Khayyam J. Math. 5 (2019), no. 2, 1–14
DOI: 10.22034/kjm.2019.84141

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

CERTAIN RESULTS ON STARLIKE AND CLOSE-TO-CONVEX FUNCTIONS

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Communicated by A.K. Mirmostafaee

Abstract. Using the technique of differential subordination, we here, obtain certain sufficient conditions for starlike and close-to-convex functions. In most of the results obtained here, the region of variability of the differential operators implying starlikeness and close-to-convexity of analytic functions has been extended. The extended regions of the operators have been shown pictorially.

1. Introduction

Let $\mathcal{A}_n(p)$ denote the class of functions f of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \text{ where } n, p \in \mathbb{N}$$

which are analytic and p-valent in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. A function $f \in \mathcal{A}_n(p)$ is said to be in the class $\mathcal{S}_n^*(p,\ \alpha)$ of p-valent starlike functions of order α in $\mathbb{E}$, if it satisfies

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (0 \leq \alpha < p;\ z \in \mathbb{E}).$$

On the other hand, a function $f \in \mathcal{A}_n(p)$ and $g \in \mathcal{S}_n^*(p,\ 0)$ is said to be in the class $\mathcal{C}_n(p,\ \alpha)$ of p-valently close-to-convex of order α in $\mathbb{E}$, if it satisfies

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha \quad (z \in \mathbb{E};\ 0 \leq \alpha < p).$$

Date: Received: 5 March 2018 ; Revised: 10 January 2019 ; Accepted: 18 January 2019.
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2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C80.
Key words and phrases. starlike function, close-to-convex function, Bazilevič function, differential subordination.

In particular, we write $\mathcal{S}_1^*(1,\ 0)=\mathcal{S}^*$ and $\mathcal{C}_1(1,\ 0)=\mathcal{C}$, where $\mathcal{S}^*$ and $\mathcal{C}$ are the usual subclasses of $\mathcal{A}(=\mathcal{A}_1(1))$ consisting of functions which are starlike and close-to-convex, respectively. We denote $\mathcal{S}_n^*(p,\ 0)=\mathcal{S}_n^*(p)$ and $\mathcal{C}_n(p,\ 0)=\mathcal{C}_n(p)$.

Let $\Phi:\mathbb{C}^2\times\mathbb{E}\to\mathbb{C}$ be an analytic function, let p be an analytic function in $\mathbb{E}$ with $(p(z),zp'(z);z)\in\mathbb{C}^2\times\mathbb{E}$ for all $z\in\mathbb{E}$, and let h be univalent in $\mathbb{E}$. Then the function p is said to satisfy the first order differential subordination if

$$\Phi(p(z),zp'(z);z)\prec h(z),\qquad \Phi(p(0),0;0)=h(0). \tag{1.1}$$

A univalent function q is called a dominant of the differential subordination (1.1), if $p(0)=q(0)$ and $p\prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q}\prec q$ for all dominants q of (1.1), is said to be the best dominant of (1.1).

With the help of the Löewner–Kufarev differential equation, Bazilevič [1] in 1955, gave an explicit construction for a class of functions analytic and univalent in the open unit disk $\mathbb{E}$ and defined it by the following relation:

$$f(z)=\left(\frac{\beta}{1+\alpha^2}\int_0^{\alpha}(h(\zeta)-\alpha i)\zeta^{(-\alpha\beta i/(1+\alpha^2))-1}g(\zeta)^{\beta/(1+\alpha^2)}d\zeta\right)^{(1+\alpha i)/\beta}, \tag{1.2}$$

where $h(z)=1+\sum\limits_{n=1}^{\infty}a_nz^n$ satisfies $\Re\ h(z)>0$ in $\mathbb{E},\ g$ is starlike (w.r.t. origin) in $\mathbb{E},\ \alpha$ is any real number, and $\beta>0$. He showed that each such function is univalent in $\mathbb{E}$, and this class contains the classes $\mathcal{S}^*$ and $\mathcal{C}$. In 1967, Thomas [6] studied the subclass of functions defined in (1.2) for $\alpha=0$ given as

