_4 + (6\lambda^2+9\lambda-3)\gamma^2(s)a_2a_3 - \left[\frac{4\lambda(\lambda-1)(\lambda-2)}{3} + 10\lambda^2+2\lambda-2\right]\gamma^3 a_2^3\right\}w^3\\ &= 1 + U_1(t)d_1 w + (d_2U_1(t) + d_1^2U_2(t))w^2 + (d_3U_1(t) + 2d_1d_2U_2(t) + d_1^3U_3(t))w^3 + \cdots .\end{aligned}$$

This yields the following relations:

$$(2\lambda-1)\gamma(s)a_2 = U_1(t)c_1, \tag{2.7}$$

$$(3\lambda-1)\gamma(s)a_3 + (2\lambda^2-4\lambda+1)\gamma^2(s)a_2^2 = c_2U_1(t) + c_1^2U_2(t), \tag{2.8}$$

$$\begin{aligned}&(4\lambda-1)\gamma(s)a_4 + (6\lambda^2-11\lambda+2)\gamma^2(s)a_2a_3\\ &+\left[\frac{4\lambda(\lambda-1)(\lambda-2)}{3} + (4\lambda-2\lambda^2-1)\right]\gamma^3a_2^3 = c_3U_1(t) + 2c_1c_2U_2(t) + c_1^3U_3(t),\end{aligned} \tag{2.9}$$

$$-(2\lambda-1)\gamma(s)a_2 = U_1(t)d_1,$$

$$(2\lambda^2+2\lambda-1)\gamma^2(s)a_2^2 - (3\lambda-1)\gamma(s)a_3 = d_2U_1(t) + d_1^2U_2(t), \tag{2.10}$$

and

$$\begin{aligned}&-(4\lambda-1)\gamma(s)a_4 + (6\lambda^2+9\lambda-3)\gamma^2(s)a_2a_3\\ &-\left[\frac{4\lambda(\lambda-1)(\lambda-2)}{3} + 10\lambda^2+2\lambda-2\right]\gamma^3a_2^3 = d_3U_1(t) + 2d_1d_2U_2(t) + d_1^3U_3(t).\end{aligned} \tag{2.11}$$

From (2.7) and (2.9)

$$c_1 = -d_1, \tag{2.12}$$

$$a_2 = \frac{U_1(t)c_1}{(2\lambda-1)\gamma(s)} = \frac{-U_1(t)d_1}{(2\lambda-1)\gamma(s)}, \tag{2.13}$$

and

$$2(2\lambda-1)^2\gamma^2(s)a_2^2 = U_1^2(t)[c_1^2+d_1^2]. \tag{2.14}$$

Adding (2.8) and (2.10) and making use of (2.14), we have

$$2\lambda(2\lambda-1)\gamma^2(s) = U_1(t)(c_2+d_2) + U_2(t)[c_1^2+d_1^2]. \tag{2.15}$$

Upon simplification (2.15), we get

$$a_2^2 = \frac{U_1^3(t)(c_2+d_2)}{[U_1^2(T)(4\lambda^2-2\lambda)-(2\lambda-1)^2U_2(t)]\gamma^2(s)}.$$

By making use of Lemma 1.1, we have

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{|4t^2(2\lambda^2-\lambda)-(2\lambda-1)^2(4t^2-1)|\,\gamma^2(s)}}.$$

From (2.8), (2.11), (2.12), and (2.13), we observe that

$$a_3 = \frac{U_1^2(t)(c_1^2+d_1^2)}{2(2\lambda-1)^2\gamma(s)} + \frac{U_1(t)(c_2-d_2)}{2(3\lambda-1)\gamma(s)}. \tag{2.16}$$

Applying Lemma 1.1 once again, we obtain

$$|a_3| \le \frac{4t^2}{(2\lambda-1)^2\gamma(s)} + \frac{2t}{(3\lambda-1)\gamma(s)}.$$

Now from (2.9) and (2.11), it is evident that

$$a_4 = \frac{(-8\lambda^3-34\lambda+9)U_1^3(t)C_1^3}{6(2\lambda-1)^3(4\lambda-1)\gamma(s)} + \frac{5U_1^3(t)[c_1^2+d_1^2]}{4(2\lambda-1)^3\gamma(s)} + \frac{5U_1^2(t)[C_1C_2-c_1d_2]}{4(2\lambda-1)(3\lambda-1)\gamma(s)}$$

$$+\frac{(c_3-d_3)U_1(t)}{2(4\lambda-1)\gamma(s)} + \frac{(2C_1C_2-2d_1C_2)U_2(t)}{2(4\lambda-1)\gamma(s)} + \frac{(c_1^3-d_1^3)U_3(t)}{2(4\lambda-1)\gamma(s)}.$$

