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CONFORMAL SEMI-INVARIANT SUBMERSIONS FROM ALMOST CONTACT METRIC MANIFOLDS ONTO RIEMANNIAN MANIFOLDS

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Abstract. As a generalization of semi-invariant Riemannian submersions, we introduce conformal semi-invariant submersions from almost contact metric manifolds onto Riemannian manifolds and study such submersions from Cosymplectic manifolds onto Riemannian manifolds. Examples of conformal semi-invariant submersions in which structure vector field is vertical are given. We study geometry of foliations determined by distributions involved in definition of conformal anti-invariant submersions. We also study the harmonicity of such submersions and find necessary and sufficient conditions for the distributions to be totally geodesic.

1. Introduction

Let (M, g_M) and (N, g_N) be two Riemannian manifolds of dimension m and n respectively. A differentiable map $f : M \to N$ is called a Riemannian submersion if f has maximal rank and f_* preserves the lengths of horizontal vectors. Riemannian submersions between Riemannian manifolds were introduced by O'Neill [18] and Gray [10]. Firstly, Watson studied Riemannian submersions between Riemannian manifolds equipped with an additional structure of almost complex type [24].

Several geometers studied almost contact metric submersions [5], locally conformal Kähler submersions [16], Riemannian submersions and related topics [8],

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Riemannian submersions from quaternionic manifolds [12], h-semi-invariant submersions [19], mixed para-quaternionic 3-submersions [23], anti-invariant $\xi^{\perp}$ Riemannian submersions from almost contact manifolds [14], on para-quaternionic submersions between para-quaternionic Kähler manifolds [4] etc.

As a generalization of holomorphic submersions [11] and anti-invariant submersions [21], semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds were introduced by Sahin [22]. We see that a Riemannian submersion f from an almost Hermitian manifold (M, J_M, g_M) onto a Riemannian manifold (N, g_N) is called a semi-invariant submersion, if the fibers have differentiable distributions D and $D^{\perp}$ such that D is invariant with respect to J_M and its orthogonal complement $D^{\perp}$ is totally real distribution. Moreover, almost Hermitian submersions [24] and anti-invariant submersions [21] are semi-invariant submersions with $D = \{0\}$ and $D^{\perp} = \{0\}$, respectively.

Gudmundsson and Wood introduced conformal holomorphic submersions as a generalization of holomorphic submersions [11] and obtained necessary and sufficient conditions for conformal holomorphic submersions to be a harmonic morphism. The harmonicity of conformal holomorphic submersions is also discussed in ([6],[7]).

A horizontally conformal submersion is generalization of Riemannian submersions defined as follows [3]: Let $f : (M, g_M) \to (N, g_N)$ be a smooth submersion between Riemannian manifolds, we call f a horizontally conformal submersion, if there is a positive function λ such that

$$g_M(U, V) = \frac{1}{\lambda^2} g_N(f_* U, f_* V) \tag{1.1}$$

for every $U, V \in \Gamma(\ker f_*)^{\perp}$. It is clear that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. We note that horizontally conformal submersions are special horizontally conformal maps that were studied independently by Fuglede [9] and Ishihara [13]. Next, a horizontally conformal submersion $f : (M, g_M) \to (N, g_N)$ is said to be horizontally homothetic if the gradient of its dilation is vertical, that is,

$$\mathcal{H}(grad\lambda) = 0, \tag{1.2}$$

where $\mathcal{H}$ is the projection on the horizontal space $(\ker f_{*p})^{\perp}$.

Conformal anti-invariant submersions [1] and semi-invariant submersions [2] from almost Hermitian manifolds onto Riemannian manifolds were studied by Akyol and Sahin. We introduce here the notion of conformal semi-invariant submersions from almost contact metric manifolds onto Riemannian manifolds and study some geometric properties.

In this paper, we study conformal semi-invariant submersions as a generalization of semi-invariant Riemannian submersions. We investigate the geometry of total space and the base manifold for the existence of such submersions. The paper is organized as follows:

In Section 2, we collect the main notions and formulae for other sections. In Section 3, we introduce conformal semi-invariant submersions from almost contact metric manifolds onto Riemannian manifolds admitting the vertical structure vector field and investigate the geometry of leaves of the horizontal distribution and the vertical distribution. We also find the necessary and sufficient conditions for a conformal semi-invariant submersion to be harmonic and totally geodesic and give some examples.

2. Preliminaries

A $(2n+1)$-dimensional Riemannian manifold M is said to be an almost contact metric manifold [25], if there exist on M, a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η, and a Riemannian metric g such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \tag{2.2}$$

$$\eta(\xi) = 1, \tag{2.3}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(\phi X, Y) = -g(X, \phi Y) \tag{2.4}$$

for any vector fields X, Y on M.

Such a manifold is said to be a contact metric manifold, if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \phi Y)$, is called the fundamental 2-form on M. On the other hand, the almost contact metric structure of M is said to be normal, if $[\phi, \phi](X, Y) = -2d\eta(X, Y)\xi$, for any vector fields X, Y on M, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ given by

$$[\phi, \phi] \ \ (X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]. \tag{2.5}$$

A normal contact metric manifold is called a Cosymplectic manifold, if

$$(\nabla_X \phi)Y = 0, \tag{2.6}$$

for any vector fields X, Y on M [15]. Moreover, for a Cosymplectic manifold the following equation satisfies:

$$\nabla_X \xi = 0. \tag{2.7}$$

Example 2.1. [17]. We consider R^{2k+1} with Cartesian coordinates (x_i, y_i, z) $(i = 1, \dots, k)$ and its usual contact form $\eta = dz$.

The characteristic vector field ξ is given by $\frac{\partial}{\partial z}$, and its Riemannian metric g and tensor field ϕ are given by

$$g = \sum_{i=1}^{k}((dx_i)^2 + (dy_i)^2) + (dz)^2, \quad \phi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad i, j = 1, \dots, k.$$

This gives a Cosymplectic structure on R^{2k+1}. The vector fields $E_i = \frac{\partial}{\partial y_i}, E_{k+i} = \frac{\partial}{\partial x_i}, \xi = \frac{\partial}{\partial z}$ form a ϕ-basis for the Cosymplectic structure. On the other hand, it can be shown that $(R^{2k+1}, \phi, \xi, \eta, g)$ is a Cosymplectic manifold.

Definition 2.2. [3]. Let $f : (M, g_M) \to (N, g_N)$ be a smooth map between Riemannian manifolds and let $x \in M$. Then f is called horizontally weakly conformal or semi-conformal at x if either

(i) $df_x = 0$, or

(ii) df_x maps the horizontal space $\mathcal{H}_x = (\ker(df_x))^{\perp}$ conformally onto $T_{f(x)}N$; that is, df_x is surjective and there exists a number $\Lambda(x) \neq 0$ such that

$$g_N(dfU, dfV) = \Lambda(x) g_M(U, V), \qquad (U, V \in \mathcal{H}_x). \tag{2.8}$$

Note that we can write the last equation more succinctly as

$$(f^* g_N)_x|_{\mathcal{H}_x \times \mathcal{H}_x} = \Lambda(x)(g_M)_x|_{\mathcal{H}_x \times \mathcal{H}_x}.$$

The fundamental tensors of a submersion were introduced in [18]. They play a similar role to that of the second fundamental form of an immersion. More precisely, O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ defined for vector fields E, F on M by

$$\mathcal{A}_E F = \mathcal{V}\nabla^M_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla^M_{\mathcal{H}E}\mathcal{V}F, \tag{2.9}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla^M_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{V}E}\mathcal{H}F, \tag{2.10}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections [9], respectively.