$$f(z)=\left(\beta\int_0^{z}h(\zeta)\zeta^{-1}g^{\beta}(\zeta)d\zeta\right)^{1/\beta},$$

by differentiating this expression. He obtained

$$zf'(z)=(f(z))^{1-\beta}(g(z))^{\beta}h(z),$$

where

$$\Re\left(\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}\right)>0,\quad z\in\mathbb{E}, \tag{1.3}$$

and called a function satisfying (1.3) a Bazilevič function of type β and denoted this subclass by $\mathcal{B}(\beta)$. Thereafter, Singh [5] studied the class $\mathcal{B}(\alpha)$ of Bazilevič functions of type α and a subclass $\mathcal{B}_1(\alpha)$ obtained when the starlike function $g(z)=z$ and gave sharp estimates for the modules of the coefficients. Later on, in 2004, Irmak et al. [3] studied the following class of p-valently Bazilevič functions of type β and order γ.

A function $f\in\mathcal{A}_n(p)$ is said to be p-valently Bazilevič functions of type β $(\beta\geq 0)$ and order γ $(0\leq\gamma<p,\ p\in\mathbb{N})$, if there exists a function g belonging to the class

$\mathcal{S}_n^*(p)$ such that

$$\Re\left(\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}\right) > \gamma, \quad z \in \mathbb{E}.$$

They denoted the class of all such functions by $\mathcal{B}_n(p,\ \beta,\ \gamma)$. In particular, $f \in \mathcal{B}_n(p,1,\alpha) = \mathcal{C}_n(p,\ \gamma)$ the class of p-valently close-to-convex of order γ and $\mathcal{B}_n(p,0,\gamma) = \mathcal{S}_n^*(p,\ \gamma)$. Also note that $\mathcal{B}_1(1,\beta,0) = \mathcal{B}(\beta)$ and that $\mathcal{B}_1(1,\beta,0) = \mathcal{B}_1(\beta)$, when $g(z) = z$. The class $\mathcal{B}_n(p,\beta,\gamma)$ was also studied by Goswami and Bansal in [2] to obtain certain sufficient conditions and angular properties for starlikeness and convexity of the members of this class. They obtained the following result.

Theorem 1.1. *Let $z \in \mathbb{E},\ \beta \geq 0,\ 0 \leq \alpha < p,\ n,p \in \mathbb{N}$. Suppose that $f \in \mathcal{A}_n(p)$ and $g \in \mathcal{S}_n^*(p)$ satisfy anyone of the following inequalities:*

$$\left|\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}\left(1+\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}\right)\right| < p-\alpha, \tag{1.4}$$

$$\left|\frac{1+\dfrac{zf''(z)}{f'(z)}-(1-\beta)\dfrac{zf'(z)}{f(z)}-\beta\dfrac{zg'(z)}{g(z)}}{\dfrac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}}\right| < \frac{p-\alpha}{(2p-\alpha)^2}, \tag{1.5}$$

$$\left|\frac{1+\dfrac{zf''(z)}{f'(z)}-(1-\beta)\dfrac{zf'(z)}{f(z)}-\beta\dfrac{zg'(z)}{g(z)}}{\dfrac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}-p}\right| < \frac{1}{2p-\alpha}, \tag{1.6}$$

$$\left|\frac{\dfrac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}\left(1+\dfrac{zf''(z)}{f'(z)}-(1-\beta)\dfrac{zf'(z)}{f(z)}-\beta\dfrac{zg'(z)}{g(z)}\right)}{\dfrac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}-p}\right| < 1, \tag{1.7}$$

$$\begin{aligned}\left|\delta\left(\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}-p\right)+1+\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}\right|&\\ < \frac{(p-\alpha)(1+\delta(2p-\alpha))}{2p-\alpha}.&\end{aligned} \tag{1.8}$$

Then $f \in \mathcal{B}_n(p,\beta,\alpha)$.

