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ON GENERAL (α, β)-METRICS WITH SOME CURVATURE PROPERTIES

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Abstract. In this paper, we study a class of Finsler metric called general (α, β) metrics and obtain an equation that characterizes these Finsler metrics of almost vanishing H-curvature. As a consequence of this result, we prove that a general (α, β)-metric has almost vanishing H-curvature if and only if it has almost vanishing Ξ-curvature.

1. Introduction

One of the finest differential geometer of twentieth century, Chern, used to say "Finsler geometry is just Riemannian geometry without the quadratic restriction on its metrics" [7]. In the study of Finsler geometry, we often encounter long and complicated calculations. However, when we consider Finsler metrics with certain symmetries, that would make things much easier. In 1996, Rutz [14] introduced a special class of Finsler metrics called spherically symmetric which is invariant under rotation. In general relativity, the solution of vacuum Einstein field equations describing the gravitational field, which is spherically symmetric, we obtain the Schwarzschild metric in four-dimensional space-time [21]. A Finsler metric F on $B^n(\delta)$ is called spherically symmetric if $F(Ax, Ay) = F(x, y)$, for all $n \times n$ orthogonal matrix A, $x = (x^i) \in B^n(\delta)$ and $y = (y^i) \in T_xB^n(\delta)$. Here $B^n(\delta)$ denotes the Euclidean ball of radius δ around the origin and $T_xB^n(\delta)$ denotes the tangent space of $B^n(\delta)$ at the point x. Zhou [22] proved that a

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Finsler metric F on $B^n(\delta)$ is spherically symmetric if and only if there exists a function $\phi : [0,\delta)\times\mathbb{R}\to\mathbb{R}$ such that

$$F(x,y) = |y|\phi\left(|x|, \frac{\langle x,y\rangle}{|y|}\right), \tag{1.1}$$

where $|.|$ denotes the Euclidean norm and $\langle,\rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$.

The concept of (α,β)-metrics was introduced by Matsumoto in 1972 [9] as a generalization of Randers metrics, and the Randers metrics was introduced by Randers [13]. The (α,β)-metrics are of the form $F=\alpha\phi(s)$, where ϕ is a C^∞ positive function and $s=\frac{\beta}{\alpha}$. In 2012, Yu and Zhu [20] introduced a new class of Finsler metrics, called general (α,β)-Finsler metrics given by $F=\alpha\phi(b^2,s)$, where $\phi=\phi(b^2,s)$ is a C^∞ positive function and $b^2:=\|\beta\|_\alpha^2$. This class of Finsler metrics not only generalize (α,β)-metrics in a natural way, but also includes spherically symmetric Finsler metrics. It is interesting to note that the general (α,β) metric includes an interesting family of Finsler metric constructed by Bryant [5, 4, 3]. Bryant metrics are rectilinear Finsler metrics on the unit sphere S^n with flag curvature $K=1$ defined by

$$F(X,Y) = \Re\left\{\frac{\sqrt{Q(X,X)Q(Y,Y)-Q(X,Y)^2}}{Q(X,X)} - i\frac{Q(X,Y)}{Q(X,X)}\right\},$$

where $Q(X,Y)=x_0y_0+e^{ip_1}x_1y_1+e^{ip_2}x_2y_2+\cdots+e^{ip_n}x_ny_n$ is a complex quadratic form on $\mathbb{R}^{n+1}$ for $n\geq 2$ with the parameters satisfying $0\leq p_1\leq p_2\leq\cdots\leq p_n<\pi$ and $X=(x_0,\ldots,x_n)\in S^n$, $Y=(y_0,\ldots,y_n)\in T_XS^n$.

Shen et al. showed that a Randers metric is locally projectively flat and of constant flag curvature if and only if α is locally projectively flat and β is closed and homothetic with respect to α [2, 15]. The Beltrami theorem says that a Riemannian metric is locally projectively flat if and only if it has a constant sectional curvature. Thus in this case α and β have to satisfy the following:

$${}^\alpha R^i_j = \mu(\alpha^2\delta^i_j - y^iy_j), \quad b_{i|j} = c(x)a_{ij}, \tag{1.2}$$

where ${}^\alpha R^i_j$ denotes the Riemann curvature of the Riemannian metric α and μ is the Ricci constant. Shen [18] also showed that a general (α,β)-metric satisfying (1.2) will be projectively flat if and only if ϕ satisfies $\phi_{ss}=2(\phi_{b^2}-s\phi_{b^2s})$. The spherically symmetric metric F given by (1.1) satisfies the property (1.2) automatically.