It is easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$.

On the other hand, from equations (2.9) and (2.10), we have

$$\nabla^M_X Y = \mathcal{T}_X Y + \widehat{\nabla}_X Y, \tag{2.11}$$

$$\nabla^M_X V = \mathcal{H}\nabla^M_X V + \mathcal{T}_X V, \tag{2.12}$$

$$\nabla^M_V X = \mathcal{A}_V X + \mathcal{V}\nabla^M_V X, \tag{2.13}$$

$$\nabla^M_U V = \mathcal{H}\nabla^M_U V + \mathcal{A}_U V, \tag{2.14}$$

for $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^{\perp}$, where $\mathcal{V}\nabla_X Y = \widehat{\nabla}_X Y$. If X is basic, then $\mathcal{A}_X V = \mathcal{H}\nabla_V X$.

For $U, V \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^{\perp}$, the tensors $\mathcal{T}, \mathcal{A}$ satisfy:

$$\mathcal{T}_U V = \mathcal{T}_V U, \mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y].$$

It is easily seen that for $x \in M$, $X \in \mathcal{V}_x$ and $U \in \mathcal{H}_x$ the linear operators

$$\mathcal{A}_U,\ \mathcal{T}_X : T_x M \to T_x M$$

are skew-symmetric; that is,

$$g_M(\mathcal{A}_U E, F) = -g_M(E, \mathcal{A}_U F) \text{ and } g_M(\mathcal{T}_X E, F) = -g_M(E, \mathcal{T}_X F) \tag{2.15}$$

for all $E, F \in \Gamma(T_x M)$. We also see that the restriction of $\mathcal{T}$ to the vertical distribution $\mathcal{T}|_{\mathcal{V}\times\mathcal{V}}$ is exactly the second fundamental form of the fibers of f. Since $\mathcal{T}_X$ is skew-symmetric, we get that f has totally geodesic fibers if and only if $\mathcal{T} \equiv 0$.

Now we recall the notion of harmonic maps between Riemannian manifolds (M, g_M) and (N, g_N). Let $f : (M, g_M) \to (N, g_N)$ be a smooth map. Then differential f_* of f can be observed as a section of the bundle $Hom(TM, f^{-1}TN) \to M$, where $f^{-1}TN$ is the bundle which has the fibers $(f^{-1}TN)_p = T_{f(p)}N$, $p \in M$. $Hom(TM, f^{-1}TN)$ has a connection ∇ induced from the Riemannian connection (Levi-Civita connection) ∇^M and the pullback connection. Then the second fundamental form of f is given by

$$(\nabla f_*)(U, V) = \nabla^f_U f_*(V) - f_*(\nabla^M_U V), \tag{2.16}$$

for vector fields $U, V \in \Gamma(TM)$, where ∇^f is the pullback connection. It is known that the second fundamental form is always symmetric. A smooth map $f : (M, g_M) \to (N, g_N)$ is said to be harmonic if $trace(\nabla f_*) = 0$. On the other hand, the tension field of f is the section $\tau(f)$ of $\Gamma(f^{-1}TN)$ defined by

$$\tau(f) = div f_* = \sum_{i=1}^{m} (\nabla f_*)(e_i, e_i), \tag{2.17}$$

where $\{e_1, \dots, e_m\}$ is the orthonormal frame field on M. Then it follows that f is harmonic if and only if $\tau(f) = 0$; for these facts, see [3].

Lastly, we recall the subsequent lemma from [3].

Lemma 2.3. *Let $f : M \to N$ be a horizontally conformal submersion. Then for any horizontal vector fields U, V and vertical vector fields X, Y, we have*

$$\begin{aligned}
(i)(\nabla df)(U, V) &= U(ln\lambda)df(V) + V(ln\lambda)df(U) - g_M(U, V)df(\mathcal{H}(gradln\lambda)),\\
(ii)(\nabla df)(X, Y) &= -df(\mathcal{T}_X Y),\\
(iii)(\nabla df)(U, X) &= -df(\nabla^M_U X) = -df((\mathcal{A}^{\mathcal{H}})^*_U X),
\end{aligned}$$

where $(\mathcal{A}^{\mathcal{H}})^*_U$ is the adjoint of ($\mathcal{A}^{\mathcal{H}}_U$) characterized by

$$\langle (\mathcal{A}^{\mathcal{H}})^*_U E, F\rangle = \langle E, \mathcal{A}^{\mathcal{H}}_U F\rangle \quad \textit{for all } E, F \in \Gamma(TM).$$

3. Conformal semi-invariant submersions admitting vertical structure vector field

In this section, we define a conformal semi-invariant submersion from an almost contact metric manifold onto a Riemannian manifold admitting vertical structure vector field. We investigate the geometry of foliations and integrability of distributions. Moreover, we also study the harmonicity of such submersions and find necessary and sufficient conditions for a conformal semi-invariant submersion to be totally geodesic.

Definition 3.1. Let M be an almost contact manifold with Riemannian metric g_M and let N be a Riemannian manifold with Riemannian metric g_N. A horizontally conformal submersion $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ with dilation λ is called a conformal semi-invariant submersion if there is a distribution $D_1 \subseteq (\ker f_*)$ such that

$$(\ker f_*) = D_1 \oplus D_2 \oplus < \xi >, \tag{3.1}$$

and $\phi(D_1) = D_1$, $\phi(D_2) \subseteq (\ker f_*)^{\perp}$, where D_1, D_2 and $< \xi >$ are mutually orthogonal distributions in $\ker f_*$.

Now, we shall give an example of a conformal semi-invariant submersion from an almost contact metric manifold onto a Riemannian manifold.

Note that for a Euclidean space R^{2n+1} with coordinates $(x_1, x_2, \ldots, x_{2n}, x_{2n+1})$, we can canonically choose an almost contact metric structure (ϕ, ξ, η, g_M) on R^{2n+1} as follows:

$$\begin{aligned}\phi(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + \cdots + a_{2n-1}\frac{\partial}{\partial x_{2n-1}} + a_{2n}\frac{\partial}{\partial x_{2n}} + a_{2n+1}\frac{\partial}{\partial x_{2n+1}})\\ = (-a_2\frac{\partial}{\partial x_1} + a_1\frac{\partial}{\partial x_2} + \cdots - a_{2n}\frac{\partial}{\partial x_{2n-1}} + a_{2n-1}\frac{\partial}{\partial x_{2n}}),\end{aligned}$$

where $\xi = \frac{\partial}{\partial x_{2n+1}}$ and $a_1, a_2, a_3, \ldots, a_{2n}, a_{2n+1}$ are C^∞-real valued functions in R^{2n+1}. Let $\eta = dx_{2n+1}$ be a 1-form, $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n+1}}, \ldots, \frac{\partial}{\partial x_{2n}}, \frac{\partial}{\partial x_{2n+1}}\}$ an orthonormal frame field and $g_M R^{2n+1}$ a Euclidean metric on R^{2n+1}.