In this paper, we investigate the sufficient conditions for starlikeness and close-to-convexity of the members of the class $\mathcal{B}_n(p,\ \beta,\ \alpha)$ of Bazilevič functions obtained by Goswami et al. in [2]. The objective of the present paper is to investigate the same operators for similar conclusions by using the technique of differential subordination. We notice that the region of variability of different operators has been extended in the present paper.

2. Preliminary

We need the following lemma to prove our main results.

Lemma 2.1 ([4, Theorem 3.4h, p. 132])**.** *Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$, and suppose that either*

(i) *h is convex, or*
(ii) *Q_1 is starlike.*

In addition, assume that

(iii) $\Re\left(\dfrac{zh'(z)}{Q_1(z)}\right) > 0.$

If p is analytic in $\mathbb{E}$, with $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p \prec q$ and q is the best dominant.

3. Main results

Theorem 3.1. *Let $\beta \geq 0$, $\psi \in \mathbb{C} \setminus \{0\}$, n, $p \in \mathbb{N}$ and let q be a univalent function in $\mathbb{E}$ with $q(z) \neq 0$ and*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} + (\psi - 1)\frac{zq'(z)}{q(z)}\right) > 0, \quad z \in \mathbb{E}.$$

Suppose that $f \in \mathcal{A}_n(p)$ and $g \in \mathcal{S}_n^*(p)$ satisfy the differential subordination

$$\left(\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}\right)^{\psi}\left(1 + \frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta\frac{zg'(z)}{g(z)}\right) \prec zq'(z)(q(z))^{\psi-1}, \tag{3.1}$$

where the complex power in (3.1) takes its principal values; then

$$\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}} \prec q(z), \quad z \in \mathbb{E},$$

and q is the best dominant.

Proof. Define $P(z)$ by

$$P(z) = \frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}.$$

Therefore

$$\frac{zP'(z)}{P(z)} = 1 + \frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta\frac{zg'(z)}{g(z)},$$

and (3.1) reduces to

$$zP'(z)(P(z))^{\psi-1} \prec zq'(z)(q(z))^{\psi-1}.$$

Define θ and ϕ by

$$\theta(w) = 0 \ \text{ and } \ \phi(w) = w^{\psi-1},$$

where θ and ϕ are analytic in $\mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Therefore

$$Q_1(z) = zq'(z)\phi(q(z)) = zq'(z)(q(z))^{\psi-1},$$

and $h(z) = \theta(q(z)) + Q_1(z) = Q_1(z)$. A little calculation yields

$$\frac{zQ_1'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + (\psi - 1)\frac{zq'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q_1(z)} = \frac{zQ_1'(z)}{Q_1(z)}.$$

In view of the given condition, we have $Q_1(z)$ is starlike in $\mathbb{E}$ and

$$\Re\left(\frac{zh'(z)}{Q_1(z)}\right) > 0.$$

The proof, now, follows from Lemma 2.1. $\square$

For $\psi = -1$ in Theorem 3.1, selecting $q(z) = \dfrac{1+z}{1-z}$ as a dominant, we can easily check that

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - 2\frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1-z}{1+z}\right) > 0.$$

Hence we have the following result.

Example 3.2. Let $n,\ p \in \mathbb{N}$. If $f \in \mathcal{A}_n(p)$ and $g \in \mathcal{S}_n^*(p)$ satisfy the differential subordination

$$\frac{1 + \dfrac{zf''(z)}{f'(z)} - (1-\beta)\dfrac{zf'(z)}{f(z)} - \beta\dfrac{zg'(z)}{g(z)}}{\dfrac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}} \prec \frac{2z}{(1+z)^2}.$$

Then

$$\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{E},$$

and hence f is a p-valently Bazilevič function of type β $(\beta \geq 0)$.