The Cartan torsion, the S-curvature, the Ξ-curvature and the H-curvature are the examples of few non-Riemannian quantities in Finsler geometry as they vanish for Riemannian metrics. The S-curvature $S(x,y)$ was introduced by Shen [6, 16] and was defined as follows:

$$S(x,y) = \frac{d}{dt}\left[\tau(\gamma(t),\gamma'(t))\right]_{t=0},$$

where $\tau(x,y)$ is the distortion of the metric F and $\gamma(t)$ is the geodesic with $\gamma(0)=x$ and $\gamma'(0)=y$ on M.

The non-Riemannian quantity Ξ-curvature is denoted by $\Xi = \Xi_j dx^j$ and is defined as

$$\Xi_j := S_{.j|i}y^i - S_{|j},$$

where "$|$" denotes the horizontal covariant derivative and "." denotes the vertical covariant derivative of F [16]. The Finsler metric F is said to have almost vanishing Ξ-curvature if

$$\Xi_j = -(n+1)F^2\left(\frac{\theta}{F}\right)_{y^j}. \tag{1.3}$$

In 1988, Akbar-Zadeh introduced H-curvature which is closely related to the S-curvature [1]. The H-curvature $H_y = H_{ij}dx^i \otimes dx^j$ is defined by

$$H_{ij} = \frac{1}{4}(\Xi_{i.j} + \Xi_{j.i}). \tag{1.4}$$

Also F is said to have almost vanishing H-curvature if

$$H_{ij} = \frac{n+1}{2}\theta F_{y^iy^j}. \tag{1.5}$$

Several authors studied the H-curvature of different class of Finsler metrics [10, 12]. In [11], Mo proved that all spherically symmetric Finsler metrics of almost vanishing H-curvature are of almost vanishing Ξ-curvature and corresponding one forms are exact, generalizing a result previously only known in the case of metrics with vanishing H-curvature. In general, it is difficult to find the Riemann curvature tensor for general (α,β)-metrics. In this paper, we further generalize Mo's result for general (α,β)-metrics under the assumption (1.2) and prove the following results.

Theorem 1.1. *The general (α,β)-metric $F = \alpha\phi(b^2,s)$ satisfying (1.2) has almost vanishing H-curvature if and only if*

$$\alpha s\left[(n+1)\frac{\partial R_1}{\partial s} + 3(b^2-s^2)\frac{\partial R_2}{\partial s} + 2(n+1)R_3\right] = 3(n+1)\theta(\phi - s\phi_s), \quad \theta = \theta_j(x)y^j, \tag{1.6}$$

where R_1, R_2, and R_3 are given in (2.6), (2.9), and (2.8), respectively.

As an application of Theorem 1.1, we have the following corollary.

Corollary 1.2. *For the general (α,β)-metric $F = \alpha\phi(b^2,s)$ satisfying (1.2) the H-curvature almost vanishes if and only if the Ξ-curvature almost vanishes. In this case, the corresponding 1-form θ is an exact form.*

As a consequence of Corollary 1.2, for $\theta = 0$, we get the following corollary.

Corollary 1.3. *For the general (α,β)-metric $F = \alpha\phi(b^2,s)$ satisfying (1.2) the H-curvature vanishes if and only if the Ξ-curvature vanishes.*

A Finsler metric is said to be R-quadratic if its Riemann curvature R_y is quadratic in $y \in T_xM$. These R-quadratic Finsler metrics always have vanishing H-curvature [10]. Together with Corollary 1.3, we have the following.

Corollary 1.4. *The Ξ-curvature of a R-quadratic general (α,β)-metric always vanishes.*

2. PRELIMINARIES

Let M be an n-dimensional smooth manifold. T_xM denotes the tangent space of M at x. The tangent bundle of M is the union of tangent spaces $TM := \bigcup_{x\in M} T_xM$. We denote the elements of TM by (x,y) where $y \in T_xM$ and $TM_0 := TM \setminus \{0\}$.

Definition 2.1 (see [8]). A Finsler metric on M is a function $F : TM \to [0,\infty)$ satisfying the following conditions:

(i) F is C^∞ on TM_0.
(ii) F is a positively 1-homogeneous on the fibers of tangent bundle TM.
(iii) The Hessian of $\frac{F^2}{2}$ with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.