Example 3.2. Let R^7 have an almost contact metric structure defined above. Let $f : R^7 \to R^2$ be a Riemannian submersion defined by

$$f(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (e^{x_3}\sin x_2, e^{x_3}\cos x_2),$$

where $x_2 \in R - \{\frac{k\pi}{2}, k\pi\}$, $k \in R$. Then,

$$\begin{aligned}(\ker f_*) =&\text{span}\{E_1 = \frac{\partial}{\partial x_1}, E_4 = \frac{\partial}{\partial x_4}, E_5 = \frac{\partial}{\partial x_5}, E_6 = \frac{\partial}{\partial x_6}, E_7 = \frac{\partial}{\partial x_7}\},\\ (\ker f_*)^\perp =&\text{span}\{E_2 = e^{x_3}\cos x_2\frac{\partial}{\partial x_2} + e^{x_3}\sin x_2\frac{\partial}{\partial x_3},\\ &E_3 = -e^{x_3}\sin x_2\frac{\partial}{\partial x_2} + e^{x_3}\cos x_2\frac{\partial}{\partial x_3}\}.\end{aligned}$$

Thus it follows that $D_1 = \text{span}\{E_5, E_6\}$ and $D_2 = \text{span}\{E_1, E_4\}$. Let y_1, y_2 be local coordinates in R^2. Also by direct computations, we get

$$f_*E_2 = (e^{x_3})^2\frac{\partial}{\partial y_1}, f_*E_3 = (e^{x_3})^2\frac{\partial}{\partial y_2}.$$

Hence, we have

$$g_2(f_*E_2, f_*E_2) = (e^{x_3})^2 g_7(E_2, E_2), g_2(f_*E_3, f_*E_3) = (e^{x_3})^2 g_7(E_3, E_3),$$

where g_7 and g_2 denote the Euclidean metrics on R^7 and R^2 respectively . Thus f is a conformal semi-invariant submersion with $\lambda = e^{x_3}$.

Let $(M, \phi, \xi, \eta, g_M)$ be an almost contact metric manifold and (N, g_N) a Riemannian manifold. Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal semi-invariant submersion. Then there is a distribution $D_1 \subseteq (\ker f_*)$ such that

$$\begin{aligned}(\ker f_*) = D_1 \oplus D_2 \oplus <\xi>, \quad \phi(D_1) = D_1, \qquad (3.2)\\ \phi(D_2) \subseteq (\ker f_*)^\perp, \quad (\ker f_*)^\perp = \phi(D_2) \oplus \mu.\end{aligned}$$

We denote the complementary distribution to $\phi(D_2)$ in $(\ker f_*)^\perp$ by μ. Then for $X \in \Gamma(\ker f_*)$, we get

$$X = PX + QX + \eta(X)\xi, \qquad (3.3)$$

where $PX \in \Gamma(D_1)$ and $QX \in \Gamma(D_2)$.

For $Y \in \Gamma(\ker f_*)$, we get

$$\phi Y = \psi Y + \omega Y, \tag{3.4}$$

where $\psi Y \in \Gamma(D_1)$ and $\omega Y \in \Gamma(\phi D_2)$. Also, for $U \in \Gamma(\ker f_*)^\perp$, we have

$$\phi U = BU + CU, \tag{3.5}$$

where $BU \in \Gamma(D_2)$ and $CU \in \Gamma(\mu)$.

For $X, Y \in \Gamma(\ker f_*)$, define

$$(\nabla^M_X \psi)Y = \widehat{\nabla}_X \psi Y - \psi \widehat{\nabla}_X Y, \tag{3.6}$$

$$(\nabla^M_X \omega)Y = \mathcal{H}\nabla^M_X \omega Y - \omega \widehat{\nabla}_X Y. \tag{3.7}$$

Then it is easy to obtain the following result.

Lemma 3.3. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then*

$$(\nabla^M_X \psi)Y = B\mathcal{T}_X Y - \mathcal{T}_X \omega Y, \tag{3.8}$$

$$(\nabla^M_X \omega)Y = C\mathcal{T}_X Y - \mathcal{T}_X \psi Y, \tag{3.9}$$

for $X, Y \in \Gamma(\ker f_*)$.

Lemma 3.4. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then*

(i) the distribution D_1 is integrable if and only if $(\nabla f_*)(V, \phi U) - (\nabla f_*)(U, \phi V) \in \Gamma(f_*\mu)$, for $U, V \in \Gamma(D_1)$.

(ii) the distribution D_2 is always integrable.

Lemma 3.5. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then*

$$g_M(BV, \phi X) = 0, \quad g_M(BV, \phi Y) = 0, \quad g_M(BV, \phi Z) = 0, \tag{3.10}$$

and

$$\begin{aligned} g_M(\nabla^M_U BV, \phi X) &= -g_M(BV, \nabla^M_U \phi X), \\ g_M(\nabla^M_U BV, \phi Y) &= -g_M(BV, \nabla^M_U \phi Y), \\ g_M(\nabla^M_U BV, \phi Z) &= -g_M(BV, \nabla^M_U \phi Z) \end{aligned} \tag{3.11}$$

for $X \in \Gamma(D_1)$, $Y \in \Gamma(D_2)$, $Z \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(D_1)$, $Y \in \Gamma(D_2)$, $Z \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, since $\phi X \in \Gamma(D_1)$, $\phi Y \in \Gamma(\ker f_*)^\perp$, $BV \in \Gamma(D_2)$, $\psi Z \in \Gamma(D_1)$ and $\omega Z \in \Gamma(\phi D_2)$, using equations (3.4) and (3.5), we get equation (3.10).

Now, using equations (3.10), (2.15), and (2.16), we get equation (3.11). □

Theorem 3.6. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then the distribution $(\ker f_*)^\perp$ is integrable if and only if*

$$\mathcal{A}_V \omega BU - \mathcal{A}_U \omega BV + \phi(\mathcal{A}_V CU - \mathcal{A}_U CV) \notin \Gamma(D_1),$$

and

$$\begin{aligned}\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi Z)\\ = g_M(\mathcal{A}_V BU - \mathcal{A}_U BV - CV(ln\lambda)U\\ +CU(ln\lambda)V + 2g_M(U, CV)\mathcal{H} gradln\lambda, \phi Z)\end{aligned}$$

for $Z \in \Gamma(D_2)$ and $U, V \in \Gamma(\ker f_*)^\perp$.