We have the following observations regarding the above result.

Remark 3.3. (i) Taking $n = 1 = p$ and $\beta = 0$, in Example 3.2, we have the following:
If $f \in \mathcal{A}$ satisfies

$$\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{2z}{(1+z)^2}, \tag{3.2}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{E},$$

and hence we conclude that f is starlike.
By putting $p = 1 = n$ and $\alpha = \beta = 0$, the corresponding result can be obtained from (1.5), as below:
If $f \in \mathcal{A}$ satisfies

$$\left|\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| < \frac{1}{4}, \quad z \in \mathbb{E}, \tag{3.3}$$

then f is starlike.

(ii) Putting $p = 1 = n = \beta$, in Example 3.2, we conclude the following result: If $f \in \mathcal{A}$ and $g \in \mathcal{S}^*$ satisfy the differential subordination

$$\frac{g(z)}{zf'(z)}\left(\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + 1\right) \prec \frac{2z}{(1+z)^2}, \tag{3.4}$$

then

$$\frac{zf'(z)}{g(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{E},$$

hence $f \in \mathcal{C}$.

For $\alpha = 0$ and $\beta = 1 = p = n$, the result below can be obtained from (1.5).

If $f \in \mathcal{A}$ and $g \in \mathcal{S}^*$ satisfy

$$\left|\frac{g(z)}{zf'(z)}\left(\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + 1\right)\right| < \frac{1}{4}, \quad z \in \mathbb{E}, \tag{3.5}$$

then $f \in \mathcal{C}$.

The region of variability of the operator $\dfrac{f(z)}{zf'(z)}\left(1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}\right)$ given in (3.2), is the complex plane except the slit from $\dfrac{1}{2}$ to ∞ to conclude that f is starlike, whereas the region, given in (3.3) for the same operator is the disk centered at origin with radius $\dfrac{1}{4}$ for conclusion f to be starlike. Thus as shown in Figure 1, the region extends in (3.2) over (3.3) to conclude function f to be starlike. There is an extension of the region for close-to-convexity of the operator $\dfrac{g(z)}{zf'(z)}\left(\dfrac{zf''(z)}{f'(z)} - \dfrac{zg'(z)}{g(z)} + 1\right)$ given in (3.4) over the region given in (3.5) as shown in Figure 1.

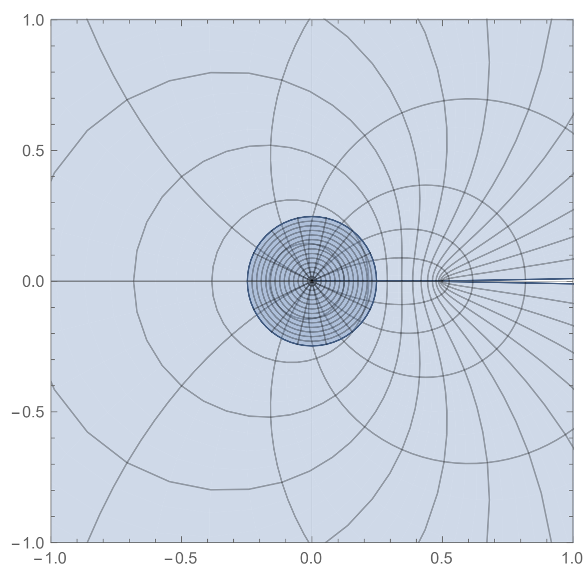

FIGURE 1.