The pair (M,F) is called a Finsler space. The metric F is called the fundamental function and g_{ij} is called the fundamental tensor.

The spray coefficients of the Finsler metric F is defined by

$$G^i = \frac{1}{4}g^{il}\left\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\right\}, \tag{2.1}$$

where $g^{ij} = (g_{ij})^{-1}$.

Definition 2.2 (see[20]). A Finsler metric F on a manifold M is called a general (α,β)-metric, if it can be expressed in the form

$$F = \alpha\phi\left(b^2, s\right), \tag{2.2}$$

for some C^∞ function $\phi(b^2,s)$, where α is a Riemannian metric and β is a 1-form. Also F is called an (α,β)-metric, if F can be expressed as $F = \alpha\phi(s)$ for some C^∞ function $\phi(s)$, Riemannian metric α, and 1-form β.

You and Zhu [20] have proved that a general (α,β)-metric $F = \alpha\phi(b^2,s)$ satisfies

$$\phi - s\phi_s > 0, \quad \phi - s\phi_s + (b^2 - s^2)\phi_{ss} > 0, \quad \text{for } , n \geq 3,$$

or

$$\phi - s\phi_s + (b^2 - s^2)\phi_{ss} > 0, \quad \text{for } n = 2,$$

where s and b are arbitrary numbers with $|s| \leq b < b_0$.

Here ϕ_s denotes the differentiation of ϕ with respect to s.

For a general (α,β)-metric $F = \alpha\phi(b^2,s)$, the fundamental tensor g_{ij} is given by [20]

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1\left(b_i\alpha_{y^j} + b_j\alpha_{y^i}\right) - s\rho_1\alpha_{y^i}\alpha_{y^j}, \tag{2.3}$$

where $\rho = \phi(\phi - s\phi_s)$, $\quad \rho_0 = \phi\phi_{ss} + \phi_s\phi_s, \quad \rho_1 = (\phi - s\phi_s)\phi_s - s\phi\phi_{ss}$.

Moreover,

$$det(g_{ij}) = \phi^{n+1}(\phi - s\phi_s)^{n-2}\left(\phi - s\phi_s + (b^2 - s^2)\phi_{ss}\right)det(a_{ij}),$$

and the inverse metric tensor g^{ij} is given by

$$g^{ij} = \rho^{-1}\left\{a^{ij} + \eta b^i b^j + \eta_0\alpha^{-1}\left(b^i y^j + b^j y^i\right) + \eta_1\alpha^{-2}y^i y^j\right\}, \tag{2.4}$$

where $(a^{ij}) = (a_{ij})^{-1}$, $\quad b^i = a^{ij}b_j$,

$$\eta = -\frac{\phi_{ss}}{\phi - s\phi_s + (b^2 - s^2)\,\phi_{ss}}, \qquad \eta_0 = -\frac{(\phi - s\phi_s)\,\phi_s - s\phi\phi_{ss}}{\phi\,(\phi - s\phi_s + (b^2 - s^2)\,\phi_{ss})},$$

$$\eta_1 = \frac{(s\phi + (b^2 - s^2)\,\phi_s)\,((\phi - s\phi_s)\,\phi_s - s\phi\phi_{ss})}{\phi^2\,(\phi - s\phi_s + (b^2 - s^2)\,\phi_{ss})}.$$

For any $x \in M$ and $y \in T_x(M) \setminus \{0\}$ the Riemann curvature $R_y = R^i_k \frac{\partial}{\partial x^i} \otimes dx^k$ of F is defined by

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k}y^m + 2G^m \frac{\partial^2 G^i}{\partial x^k \partial y^m} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}.$$