Proof. Let $U, V \in \Gamma(\ker f_*)^\perp$. Now consider

$$\begin{aligned} g_M([U, V], \xi) &= g_M(\nabla^M_U V, \xi) - g_M(\nabla^M_V U, \xi), \\ &= -g_M(V, \nabla^M_U \xi) + g_M(U, \nabla^M_V \xi). \end{aligned}$$

Using equation (2.7), we have

$$g_M([U, V], \xi) = 0.$$

The distribution $\Gamma(\ker f_*)^\perp$ is integrable if and only if

$$g_M([U, V], X) = 0, \textit{and} \quad g_M([U, V], Z) = 0,$$

for $X \in \Gamma(D_1), Z \in \Gamma(D_2)$ and $U, V \in \Gamma(\ker f_*)^\perp$. Using equations (2.4), (2.6), and (3.5), we get

$$\begin{aligned} g_M([U, V], X) &= g_M(\nabla^M_U BV, \phi X) + g_M(\nabla^M_U CV, \phi X) \\ &\quad -g_M(\nabla^M_V BU, \phi X) - g_M(\nabla^M_U BV, \phi X). \end{aligned}$$

From equations (2.6), (3.11), and (2.14), we have

$$\begin{aligned} g_M([U, V], X) &= -g_M(BV, \phi \nabla^M_U X) + g_M(\mathcal{A}_U CV, \phi X) \\ &\quad +g_M(BU, \phi \nabla^M_V X) - g_M(\mathcal{A}_V CU, \phi X). \end{aligned}$$

Using equation (2.4), (2.6), and (3.4), one has that

$$g_M([U, V], X) = g_M(\mathcal{A}_V \omega BU - \mathcal{A}_U \omega BV - \phi \mathcal{A}_U CV + \phi \mathcal{A}_V CU, X). \tag{3.12}$$

On the other hand, equations (2.4), (2.6), and (3.5), imply that

$$\begin{aligned} g_M([U, V], Z) &= g_M(\nabla^M_U BV, \phi Z) + g_M(\nabla^M_U CV, \phi Z) \\ &\quad -g_M(\nabla^M_V BU, \phi Z) - g_M(\nabla^M_V CU, \phi Z). \end{aligned}$$

Since f is a conformal submersion, by equations (2.11), (2.16), and (3.10) and Lemma 2.3(i), we get

$$\begin{aligned} g_M([U, V], Z) = g_M(&\mathcal{A}_V BU - \mathcal{A}_U BV - CV(ln\lambda)U + CU(ln\lambda)V \\ &+ 2g_M(U, CV)\mathcal{H} gradln\lambda, \phi Z) \\ &- \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi Z). \end{aligned} \tag{3.13}$$

□

Theorem 3.7. *Let f be a conformal semi-invariant submersion from a Cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with integrable distribution $(\ker f_*)^\perp$. Then f is a horizontally homothetic map if and only if*

$$\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_*\phi Z) = g_M(\mathcal{A}_V BU - \mathcal{A}_U BV, \phi Z), \qquad (3.14)$$

for $Z \in \Gamma(D_2)$ and $U, V \in \Gamma(\ker f_*)^\perp$.

Proof. For $Z \in \Gamma(D_2)$ and $U, V \in \Gamma(\ker f_*)^\perp$, from equation (3.13), we have

$$\begin{aligned} g_M([U,V], Z) &= g_M(\mathcal{A}_V BU - \mathcal{A}_U BV - CV(ln\lambda)U + CU(ln\lambda)V \\ &\quad + 2g_M(U, CV)\mathcal{H}gradln\lambda, \phi Z) \\ &\quad - \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_*\phi Z). \end{aligned}$$

If f is a horizontally homothetic map, then

$$\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_*\phi Z) = g_M(\mathcal{A}_V BU - \mathcal{A}_U BV, \phi Z).$$

Conversely, if (3.14) is satisfied, then

$$\begin{aligned} 0 = g_M(V, \phi Z) g_M(\mathcal{H}gradln\lambda, CU) - g_M(U, \phi Z) g_M(\mathcal{H}gradln\lambda, CV) \qquad (3.15) \\ + 2g_M(U, CV) g_M(\mathcal{H}gradln\lambda, \phi Z). \end{aligned}$$

Now, putting $V = \phi Z$, for $Z \in \Gamma(D_2)$ in equation (3.15), we have

$$g_M(\phi Z, \phi Z) g_M(\mathcal{H}gradln\lambda, CU) = 0.$$

Thus λ is constant on $\Gamma(\mu)$. On the other hand, taking $V = CU$, for $U \in \Gamma(\mu)$ in (3.15), we have

$$g_M(U, U) g_M(\mathcal{H}gradln\lambda, \phi Z) = 0.$$

From the above equation, λ is constant on $\Gamma(\phi D_2)$. □

As a conformal version of anti-holomorphic semi-invariant submersion [20], a conformal semi-invariant submersion is called a conformal anti-holomorphic semi-invariant submersion if $\phi D_2 = (\ker f_*)^\perp$.

Corollary 3.8. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal anti-holomorphic semi-invariant submersion, then the following assertions are equivalent to each other:*

(i) $(\ker f_*)^\perp$ is integrable,
(ii) $g_N(f_*\phi U, (\nabla f_*)(Z, \phi V)) = g_N(f_*\phi V, (\nabla f_*)(Z, \phi U))$, for $U, V \in \Gamma(D_2)$ and $Z \in \Gamma(\ker f_*)$.

Proof. For $U, V \in \Gamma(D_2)$ and $Z \in \Gamma(\ker f_*)$, using equations (1.1), (2.4), (2.6) and (2.7), we have

$$g_M([\phi U, \phi V], Z) = -g_M(\phi V, \nabla^M_Z \phi U) + g_M(\phi U, \nabla^M_Z \phi V).$$

Since f is a conformal subnersion by using Lemma 2.3, we have

$$g_M([\phi U, \phi V], Z) = \frac{1}{\lambda^2}\{g_N(f_*\phi U, (\nabla f_*)(Z, \phi V)) - g_N(f_*\phi V, (\nabla f_*)(Z, \phi U))\},$$

which completes the proof. □

Theorem 3.9. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then the distribution $(\ker f_*)^\perp$ defines a totally geodesic foliation on M if and only if*

$$\mathcal{A}_V CW + \mathcal{V}\nabla^M_V BW \in \Gamma(D_2), \tag{3.16}$$

and

$$\begin{aligned}&\frac{1}{\lambda^2} g_N(\nabla^f_V f_*\phi Z, f_*CW) \\ &\quad = g_M(\mathcal{A}_V BW - CW(ln\lambda)V + g_M(V, CW)\mathcal{H}gradln\lambda, \phi Z)\end{aligned} \tag{3.17}$$

for $X \in \Gamma(D_1), Z \in \Gamma(D_2)$ and $V, W \in \Gamma(\ker f_*)^\perp$.