Theorem 3.4. *Let q be a univalent function in $\mathbb{E}$, with $q(z) \neq \{0,\ p\}$ and*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}-\frac{zq'(z)}{q(z)-p}\right)>0.$$

If $f \in \mathcal{A}_n(p)$ and $g \in \mathcal{S}_n^*(p),\ n, p \in \mathbb{N}$, satisfy

$$\frac{1+\dfrac{zf''(z)}{f'(z)}-(1-\beta)\dfrac{zf'(z)}{f(z)}-\beta\dfrac{zg'(z)}{g(z)}}{\dfrac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}-p} \prec \frac{zq'(z)}{q(z)(q(z)-p)}, \tag{3.6}$$

then

$$\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}} \prec q(z), \quad z \in \mathbb{E}, \quad \beta \geq 0,$$

and q is the best dominant.

Proof. Define $P(z)$ by

$$P(z)=\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}.$$

Hence (3.6) reduces to

$$\frac{zP'(z)}{P(z)(P(z)-p)} \prec \frac{zq'(z)}{q(z)(q(z)-p)}.$$

Define θ and ϕ by

$$\theta(w)=0 \ \text{ and } \ \phi(w)=\frac{1}{w(w-p)},$$

where θ and ϕ are analytic in $\mathbb{C}\setminus\{0,p\}$ and $\phi(w)\neq 0,\ w \in \mathbb{C}\setminus\{0,p\}$. Now, the rest of the proof is on the similar lines as the proof of Theorem 3.1. $\square$

Selecting $q(z)=1+\dfrac{2}{3}z^2$ as a dominant in the above result, we can easily check that for $p \geq 2$,

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}-\frac{zq'(z)}{q(z)-p}\right)=\Re\left(\frac{18(1-p)-8z^4}{9(1-p)+6z^2(2-p)+4z^4}\right)>0.$$

Thus, we have the following result.

Example 3.5. For $n, p \in \mathbb{N}$ *and* $\beta \geq 0$, if $f \in \mathcal{A}_n(p)$ and $g \in \mathcal{S}_n^*(p)$ satisfy the following differential subordination:

$$\frac{1+\dfrac{zf''(z)}{f'(z)}-(1-\beta)\dfrac{zf'(z)}{f(z)}-\beta\dfrac{zg'(z)}{g(z)}}{\dfrac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}-p} \prec \frac{12z^2}{(3+2z^2)(3+2z^2-3p)},$$

then

$$\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}} \prec 1+\frac{2}{3}z^2, \quad z \in \mathbb{E};$$

hence $f \in \mathcal{B}_n(p,\beta,0)$ for $p \geq 2$.

We have the following observations.

Remark 3.6. Taking $p = 2$ and $\beta = 0$, in Example 3.5, we have the following result:
If $f \in \mathcal{A}_n(2),\ n \in \mathbb{N}$, satisfies

$$\frac{1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}}{\dfrac{zf'(z)}{f(z)} - 2} \prec \frac{12z^2}{4z^4 - 9}, \tag{3.7}$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{3}z^2, \quad z \in \mathbb{E};$$

hence $f \in \mathcal{S}_n^*(2)$.

The corresponding result can be obtained from (1.6) by setting $\beta = 0,\ \alpha = 0$ and $p = 2$, as below:
If $f \in \mathcal{A}_n(2),\ n \in \mathbb{N}$ satisfies

$$\left| \frac{1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}}{\dfrac{zf'(z)}{f(z)} - 2} \right| < \frac{1}{4}, \tag{3.8}$$

then $f \in \mathcal{S}_n^*(2),\ z \in \mathbb{E}$.

The operator $\dfrac{1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}}{\dfrac{zf'(z)}{f(z)} - 2}$ given in (3.7) takes values in the total shaded region shown in Figure 2, whereas according to (3.8), the same operator takes values in the disk centered at origin with radius $1/4$ to conclude that $f \in \mathcal{S}_n^*(2)$. Thus the region of variability has been extended in (3.7) over (3.8) for $f \in \mathcal{S}_n^*(2)$. Similarly the extension of region can be observed for the operator $\dfrac{1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zg'(z)}{g(z)}}{\dfrac{zf'(z)}{g(z)} - 2}$ to conclude that $f \in \mathcal{C}_n(2)$.