The Riemann curvature tensor R^i_j of the general (α, β)-metric under the assumption (1.2) is given by [19]

$$R^i_j = R_1\alpha^2\delta^i_j - sR_2 y_j b^i + R_2\alpha^2 b_j b^i + R_3 b_j y^i + R_4 y_j y^i, \tag{2.5}$$

where

$$R_1 = \mu(1 + s\psi) + c^2[\psi^2 - 2s\psi_{b^2} - \psi_2 + 2\chi(1 + s\psi + u\psi_s)], \tag{2.6}$$

$$R_2 = -\mu(2\chi - s\chi_s) + c^2[2(2\psi_{b^2} - s\psi_{b^2 s}) - \chi_{ss} + 2\chi(2\chi - s\chi_s) + u(2\chi\chi_{ss} - \chi_s^2)] \tag{2.7}$$

$$\begin{aligned} R_3 = -\mu(2\psi - s\psi_s) + c^2[&2(2\psi_{b^2} - s\psi_{b^2 s}) - \psi\psi_s - \psi_{ss} \\ &+2\chi(\psi - s\psi_s + u\psi_{ss}) - \chi_s(1 + ss\psi + u\psi_s)], \end{aligned} \tag{2.8}$$

$$\begin{aligned} R_4 = -\mu[1 - s(\psi - s\psi_s)] &+ c^2[\psi_s + s\psi_{ss} - \psi(\psi - s\psi_s) - 2s(\psi_{b^2} - s\psi_{b^2 s}) \\ -2\chi(1 + s\psi + u\psi_s) &+ s\chi_2(1 + s\psi + u\psi_s) - 2s\chi(\psi - s\psi_s\psi_{ss})], \end{aligned} \tag{2.9}$$

with

$$\chi = \frac{\phi_{ss} - 2(\phi_1 - s\phi_{b^2 s})}{2(\phi - s\phi_s) + (b^2 - s^2)\phi_{ss}}, \qquad \psi = \frac{\phi_s + 2s\phi_{b^2}}{2\phi} - \frac{\chi}{\phi}[s\phi + (b^2 - s^2)\phi_s].$$

One can observe that here

$$c^2 = k - \mu b^2 \qquad \text{and} \qquad R_1 + R_4 + sR_3 = 0 \tag{2.10}$$

for some constant k.
Therefore, using (2.10) in (2.5), we have

$$R^i_j = R_1(\alpha^2\delta^i_j - y_j y^i) + R_2(\alpha b_j - s y_j)\alpha b^i + R_3(\alpha b_j - s y_j)y^i. \tag{2.11}$$

3. The Ξ-curvature and H-curvature of a general (α, β)-metric

In this section, we find the expressions of non-Riemannian quantities Ξ and H of general (α, β) Finsler metrics.

The Ricci curvature Ric is defined by $Ric = R^i_i$, and for a general (α, β)-metrics, Ric can be obtained as

$$\begin{aligned} Ric = R^i_i &= R_1(\alpha^2\delta^i_i - y_i y^i) + R_2(\alpha b_i - s y_i)\alpha b^i + R_3(\alpha b_i - s y_i)y^i \\ &= \alpha^2[(n-1)R_1 + (b^2 - s^2)R_2] \\ &= \alpha^2 R, \end{aligned} \tag{3.1}$$

where

$$R = (n-1)R_1 + (b^2 - s^2)R_2. \tag{3.2}$$

By some simple calculations, we can obtain the following results:

$$\frac{\partial \alpha^2}{\partial y^j} = 2y_j, \quad s_{y^j} = \frac{\alpha b_j - s y_j}{\alpha^2} \tag{3.3}$$

and

$$s_{y^j} y^j = 0, \quad s_{y^j} b^j = \frac{b^2 - s^2}{\alpha}. \tag{3.4}$$

Using (3.2) and (3.3), we have

$$\frac{\partial}{\partial y^j} Ric = \frac{\partial}{\partial y^j} \alpha^2 R = \alpha R_s b_j + (2R - sR_s) y_j, \tag{3.5}$$

where $R_s = \frac{\partial R}{\partial s}$.

Now differentiating (2.11) with respect to y^i and using (3.3) and (3.4) and then taking the summation over i, we have

$$\begin{aligned}\sum_i \frac{\partial R^i_j}{\partial y^i} = {} & \left[R_{1s} + sR_2 + (b^2 - s^2)R_{2s} + (n+1)R_3\right] \alpha b_j \\ & + \left[(1-n)R_1 - sR_{1s} - b^2 R_2 - s(b^2 - s^2)R_{2s} - (n+1)sR_3\right] y_j.\end{aligned} \tag{3.6}$$

Let

$$M = R_{1s} + sR_2 + (b^2 - s^2)R_{2s} + (n+1)R_3$$

and

$$N = (1-n)R_1 - sR_{1s} - b^2 R_2 - s(b^2 - s^2)R_{2s} - (n+1)sR_3.$$

Therefore, (3.6) becomes

$$\sum_i \frac{\partial R^i_j}{\partial y^i} = M\alpha b_j + N y_j. \tag{3.7}$$

We will use the following lemma to calculate the Ξ-curvature.