On the application of Lemma 1.1, the above equation yields

$$|a_4| \le \frac{4t^3(9-34\lambda-8\lambda^3)}{3(2\lambda-1)^3(4\lambda-1)\gamma(s)} + \frac{20t^3}{(2\lambda-1)^3\gamma(s)} + \frac{10t^2}{(2\lambda-1)(3\lambda-1)\gamma(s)}$$

$$+\frac{2t}{(4\lambda-1)\gamma(s)} + \frac{8t^2-2}{(4\lambda-1)\gamma(s)} + \frac{8t^3-4t}{(4\lambda-1)\gamma(s)},$$

which completes the proof. □

Corollary 2.5. *Let $f \in HT_\Sigma^\gamma(\lambda, \phi(z,t), \gamma(0))$. Then*

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{|4t^2(2\lambda^2-\lambda)-(2\lambda-1)^2(4t^2-1)|}},$$

$$|a_3| \le \frac{4t^2}{(2\lambda-1)^2} + \frac{2t}{(3\lambda-1)},$$

$$|a_4| \le \frac{4t^3(9-34\lambda-8\lambda^3)}{3(2\lambda-1)^3(4\lambda-1)} + \frac{20t^3}{(2\lambda-1)^3} + \frac{10t^2}{(2\lambda-1)(3\lambda-1)}$$

$$+\frac{2t}{(4\lambda-1)} + \frac{8t^2-2}{(4\lambda-1)} + \frac{8t^3-4t}{(4\lambda-1)}.$$

Corollary 2.6. *Let $f \in HT_{\Sigma}^{\gamma}(1, \phi(z,t), \gamma(s))$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\gamma(s)},$$

$$|a_3| \leq \frac{4t^2}{\gamma(s)} + \frac{t}{\gamma(s)},$$

$$|a_4| \leq \frac{24t^3 + 23t^2 - 2t - 2}{3\gamma(s)}.$$

Corollary 2.7. *Let $f \in HT_{\Sigma}^{\gamma}(1, \phi(z,t), 0)$. Then*

$$|a_2| \leq 2t\sqrt{2t},$$

$$|a_3| \leq 4t^2 + t,$$

$$|a_4| \leq \frac{24t^3 + 23t^2 - 2t - 2}{3}.$$

Corollary 2.8. *Let $f \in HT_{\Sigma}^{\gamma}(1, \phi(z,t), \gamma(1))$. Then*

$$|a_2| \leq \frac{3.718t\sqrt{2t}}{2.718},$$

$$|a_3| \leq \frac{3.718t^2}{2.718} + \frac{3.718t}{5.436},$$

$$|a_4| \leq \frac{3.718[24t^3 + 23t^2 - 2t - 2]}{16.308}.$$

Theorem 2.9. *Let $f \in HT_{\Sigma}^{\gamma}(\lambda, \phi(z,t), \gamma(s))$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{(-56\lambda^3 - 12\lambda^2 + 40\lambda - 33)t^4}{3(2\lambda-1)^4(4\lambda-1)\gamma^2(s)} + \frac{4t^3(8\lambda-3)}{(2\lambda-1)^2(3\lambda-1)\gamma^2(s)}$$
$$+ \frac{t^2(-68\lambda^2 + 44\lambda - 8)}{3(2\lambda-1)(3\lambda-1)^2(4\lambda-1)\gamma^2(s)} - \frac{4t}{(2\lambda-1)(4\lambda-1)\gamma^2(s)}.$$

Proof. Proceed from Theorem 2.4, we have

$$a_2 = \frac{U_1(t)c_1}{(2\lambda-1)\gamma(s)},$$

$$a_3 = \frac{U_1^2(c_1^2 + d_1^2)}{2(2\lambda-1)^2\gamma(s)} + \frac{U_1(t)(c_2 - d_2)}{2(3\lambda-1)\gamma(s)},$$

$$a_4 = \frac{(-8\lambda^3 - 34\lambda + 9)U_1^3(t)C_1^3}{6(2\lambda-1)^3(4\lambda-1)\gamma(s)} + \frac{5U_1^3(t)[c_1^2 + d_1^2]}{4(2\lambda-1)^3\gamma(s)} + \frac{5U_1^2(t)[C_1C_2 - c_1d_2]}{4(2\lambda-1)(3\lambda-1)\gamma(s)}$$
$$+ \frac{(c_3 - d_3)U_1(t)}{2(4\lambda-1)\gamma(s)} + \frac{(2C_1C_2 - 2d_1C_2)U_2(t)}{2(4\lambda-1)\gamma(s)} + \frac{(c_1^3 - d_1^3)U_3(t)}{2(4\lambda-1)\gamma(s)}.$$