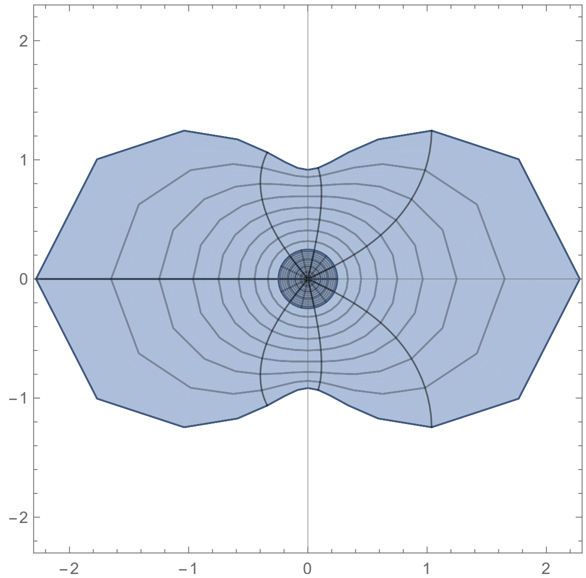

FIGURE 2.

Theorem 3.7. *Let $\beta \geq 0,$ let $n,\ p \in \mathbb{N}$, and let q be a univalent function in $\mathbb{E}$ with $q(z) \neq p$ and*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)-p}\right) > 0.$$

Suppose that $f \in \mathcal{A}_n(p)$ and $g \in \mathcal{S}^*_n(p)$ satisfy

$$\frac{\dfrac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}\left(1 + \dfrac{zf''(z)}{f'(z)} - (1-\beta)\dfrac{zf'(z)}{f(z)} - \beta\dfrac{zg'(z)}{g(z)}\right)}{\dfrac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}} - p} \tag{3.9}$$

$$\prec \frac{zq'(z)}{q(z)-p}, \quad z \in \mathbb{E};$$

then

$$\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}} \prec q(z),$$

and q is the best dominant.

Proof. Define $P(z)$ by

$$P(z) = \frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}.$$

Thus (3.9) reduces to

$$\frac{zP'(z)}{P(z)-p} \prec \frac{zq'(z)}{q(z)-p}.$$

Define θ and ϕ by

$$\theta(w) = 0 \ \text{ and } \ \phi(w) = \frac{1}{w-p},$$

where θ and ϕ are analytic in $\mathbb{C}\setminus\{p\}$ and $\phi(w) \neq 0,\ w \in \mathbb{C}\setminus\{p\}$. Now, the proof of this theorem is the same as the proof of Theorem 3.1. □

Selecting $q(z) = 1 + \frac{2}{3}z^2$ as a dominant in the above theorem, we observe that

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z) - p}\right) = \Re\left(\frac{6(1-p)}{3(1-p) + 2z^2}\right).$$

Obviously, $\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z) - 1}\right) = 0$ for $p = 1$. Selecting $p \geq 2$, we have

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z) - p}\right) > 0.$$

Hence, we get the following result.

Example 3.8. If $f \in \mathcal{A}_n(p)$ and $g \in \mathcal{S}_n^*(p),$ *where* $n, p \in \mathbb{N}$, satisfy the differential subordination

$$\frac{\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}\left(1 + \frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta\frac{zg'(z)}{g(z)}\right)}{\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}} - p} \prec \frac{4z^2}{2z^2 + 3(1-p)},$$

then

$$\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}} \prec 1 + \frac{2}{3}z^2, \quad z \in \mathbb{E},$$

and $f \in \mathcal{B}_n(p, \beta, 0),$ for $p \geq 2$ and $\beta \geq 0$.

We have the following observations.