Lemma 3.1 (see[10, 16])**.** *The Ξ-curvature of a Finsler metric F is given by*

$$\Xi_j = -\frac{1}{3}\left(2\sum_i \frac{\partial R^i_j}{\partial y^i} + \frac{\partial}{\partial y^j} Ric\right). \tag{3.8}$$

Plugging (3.5) and (3.7) into (3.8), we obtain

$$\Xi_j = -\frac{1}{3}[(2M + R_s)\alpha b_j + (2N + 2R - sR_s) y_j]. \tag{3.9}$$

From (3.2), we have

$$R_s = (n-1)R_{1s} + (b^2 - s^2)R_{2s} - 2sR_s.$$

Then we can have

$$(2M + R_s) = (n+1)R_{1s} + 3(b^2 - s^2)R_{2s} + 2(n+1)R_3 := \kappa. \tag{3.10}$$

and

$$2N + 2R - sR_s = -s\kappa. \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), we obtain the following formula for Ξ as:

$$\Xi_j = -\frac{\kappa}{3}(\alpha b_j - s y_j), \tag{3.12}$$

where κ is given as in (3.10).

Now differentiating (3.12) with respect to y^i and using (3.3), we have

$$\Xi_{j.i} = -\frac{\kappa_s}{3\alpha^2}(\alpha b_i - s y_i)(\alpha b_j - s y_j) - \frac{\kappa}{3}\left(\frac{b_j y_i - b_i y_j}{\alpha} + \frac{s}{\alpha^2} y_i y_j - s a_{ij}\right),$$

where $\kappa_s := \frac{\partial \kappa}{\partial s}$.

Therefore, from (1.4), we have

$$\begin{aligned} H_{ij} &= -\frac{\kappa_s}{6\alpha^2}(\alpha b_i - s y_i)(\alpha b_j - s y_j) - \frac{s\kappa}{6}\left(\frac{y_i y_j}{\alpha^2} - a_{ij}\right) \\ &= \frac{1}{6}\left[s\kappa a_{ij} - \kappa_s b_i b_j + \frac{s\kappa_s}{\alpha}(b_j y_i + b_i y_j) - \frac{s}{\alpha^2}(\kappa + s\kappa_s) y_i y_j\right]. \end{aligned}$$

4. Almost vanishing H-curvature

In this section, we prove Theorem 1.1 and Corollary 1.2. Using (3.3), we obtain

$$\alpha_{y^i y^j} = \frac{\alpha^2 a_{ij} - y_i y_j}{\alpha^3}. \tag{4.1}$$

$$s_{y^i y^j} = \frac{3 s y_i y_j - \alpha b_i y_j - \alpha b_j y_i - s\alpha^2 a_{ij}}{\alpha^4}. \tag{4.2}$$

Proof of Theorem 1.1. Differentiating (2.2) with respect to y^i, we have

$$F_{y^i} = \alpha_{y^i}\phi + \alpha\phi_s s_{y^i}. \tag{4.3}$$

Differentiating again (4.3) with respect to y^j yields

$$F_{y^i y^j} = \alpha_{y^i y^j}\phi + (\alpha_{y^i} s_{y^j} + \alpha_{y^j} s_{y^i})\phi_s + \alpha s_{y^i} s_{y^j}\phi_{ss} + \alpha s_{y^i y^j}\phi_s. \tag{4.4}$$

Plugging (3.3),(4.1),(4.2) into (4.4) yields

$$F_{y^i y^j} = \frac{1}{\alpha^3}\left[(\phi - s\phi_s)\alpha^2 a_{ij} + \alpha^2\phi_{ss} b_i b_j - \alpha s\phi_{ss}(b_i y_j + b_j y_i) - (\phi - s\phi_s - s^2\phi_{ss}) y_i y_j\right].$$