Proof. Let $U, V \in \Gamma(\ker f_*)^\perp$. Similarly to as in the proof of Theorem 3.6, we have

$$g_M([U, V], \xi) = 0.$$

The distribution $(\ker f_*)^\perp$ defines a totally geodesic foliation on M if and only if

$$g_M(\nabla^M_V W, X) = 0, \text{ and } g_M(\nabla^M_V W, Z) = 0,$$

for $X \in \Gamma(D_1), Z \in \Gamma(D_2)$ and $V, W \in \Gamma(\ker f_*)^\perp$. Then by using equations (2.1), (2.4), and (2.6), we get

$$g_M(\nabla^M_V W, X) = -g_M(\phi\nabla^M_V \phi W, X).$$

Equations (3.4), (3.5), (2.15), and (2.16), imply that

$$g_M(\nabla^M_V W, X) = -g_M(\psi(\mathcal{A}_V CW + \mathcal{V}\nabla^M_V BW), X). \tag{3.18}$$

On the other hand, from equations (2.4), (3.5), and (3.11), we have

$$g_M(\nabla^M_V W, Z) = -g_M(BW, \nabla^M_V \phi Z) - g_M(CW, \nabla^M_V \phi Z).$$

Since f is a conformal semi-invariant submersion, equations (2.13), (2.16), and (3.10) and Lemma 2.3(i), ensure that

$$\begin{aligned}g_M(\nabla^M_V W, Z) &= g_M(\mathcal{A}_V BW - CW(ln\lambda)V + g_M(V, CW)\mathcal{H}gradln\lambda, \phi Z) \\ &\quad -\frac{1}{\lambda^2} g_N(\nabla^f_V f_*\phi Z, f_*CW).\end{aligned}$$

□

Definition 3.10. Let f be a conformal semi-invariant submersion from a Cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then we say that D_2 is parallel along $(\ker f_*)^\perp$ if $\nabla^M_U Z \in \Gamma(D_2)$, for $Z \in \Gamma(D_2)$ and $U \in \Gamma(\ker f_*)^\perp$.

Corollary 3.11. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then f is a horizontally homothetic map if and only if*

$$\frac{1}{\lambda^2} g_N(\nabla_U^f f_* \phi Z, f_* CV) = g_M(\mathcal{A}_U BV, \phi Z) \tag{3.19}$$

for $Z \in \Gamma(D_2)$ and $U, V \in \Gamma(\ker f_*)^\perp$.

Proof. For $Z \in \Gamma(D_2)$ and $U, V \in \Gamma(\ker f_*)^\perp$, using equations (3.16) and (3.19), we have

$$-g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi Z) + g_M(U, CV) g_M(\mathcal{H} gradln\lambda, \phi Z) = 0. \tag{3.20}$$

Now, putting $U = \phi Z$, for $Z \in \Gamma(D_2)$ in the equation (3.19) and using equation (3.10), we get

$$g_M(\mathcal{H} gradln\lambda, CV) g_M(\phi Z, \phi Z) = 0.$$

Thus λ is constant on $\Gamma(\mu)$. On the other hand, putting $U = CV$ in equation (3.19) for $U \in \Gamma(\mu)$ and using equation (3.10), we have

$$g_M(\mathcal{H} gradln\lambda, \phi Z) g_M(CV, CV) = 0.$$

From the above equation, λ is constant on $\Gamma(\phi D_2)$. □

Corollary 3.12. *Let f be a conformal antiholomorphic semi-invariant submersion from a Cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other:*

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on M,
(ii) $(\nabla f_*)(Z, \phi X) \in \Gamma f_*(\mu)$ for $X \in \Gamma(D_2)$ and $Z \in \Gamma(\ker f_*)^\perp$.

Theorem 3.13. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then the distribution $(\ker f_*)$ defines a totally geodesic foliation on M if and only if*

$$\frac{1}{\lambda^2} g_N(f_* \omega X, \nabla_{\omega Y}^f f_* U) \\ = g_M(C\mathcal{T}_X \psi Y + \mathcal{A}_{\omega Y} \psi X + g_M(\omega X, \omega Y)(\mathcal{H} gradln\lambda), U),$$

and

$$\widehat{\nabla}_X \psi Y + \mathcal{T}_X \omega Y \in \Gamma(D_1),$$

$X, Y \in \Gamma(\ker f_*), U \in \Gamma(\mu)$ and $Z \in \Gamma(D_2)$.

Proof. For $X, Y \in \Gamma(\ker f_*), U \in \Gamma(\mu)$ and $Z \in \Gamma(D_2)$, the distribution $(\ker f_*)$ defines a totally geodesic foliation on M if and only if

$$g_M(\nabla_X^M Y, U) = 0 \text{ and } g_M(\nabla_X^M Y, \phi Z) = 0.$$

Using equations (2.4), (2.6) and (3.4), we have

$$g_M(\nabla_X^M Y, U) = g_M(\nabla_X^M \psi Y, \phi U) + g_M(\nabla_X^M \omega Y, \phi U).$$

Since $[X, \omega Y] \in \Gamma(D_2)$, hence

$$g_M(\nabla_X^M Y, U) = g_M(\nabla_X^M \psi Y, \phi U) + g_M(\nabla_{\omega Y}^M X, \phi U).$$

From equations (2.6), (2.4), (3.4), and (3.10), we get

$$g_M(\nabla^M_X Y, U) = g_M(\nabla^M_X \psi Y, \phi U) + g_M(\psi X, \nabla^M_{\omega Y} U) + g_M(\omega X, \nabla^M_{\omega Y} U).$$

Since f is a conformal submersion, from equations (2.12), (2.13), (2.16) and Lemma 2.3(i), we have

$$\begin{aligned} & g_M(\nabla^M_X Y, U) \\ & = g_M(\mathcal{T}_X \psi Y, \phi U) + g_M(\psi X, \mathcal{A}_{\omega Y} U) \\ & -\frac{1}{\lambda^2} g_N(f_*\omega X, f_* U) g_M(\mathcal{H} gradln\lambda, \omega Y) \\ & -\frac{1}{\lambda^2} g_N(f_*\omega X, f_*\omega Y) g_M(\mathcal{H} gradln\lambda, U) \\ & +\frac{1}{\lambda^2} g_M(U, \omega Y) g_N(f_*\mathcal{H} gradln\lambda, f_*\omega X) + \frac{1}{\lambda^2} g_N(f_*\omega X, \nabla^f_{\omega Y} f_* U). \end{aligned}$$

Hence, we obtain:

$$\begin{aligned} g_M(\nabla^M_X Y, U) = & g_M(-C\mathcal{T}_X \psi Y - \mathcal{A}_{\omega Y} \psi X - g_M(\omega X, \omega Y)(\mathcal{H} gradln\lambda), U) \\ & + \frac{1}{\lambda^2} g_N(f_*\omega X, \nabla^f_{\omega Y} f_* U). \end{aligned} \tag{3.21}$$

On the other hand, since $[X, Y] \in \Gamma(\ker f_*)$ and using equations (2.1), (2.6), and (3.5), we get

$$g_M(\nabla^M_X Y, \phi Z) = -\ g_M(\omega \nabla^M_X \psi Y, \phi Z) -\ g_M(\omega \nabla^M_X \omega Y, \phi Z).$$

Again using equations (2.13) and (2.14), we have

$$g_M(\nabla^M_X Y, \phi Z) = -\ g_M(\omega \widehat{\nabla}_X \psi Y, \phi Z) -\ g_M(\omega \mathcal{T}_X \omega Y, \phi Z). \tag{3.22}$$

□

Next, we give certain conditions for dilation λ to be constant on μ. We first give the following definition.