Remark 3.9. Taking $p = 2$ and $\beta = 0$ in Example 3.8, we have:

If $f \in \mathcal{A}_n(2),$ $n \in \mathbb{N}$ and $f'(z) \neq 0$ satisfies

$$\frac{\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)}{\frac{zf'(z)}{f(z)} - 2} \prec \frac{4z^2}{2z^2 - 3}, \tag{3.10}$$

then $\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{3}z^2,$ $z \in \mathbb{E}$, and hence $f \in \mathcal{S}_n^*(2)$.

By setting $\beta = 0$ and $p = 2$, the result from (1.7) can be obtained as below:

If $f \in \mathcal{A}_n(2),$ $n \in \mathbb{N}$ satisfies

$$\left|\frac{\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)}{\frac{zf'(z)}{f(z)} - 2}\right| < 1, \quad z \in \mathbb{E}. \tag{3.11}$$

then $f \in \mathcal{S}_n^*(2)$.

We observe that according to (3.10), the operator $\dfrac{\dfrac{zf'(z)}{f(z)}\left(1+\dfrac{zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}\right)}{\dfrac{zf'(z)}{f(z)}-2}$ takes the values in the total shaded region except a small portion of unit disk as shown in Figure 3, whereas according to (3.11), same operator takes the values in the unit disk to conclude that $f\in\mathcal{S}_n^*(2)$. Thus combining the results given in (3.10) and (3.11) gives an extension of region for the same conclusion. Similarly there is an extension of the region for operator $\dfrac{\dfrac{zf'(z)}{g(z)}\left(1+\dfrac{zf''(z)}{f'(z)}-\dfrac{zg'(z)}{g(z)}\right)}{\dfrac{zf'(z)}{g(z)}-2}$ to conclude that $f\in\mathcal{C}_n(2)$.

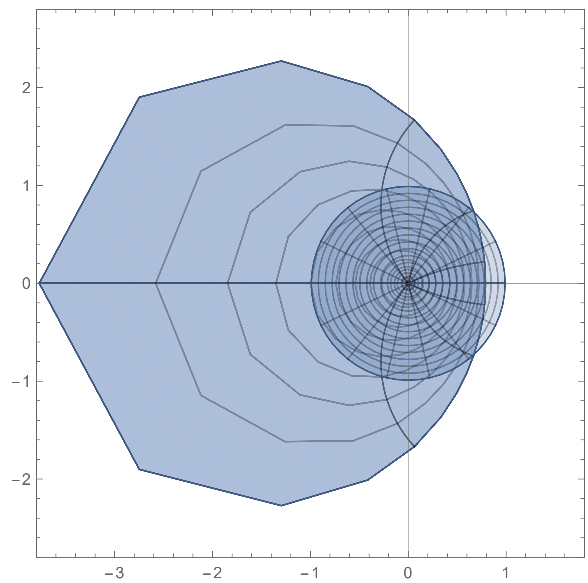

FIGURE 3.

Theorem 3.10. *Let $\beta\geq 0,\ n,p\in\mathbb{N},\ \delta>0$ and q be a univalent function in $\mathbb{E}$, with $\Re q(z)>0$ and*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)>0.$$

If $f\in\mathcal{A}_n(p)$ and $g\in\mathcal{S}_n^*(p)$ satisfy the differential subordination

$$\delta\left(\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}-p\right)+1+\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}$$
$$\prec\delta(q(z)-p)+\frac{zq'(z)}{q(z)},\quad z\in\mathbb{E},\tag{3.12}$$

then

$$\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}\prec q(z),$$

and q is the best dominant.