In the view of (1.5), the general (α, β)-metric is of almost vanishing H-curvature if and only if

$$\begin{aligned} & s\kappa a_{ij} - \kappa_s b_i b_j + \frac{s\kappa_s}{\alpha}(b_j y_i + b_i y_j) - \frac{s}{\alpha^2}(\kappa + s\kappa_s) y_i y_j \\ &= \frac{3(n+1)\theta}{\alpha^3}\left[(\phi - s\phi_s)\alpha^2 a_{ij} + \alpha^2\phi_{ss} b_i b_j - \alpha s\phi_{ss}(b_i y_j + b_j y_i)\right. \\ &\qquad \left. -(\phi - s\phi_s - s^2\phi_{ss}) y_i y_j\right]. \end{aligned} \tag{4.5}$$

Now equating the similar coefficients of both sides of (4.5), we have the following equations:

$$s\kappa = \frac{3(n+1)\theta}{\alpha}(\phi - s\phi_s), \tag{4.6}$$

$$-\kappa_s = \frac{3(n+1)\theta}{\alpha}\phi_{ss}, \tag{4.7}$$

$$-s\kappa_s=\frac{3(n+1)\theta}{\alpha}s\phi_{ss}, \tag{4.8}$$

$$s(\kappa+s\kappa_s)=\frac{3(n+1)\theta}{\alpha}(\phi-s\phi_s-s^2\phi_{ss}). \tag{4.9}$$

At first we show that (4.6) implies (4.7), (4.8), and (4.9).

Suppose (4.6) holds. Since F is a Finsler metric, we have $\phi-s\phi_s>0$. Since $s=\frac{\beta}{\alpha}$, the 1-form θ can be expressed by

$$\theta=\frac{\kappa}{3(n+1)(\phi-s\phi_s)}\beta. \tag{4.10}$$

Furthermore, $\frac{\kappa}{3(n+1)(\phi-s\phi_s)}$ is independent of y. In fact, it depends only on b^2. Let

$$\frac{\kappa}{3(n+1)(\phi-s\phi_s)}:=\sigma\left(\frac{b^2}{2}\right). \tag{4.11}$$

Therefore, from (4.10) and (4.11), we have

$$\theta=\sigma\left(\frac{b^2}{2}\right)\beta. \tag{4.12}$$

As $s=\frac{\beta}{\alpha}$, we have

$$\frac{\theta}{\alpha}=s\sigma\left(\frac{b^2}{2}\right). \tag{4.13}$$

By using (4.11) and (4.13), we obtain

$$\kappa_s=\left[3(n+1)\sigma\left(\frac{b^2}{2}\right)(\phi-s\phi_s)\right]_s=-3(n+1)\frac{\theta}{\alpha}\phi_{ss}.$$

Thus, we get (4.7). Now multiplying (4.7) by s yields (4.8). Equation (4.9) can be obtained easily from (4.6) and (4.7).

Now substitute the value of κ from (3.10) into (4.6), we get (1.6), which proves the theorem. □

Proof of Corollary 1.2. It is sufficient to show that Ξ-curvature almost vanishes if the H-curvature almost vanishes and in this case corresponding 1-form is exact. Suppose that a general (α,β)-metric $F=\alpha\phi(b^2,\frac{\beta}{\alpha})$ has almost vanishing H-curvature. Then (4.6), (4.11), and (4.12) hold. By using (4.12), we have

$$d\left[f\left(\frac{b^2}{2}\right)\right]=f'\left(\frac{b^2}{2}\right)d\left(\frac{b^2}{2}\right)=\sigma\left(\frac{b^2}{2}\right)\sum_j b_j db_j=\theta,$$

where $f(t):=\int\sigma(t)dt$. Hence θ is an exact form. Using (3.3) into (4.3) yields

$$F_{y^j}=\phi_s b_j+\frac{\phi-\phi_s}{\alpha}y_j.$$

Using (4.12), we get

$$\left(\frac{\theta}{F}\right)_{y^j}=\frac{\sigma\left(\frac{b^2}{2}\right)}{F^2}(\phi-s\phi_s)(\alpha b_j-sy_j). \tag{4.14}$$

Now we have

$$\begin{aligned}
\Xi_j &= -\frac{\kappa}{3}(\alpha b_j - s y_j) && \text{(from (3.12))}\\
&= -(n+1)(\phi - s\phi_s)\sigma(\frac{b^2}{2})(\alpha b_j - s y_j) && \text{(from (4.11))}\\
&= -(n+1)F^2\left(\frac{\theta}{F}\right)_{y^j}. && \text{(using (4.14))}
\end{aligned}$$

Hence we have the proof of Corollary 1.2. □

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