Definition 3.14. Let f be a conformal semi-invariant submersion from a Cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then we say that μ is parallel along $(\ker f_*)$, if $\nabla^M_X U \in \Gamma(\mu)$, for $U \in \Gamma(\mu)$ and $X \in \Gamma(\ker f_*)$.

Corollary 3.15. *Let f be a conformal semi-invariant submersion from a Cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N) such that μ is parallel along $(\ker f_*)$. Then f is constant on μ if and only if*

$$\frac{1}{\lambda^2} g_N(\nabla^f_{\omega Y} f_* U, f_*\omega X) = g_M(C\mathcal{T}_X \psi Y + \mathcal{A}_{\omega Y} \psi X, U) \tag{3.23}$$

for $U \in \Gamma(\mu)$ and $X, Y \in \Gamma(\ker f_*)$.

Proof. For $U \in \Gamma(\mu)$ and $X, Y \in \Gamma(\ker f_*)$, from equation (3.21), we have

$$\begin{aligned} g_M(\nabla^M_X Y, U) \ & = g_M(-C\mathcal{T}_X \psi Y - \mathcal{A}_{\omega Y} \psi X - g_M(\omega X, \omega Y)(\mathcal{H} gradln\lambda), U) \\ & +\frac{1}{\lambda^2} g_N(f_*\omega X, \nabla^f_{\omega Y} f_* U). \end{aligned}$$

Using equation (1.1), we get

$$g_M(\omega X, \omega Y)g_M(\mathcal{H}gradln\lambda, U) = 0.$$

From the above equation λ is constant on $\Gamma(\mu)$. The converse comes from equation (3.21). □

By Theorems 3.9 and 3.13, the following theorem can be followed.

Theorem 3.16. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then M is a locally product manifold of the form $M_{(\ker f_*)} \times_\lambda M_{(\ker f_*)^\perp}$ if and only if*

$$\mathcal{A}_U CV + \mathcal{V}\nabla^M_U BV \in \Gamma(D_2),$$

$$\frac{1}{\lambda^2} g_N(\nabla^f_U f_*\phi Z, f_*CV) = g_M(\mathcal{A}_U BV - CV(ln\lambda)U + g_M(U, CV)\mathcal{H}gradln\lambda, \phi Z),$$

and

$$\frac{1}{\lambda^2} g_N(f_*\omega X, \nabla^f_{\omega Y} f_*U) = g_M(C\mathcal{T}_X\psi Y + \mathcal{A}_{\omega Y}\psi X + g_M(\omega X, \omega Y)(\mathcal{H}gradln\lambda), U),$$

$$\widehat{\nabla}_X\psi Y + \mathcal{T}_X\omega Y \in \Gamma(D_1)$$

for all vector fields $X, Y, Z \in \Gamma(\ker f_*)$ and $V, U \in \Gamma(\ker f_*)^\perp$.

Theorem 3.17. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then D_1 defines a totally geodesic on M if and only if*

$$(\nabla f_*)(X, \phi Y) \in \Gamma f_*(\mu),$$

and

$$\frac{1}{\lambda^2} g_N((\nabla f_*)(X, \phi Y), f_*CU) = g_M(Y, \mathcal{T}_X\omega BU)$$

for $X, Y \in \Gamma(D_1)$ and $U \in \Gamma(\ker f_*)^\perp$.

Proof. Let $X, Y \in \Gamma(D_1)$. Similarly to as in the proof of Theorem 3.6, we have

$$g_M([X, Y], \xi) = 0.$$

The distribution D_1 defines a totally geodesic foliation on M if and only if

$$g_M(\nabla^M_X Y, Z) = 0, \text{ and } g_M(\nabla^M_X Y, U) = 0,$$

for $X, Y \in \Gamma(D_1), Z \in \Gamma(D_2)$ and $U \in \Gamma(\ker f_*)^\perp$.

From equations (2.4), (2.6) and (2.14), we get

$$g_M(\nabla^M_X Y, Z) = g_M(\mathcal{H}\nabla^M_X \phi Y, \phi Z).$$

Since f is a conformal semi-invariant submersion, using equation (2.16), we have

$$g_M(\nabla^M_X Y, Z) = -\frac{1}{\lambda^2} g_N((\nabla f_*)(X, \phi Y), f_*\phi Z).$$

On the other hand, by using equations (2.4), (2.6), (3.5), and (2.13), we get

$$g_M(\nabla^M_X Y, U) = g_M(Y, \nabla^M_X \phi BU) + g_M(\nabla^M_X \phi Y, CU).$$

Since f is a conformal semi-invariant submersion, using equations (3.4), (2.12), and (2.16), one has

$$g_M(\nabla^M_X Y, U) = g_M(Y, \mathcal{T}_X \omega BU) - \frac{1}{\lambda^2} g_N((\nabla f_*)(X, \phi Y), f_* CU).$$

□

Theorem 3.18. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then D_2 defines a totally geodesic on M if and only if*

$$(\nabla f_*)(Y, \phi X) \in \Gamma f_*(\mu),$$

and

$$-\frac{1}{\lambda^2} g_N(\nabla^f_{\phi Z} f_* \phi Y, f_* \phi CU) = g_M(Z, B\mathcal{T}_Y BU) + g_M(Y, Z) g_M(\mathcal{H} gradln\lambda, \phi CU)$$

for $X \in \Gamma(D_1)$, $Y, Z \in \Gamma(D_2)$ and $U \in \Gamma(\ker f_*)^{\perp}$.

Proof. Let $Y, Z \in \Gamma(D_2)$. As in Theorem 3.6, we have

$$g_M([Y, Z], \xi) = 0.$$

The distribution D_2 defines a totally geodesic foliation on M if and only if

$$g_M(\nabla^M_Y Z, X) = 0, \text{ and } g_M(\nabla^M_Y Z, U) = 0,$$

for $X \in \Gamma(D_1), Y, Z \in \Gamma(D_2)$ and $U \in \Gamma(\ker f_*)^{\perp}$. Using equations (2.4), (2.6), and (3.11), we get

$$g_M(\nabla^M_Y Z, X) = g_M(\nabla^M_Y \phi Z, \phi X).$$

Since f is a conformal submersion, from equation (2.16), we get

$$g_M(\nabla^M_Y Z, X) = -\frac{1}{\lambda^2} g_N((\nabla f_*)(Y, \phi X), f_* \phi Z).$$

On the other hand, equations (2.4), (2.5), and (3.11), imply that

$$g_M(\nabla^M_Y Z, U) = -g_M(\phi Z, \nabla^M_Y BU) + g_M(\nabla^M_{\phi Z} \phi Y, \phi CU).$$

Since f is a conformal submersion, from equation (2.16) and Lemma 2.3(i), we get

$$\begin{aligned} g_M(\nabla^M_Y Z, U) &= g_M(Z, B\mathcal{T}_Y BU) + g_M(Z, Y) g_M(\mathcal{H} gradln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Z} f_* \phi Y, f_* \phi CU). \end{aligned}$$

□

By Theorems 3.17 and 3.18, we get the following result.