Proof. Equation (3.12) becomes

$$\delta(P(z)-p)+\frac{zP'(z)}{P(z)} \prec \delta(q(z)-p)+\frac{zq'(z)}{q(z)},$$

where $P(z)$ is defined by $P(z)=\dfrac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}$.
Define θ and ϕ by

$$\theta(w)=\delta(w-p) \ \text{ and } \ \phi(w)=\frac{1}{w},$$

where θ and ϕ are analytic in $\mathbb{C}\setminus\{0\}$ and $\phi(w)\neq 0,\ w\in\mathbb{C}\setminus\{0\}$. Thus the proof, now, follows from Lemma 2.1 on the same lines as in Theorem 3.1. □

Choose $q(z)=\dfrac{1+z}{1-z}$ as a dominant in Theorem 3.10, satisfying the condition $\Re\left(\dfrac{1+z}{1-z}\right)>0$, and

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)=\Re\left(\frac{1+z^2}{1-z^2}\right)>0.$$

Thus, we have the following result.

Example 3.11. For $n,\ p\in\mathbb{N}$ and $\delta>0$, if $f\in\mathcal{A}_n(p)$ and $g\in\mathcal{S}_n^*(p)$, satisfy

$$\delta\left(\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}}-p\right)+1+\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)} \prec \delta\left(\frac{1+z}{1-z}-p\right)+\frac{2z}{1-z^2},$$

then

$$\frac{zf'(z)}{(f(z))^{1-\beta}(g(z))^{\beta}} \prec \frac{1+z}{1-z}, \quad \beta\geq 0, \quad z\in\mathbb{E},$$

and hence $f\in\mathcal{B}_n(p,\beta,0)$.

We observed the following.

Remark 3.12. For $\delta=1=p=n$ and $\beta=0$ in Example 3.11, we get
if $f\in\mathcal{A}$ satisfies

$$\frac{zf''(z)}{f'(z)} \prec \frac{2z(2+z)}{1-z^2}, \tag{3.13}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad z\in\mathbb{E},$$

and hence f is starlike.

The corresponding result, for $\beta=\alpha=0$ and $\delta=1=p=n$, can be obtained from (1.8):
If $f\in\mathcal{A}$ satisfies

$$\left|\frac{zf''(z)}{f'(z)}\right|<\frac{3}{2}, \quad z\in\mathbb{E}, \tag{3.14}$$

then f is starlike.

The operator $\dfrac{zf''(z)}{f'(z)}$ given in (3.13) takes values in the complex plane $\mathbb{C}$ except two slits as shown in Figure 4, whereas according to (3.14), $\dfrac{zf''(z)}{f'(z)}$ takes the values within the disk centered at origin of radius $3/2$ to ensure that f is starlike. Hence the region of variation of the operator $\dfrac{zf''(z)}{f'(z)}$ extends in (3.13) over the region given in (3.14). Similarly, the region of variability of the operator $\dfrac{zf'(z)}{g(z)}+\dfrac{zf''(z)}{f'(z)}-\dfrac{zg'(z)}{g(z)}$ has been extended to conclude that f is close-to-convex.

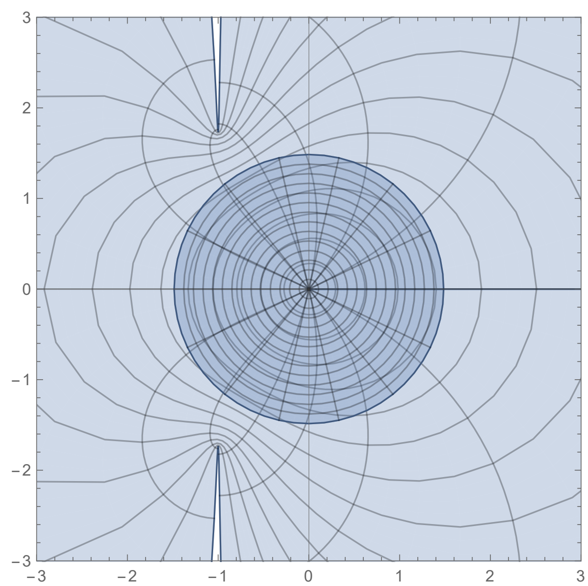

FIGURE 4.

Acknowledgement. The authors are thankful to the reviewers for their valuable comments.

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