Upon substitution for values of a_2, a_3, and a_4, we have

$$\begin{aligned}a_2a_4 - a_3^2 =& \frac{(-8\lambda^3 - 34\lambda + 9)U_1^4(t)c_1^4}{6(2\lambda-1)^4(4\lambda-1)\gamma^2(s)} + \frac{5U_1^3(t)c_1^2(c_2-d_2)}{4(2\lambda-1)^2(3\lambda-1)\gamma^2(s)}\\ &+ \frac{5U_1^4(t)c_1(c_1^2+d_1^2)}{4(2\lambda-1)^2\gamma^2(s)} + \frac{U_1^2(t)c_1[c_3-d_3]}{2(2\lambda-1)(4\lambda-1)\gamma^2(s)}\\ &+ \frac{U_1(t)U_2(t)_1c_1(2c_1c_2-2d_1d_2)}{(2\lambda-1)(4\lambda-1)\gamma^2(s)} + \frac{U_1(t)U_3(t)c_1(C_1^3-d_1^3)}{2(2\lambda-1)(4\lambda-1)\gamma^2(s)}\\ &- \frac{u_1^4(t)C_1^2[C_1^2+d_2^2]^2}{4(2\lambda-1)^4\gamma^2(s)} - \frac{u_1^2(t)[C_2-d_2]^2}{4(3\lambda-1)^2\gamma^2(s)} - \frac{U_1^3(t)[c_1^2+d_1^2][c_2-d_2]}{2(2\lambda-1)^2(3\lambda-1)\gamma^2(s)}.\end{aligned}$$

Applying Lemma 1.1 for the coefficients c_1, d_1, c_2, d_2, c_3, and d_3 yields

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| \leq& \frac{(-56\lambda^3 - 12\lambda^2 + 40\lambda - 33)t^4}{3(2\lambda-1)^4(4\lambda-1)\gamma^2(s)} + \frac{4t^3(8\lambda-3)}{(2\lambda-1)^2(3\lambda-1)\gamma^2(s)}\\ &+ \frac{t^2(-68\lambda^2+44\lambda-8)}{3(2\lambda-1)(3\lambda-1)^2(4\lambda-1)\gamma^2(s)} - \frac{4t}{(2\lambda-1)(4\lambda-1)\gamma^2(s)}\end{aligned}$$

which completes the proof. □

Corollary 2.10. *Let $f \in HT_\Sigma^\gamma(\lambda, \phi(z,t), \gamma(0))$. Then*

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| \leq& \frac{(-56\lambda^3 - 12\lambda^2 + 40\lambda - 33)t^4}{3(2\lambda-1)^4(4\lambda-1)} + \frac{4t^3(8\lambda-3)}{(2\lambda-1)^2(3\lambda-1)}\\ &+ \frac{t^2(-68\lambda^2+44\lambda-8)}{3(2\lambda-1)(3\lambda-1)^2(4\lambda-1)} - \frac{4t}{(2\lambda-1)(4\lambda-1)}\end{aligned}$$

Corollary 2.11. *Let $f \in HT_\Sigma^\gamma(1, \phi(z,t), \gamma(s))$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{39t^4}{9\gamma^2(s)} + \frac{10t^3}{\gamma^2(s)} - \frac{32t^2}{36\gamma^2(s)} - \frac{4t}{3\gamma^2(s)}$$

Theorem 2.12. *Let $f \in HT_\Sigma^\gamma(\lambda, \phi(z,t), \gamma(s))$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{4t}{(3\lambda-1)\gamma(s)}, & |\mu-1| \leq M,\\ \\ \frac{16|\mu-1|t^3}{|4(\lambda(2\lambda-1)-(2\lambda-1)^2)t^2+(2\lambda-1)^2|\gamma^2(s)}, & |\mu-1| \geq M,\end{cases}$$

where

$$M = \frac{|[\lambda(2\lambda-1)-(2\lambda-1)^2]+(2\lambda-1)^2|\,\gamma(s)}{16t^2(3\lambda-1)}.$$

Proof. From (2.16) and (2.13), we get

$$\begin{aligned}a_3 - \mu a_2^2 &= \frac{(1-\mu)U_1^3(t)(c_2+d_2)}{2\left[\lambda(2\lambda-1)U_1^2(t) - (2\lambda-1)^2U_2(t)\right]\gamma^2(s)} + \frac{U_1(t)(c_2-d_2)}{2(3\lambda-1)\gamma(s)}\\ &= U_1(t)\left[\left(h(\mu) + \frac{1}{2(3\lambda-1)\gamma(s)}\right)c_2 + \left(h(\mu) - \frac{1}{2(3\lambda-1)\gamma(s)}\right)d_2\right],\end{aligned}$$

where

$$h(\mu) = \frac{(1-\mu)U_1^2(t)}{2\left[\lambda(2\lambda-1)U_1^2(t) - (2\lambda-1)^2U_2(t)\right]\gamma^2(s)}.$$

□

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[1] Department of Mathematics, Emmanuel Alayande College of Education, P. M. B. 1010, Oyo, Oyo State, Nigeria.

E-mail address: awolereibrahim01@gmail.com

[2] Department of Pure and Applied Mathematics, Ladoke Akintola University of Tecchnology, Ogbomoso, Oyo State, Nigeria.
E-mail address: atoladipo@lautech.ng.edu