Theorem 3.19. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then the fibers of f are locally product manifold if and only if*

$$(\nabla f_*)(X, \phi Y) \in \Gamma f_*(\mu),$$

$$\frac{1}{\lambda^2} g_N((\nabla f_*)(X, \phi Y), f_* CU) = g_M(Y, \mathcal{T}_X \omega BU),$$

$$(\nabla f_*)(W, \phi V) \in \Gamma f_*(\mu),$$

$$-\frac{1}{\lambda^2} g_N(\nabla^f_{\phi W} f_*\phi V, f_*\phi CU) = g_M(W, B\mathcal{T}_V BU) + g_M(W, V) g_M(\mathcal{H} gradln\lambda, \phi CU)$$

for $X, Y \in \Gamma(D_1), V, W \in \Gamma(D_2)$ *and* $U \in \Gamma(\ker f_*)^\perp$.

Since $(\ker f_*)^\perp = \phi(D_2) \oplus \mu$ and f is a conformal semi-invariant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N), for $X \in \Gamma(D_2)$ and $Y \in \Gamma(\mu)$, we get

$$\frac{1}{\lambda^2} g_N(f_*\phi X, f_*Y) = g_M(\phi X, Y) = 0.$$

This implies that the distributions $f_*(\phi D_2)$ and $f_*(\mu)$ are orthogonal. Now, we investigate the geometry of the leaves of the distribution D_1 and D_2.

Lemma 3.20. *Let f be a conformal semi-invariant submersion from a Cosymplectic manifold $(M^{2p+2q+2r+1}, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N^{q+2r}, g_N). Then the tension field τ of f is $\tau(f) = -(2p+q) f_*(\mu^{\ker f_*}) + (2-q-2r) f_*(\mathcal{H} gradln\lambda)$, where $\mu^{\ker f_*}$ is the mean curvature vector field of the distribution of* $(\ker f_*)$.

Theorem 3.21. *Let f be a conformal semi-invariant submersion from a Cosymplectic manifold $(M^{2p+2q+2r+1}, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N^{q+2r}, g_N). If $q + 2r \neq 2$, then any two conditions below imply the third:*

(i) f is harmonic,
(ii) The fibers are minimal,
(iii) f is a horizontally homothetic map.
We also have the following result.

Corollary 3.22. *Let f be a conformal semi-invariant submersion from a Cosymplectic manifold $(M^{2p+2q+2r+1}, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N^{q+2r}, g_N). If $q + 2r = 2$, then f is harmonic if and only if the fibers are minimal.*

Now, we obtain necessary and sufficient condition for a conformal semi-invariant submersion to be totally geodesic. We recall that a differentiable map f between Riemannian manifolds is called totally geodesic if

$$(\nabla f_*)(U, V) = 0, \quad \text{for all } U, V \in \Gamma(TM).$$

A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total manifold into a geodesic in the base manifold in proportion to arc lengths. We now present the following definition.

Definition 3.23. Let $(M, \phi, \xi, \eta, g_M)$ be an almost contact metric manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then f is called a $(\phi D_2, \mu)$-totally geodesic map if

$$(\nabla f_*)(\phi U, X) = 0, \quad \text{for all } U \in \Gamma(D_2) \text{ and } X \in \Gamma(\ker f_*)^\perp.$$

In what follows, we show that this notion has an important effect on the geometry of the conformal submersion.

Theorem 3.24. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then f is a $(\phi D_2, \mu)$-totally geodesic map if and only if f is a horizontally homothetic map.*

Proof. For $U \in \Gamma(D_2)$ and $X \in \Gamma(\mu)$, from Lemma 2.3(i), we get

$$(\nabla f_*)(\phi U, X) = X(ln\lambda)f_*\phi U + \phi U(ln\lambda)f_*X - g_M(\phi U, X)f_*(\mathcal{H}gradln\lambda).$$

From the above equation, if f is horizontally homothetic, then $(\nabla f_*)(\phi U, X) = 0$. Conversely, if $(\nabla f_*)(\phi U, X) = 0$, for $U \in \Gamma(D_2)$ and $X \in \Gamma(\mu)$, since $\phi U \in \Gamma(\phi D_2)$, we get

$$X(ln\lambda)f_*\phi U + \phi U(ln\lambda)f_*X = 0. \tag{3.24}$$

Taking inner product in equation (3.24) with $f_*\phi U$, we have

$$g_M(\mathcal{H}gradln\lambda, \phi U)g_N(f_*X, f_*\phi U) + g_M(\mathcal{H}gradln\lambda, X)g_N(f_*\phi U, f_*\phi U) = 0.$$

Since f is a conformal submersion, hence

$$g_M(\mathcal{H}gradln\lambda, X)g_M(\phi U, \phi U) = 0.$$

Above equation, it follows that λ is constant on $\Gamma(\mu)$. On the other hand, taking inner product in equation (3.24) with f_*X and since f is a conformal submersion, we get

$$g_M(\mathcal{H}gradln\lambda, \phi U)g_N(f_*X, f_*X) + g_M(\mathcal{H}gradln\lambda, X)g_N(f_*\phi U, f_*X) = 0.$$

From the above equation, it follows that λ is constant on $\Gamma(\phi D_2)$. Thus λ is constant on $\Gamma(\ker f_*)^\perp$. □

Theorem 3.25. *Let $(M, \phi, \xi, \eta, g_M)$ be a Cosymplectic manifold and (N, g_N) a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-invariant submersion, then f is totally geodesic map if and only if*

(*i*) $C\mathcal{T}_U\phi V + \omega\widehat{\nabla}_U\phi V = 0$, for $U, V \in \Gamma(D_1)$,
(*ii*) $\omega\mathcal{T}_U\phi W + C\mathcal{H}\nabla^M_U\phi W$, for $W \in \Gamma(D_2)$ and $U \in \Gamma(\ker f_*)$,
(*iii*) f is a horizontally homothetic map,

(*iv*) $\mathcal{T}_U BZ + \mathcal{H}\nabla^M_U CZ \in \Gamma(\phi D_2)$ and $\mathcal{V}\nabla_U BZ + \mathcal{T}_V CZ \in \Gamma(D_1), U \in \Gamma(\ker f_*), Z \in \Gamma(\ker f_*)^\perp$.

Proof. (*i*) For $U, V \in \Gamma(D_1)$, using equations (2.1), (2.6), (2.11), (3.4), and (3.5), we get

$$\begin{aligned}(\nabla f_*)(U, V) &= -f_*(\nabla_U V)\\ &= f_*(B\mathcal{T}_U\phi V + C\mathcal{T}_U\phi V + \psi\widehat{\nabla}_U\phi V + \omega\widehat{\nabla}_U\phi V).\end{aligned}$$

Since $B\mathcal{T}_U\phi V + C\mathcal{T}_U\phi V \in \Gamma(\ker f_*)$, we get

$$(\nabla f_*)(U, V) = f_*(C\mathcal{T}_U\phi V + \omega\widehat{\nabla}_U\phi V).$$

Since f is a linear isometry between $\Gamma(\ker f_*)^\perp$ and TN, $(\nabla f_*)(U, V) = 0$ if and only if $C\mathcal{T}_U\phi V + \omega\widehat{\nabla}_U\phi V = 0$.

(ii) For $U \in \Gamma(\ker f_*)$ and $W \in \Gamma(D_2)$, using equations (2.1), (2.6), (2.12), (2.17), (3.4), and (3.5), we get

$$(\nabla f_*)(U,W) = f_*(\psi \mathcal{T}_U \phi W + \omega \mathcal{T}_U \phi W + B\nabla^M_U \phi W + C\mathcal{H}\nabla^M_U \phi W).$$

Since $\psi \mathcal{T}_U \phi W + B\nabla^M_U \phi W \in \Gamma(\ker f_*)$, we derive

$$(\nabla f_*)(U,W) = f_*(\omega \mathcal{T}_U \phi W + C\mathcal{H}\nabla^M_U \phi W).$$

Since f is a linear isometry between $\Gamma(\ker f_*)^\perp$ and TN, $(\nabla f_*)(U,V) = 0$ if and only if $\omega \mathcal{T}_U \phi W + C\mathcal{H}\nabla^M_U \phi W = 0$.

(iii) For $X, Y \in \Gamma(\mu)$, from Lemma 2.3(i), we get

$$(\nabla f_*)(X,Y) = X(ln\lambda) f_* Y + Y(ln\lambda) f_* X - g_M(X,Y) f_*(\mathcal{H} gradln\lambda).$$

From the above equation taking $Y = \phi X$, for $X \in \Gamma(\mu)$, we get

$$(\nabla f_*)(X, \phi X) = X(ln\lambda) f_* \phi X + \phi X(ln\lambda) f_* X.$$

If $(\nabla f_*)(X, \phi X) = 0$, we get

$$X(ln\lambda) f_* \phi X + \phi X(ln\lambda) f_* X = 0. \tag{3.25}$$

Taking inner product in equation (3.25) with $f_* X$ and by the fact that f is a conformal submersion, we get

$$g_M(\mathcal{H} gradln\lambda, \phi X) g_N(f_* \phi X, f_* X) + g_M(\mathcal{H} gradln\lambda, \phi X) g_N(f_* X, f_* X) = 0.$$

Above equation, it follows that λ is constant on $\Gamma\phi(\mu)$. On the other hand, taking inner product in equation (3.25) with $f_* \phi X$ and since f is a conformal submersion, we get

$$g_M(\mathcal{H} gradln\lambda, X) g_N(f_* \phi X, f_* \phi X) + g_M(\mathcal{H} gradln\lambda, \phi X) g_N(f_* X, f_* \phi X) = 0.$$

Above equation implies that λ is constant on $\Gamma(\mu)$. In a similar way, for $U, V \in \Gamma(D_2)$, using Lemma 2.3(i), we get

$$(\nabla f_*)(\phi U, \phi V) = \phi U(ln\lambda) f_* \phi V + \phi V(ln\lambda) f_* \phi U - g_M(\phi U, \phi V) f_*(\mathcal{H} gradln\lambda).$$

From the above equation taking $V = U$, we get

$$\begin{aligned}&(\nabla f_*)(\phi U, \phi U) \\ &= \phi U(ln\lambda) f_* \phi U + \phi U(ln\lambda) f_* \phi U - g_M(\phi U, \phi U) f_*(\mathcal{H} gradln\lambda).\end{aligned} \tag{3.26}$$

Taking inner product in equation (3.26) with $f_* \phi U$ and since f is a conformal submersion, we obtain

$$g_M(\mathcal{H} gradln\lambda, \phi U) g_M(\phi U, \phi U) = 0.$$

From the above equation, it follows that λ is constant on $\Gamma\phi(U)$. Thus λ is constant on $\Gamma(\ker f_*)^\perp$. If f is a horizontally homothetic map, then $f_*(\mathcal{H} gradln\lambda)$ vanishes; thus the converse is clear; that is, $(\nabla f_*)(X,Y) = 0$, for $X, Y \in \Gamma(\ker f_*)^\perp$.

(iv) In the same way with the proof of Theorem 4.3(d) in [2], we can show $\mathcal{T}_U BZ + \mathcal{H}\nabla^M_U CZ \in \Gamma(\phi D_2)$ and $\mathcal{V}\nabla_U BZ + \mathcal{T}_V CZ \in \Gamma(D_1)$, $U \in \Gamma(\ker f_*)$, $Z \in \Gamma(\ker f_*)^\perp$.

□

Example 3.26. Let R^7 have got a Cosymplectic structure as in Example 2.1. Let $f : R^7 \to R^2$ be a submersion defined by

$$f(x_1, x_2, x_3, y_1, y_2, y_3, z) = (e^{x_3}\cos y_2, e^{x_3}\sin y_2),$$

where $y_2 \in R - \{\frac{k\pi}{2}, k\pi\}$, $k \in R$. Then

$$\Gamma(\ker f_*) = \text{span}\{E_1 = \frac{\partial}{\partial y_1}, E_3 = \frac{\partial}{\partial y_3}, E_4 = \frac{\partial}{\partial x_1}, E_5 = \frac{\partial}{\partial x_2}, E_7 = \frac{\partial}{\partial z}\},$$

$$\Gamma(\ker f_*)^{\perp} = \text{span}\{E_2 = -e^{x_3}\sin y_2\frac{\partial}{\partial y_2} + e^{x_3}\cos y_2\frac{\partial}{\partial x_3},$$
$$E_6 = e^{x_3}\cos y_2\frac{\partial}{\partial y_2} + e^{x_3}\sin y_2\frac{\partial}{\partial x_3}\}.$$

Hence we have $\phi E_1 = E_4, \phi E_3 = E_6, \phi E_4 = -E_1, \phi E_5 = -E_2$. Thus it follows that $D_1 = \text{span}\{E_1, E_4\}$ and $D_1 = \text{span}\{E_3, E_5\}$. Let u_1, u_2 be local coordinates in R^2. Also by direct computations, we get

$$f_*E_2 = (e^{x_3})^2\frac{\partial}{\partial u_1} \text{ and } f_*E_6 = (e^{x_3})^2\frac{\partial}{\partial u_2}.$$

Hence, we have

$$g_2(f_*E_2, f_*E_2) = (e^{x_3})^2 g_7(E_2, E_2) \text{ and } g_2(f_*E_6, f_*E_6) = (e^{x_3})^2 g_7(E_6, E_6),$$

where g_7 and g_2 denote the Euclidean metrics on R^7 and R^2 respectively. Thus f is a conformal semi-invariant submersion with $\lambda = e^{x_3}$.

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