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ON A NEW CLASS OF BERNSTEIN TYPE OPERATORS BASED ON BETA FUNCTION

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Abstract. We develop Bernstein type operators using the Beta function and study their approximation properties. By using Korovkin's theorem, we achieve the uniform convergence of sequences of these operators. We obtain the rate of convergence in terms of modulus of continuity and establish the Voronovskaja type asymptotic result for these operators. At last the graphical comparison of these newly defined operators with few of the fundamental but significant operators is discussed.

1. Introduction and preliminaries

Let $B(X)$ be the class of all real-valued bounded functions defined on a nonempty set X. For approximation of a real-valued continuous function $f(x)$ defined on $[0,1]$, Bernstein [4] introduced the following positive linear operator B_n.
$B_n : B([0,1]) \to B([0,1])$ defined by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x)\ f\left(\frac{k}{n}\right) \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$ is the Bernstein basis function and $0 \leqslant k \leqslant n$. The expression in (1.1) is also called the Bernstein polynomial of degree n of the function f.

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Let $C(X)$ be the class of all real-valued continuous functions defined on a nonempty set X. If a real-valued function $f \in C([0,1])$, then $B_n(f;\cdot)$ converges to f uniformly on $[0,1]$.

Durrmeyer in [7] introduced a generalization of Bernstein operators using summation-integral type formula, as follows:

$$D_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t)\, dt. \tag{1.2}$$

In 1941, Szász and Mirakyan [20] introduced an operator, known as the Szász-Mirakyan operator, which is given below.

For $x \in [0,\infty)$, $S_n : C([0,\infty)) \to C([0,\infty))$ is defined as

$$S_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x)\, f\left(\frac{k}{n}\right), \tag{1.3}$$

where $s_{n,k}(x) = e^{-nx}\dfrac{(nx)^k}{k!}$.

Further, in 1957, Baskakov [3] introduced an operator on the space of continuous functions, known as the Baskakov operator, which is given below.

For $x \in [0,\infty)$, $V_n : C([0,\infty)) \to C([0,\infty))$ is defined as

$$V_n(f;x) = \sum_{k=0}^{\infty} v_{n,k}(x)\, f\left(\frac{k}{n}\right), \tag{1.4}$$

where $v_{n,k}(x) = \dbinom{n+k-1}{k} \dfrac{x^k}{(1+x)^{n+k}}$.

Many researchers have done lots of work using the Bernstein operator (1.1). Several generalization and development are studied in [1,2,5,6,9–16,18,19,21–23]. With the motivation from the above development, we introduce the following operator which provides better approximation than few of the above operators.

For $f \in C([0,1])$, we define a Beta-Bernstein operator as follows:
For $x \in [0,1]$, $\mathfrak{B}_n : C([0,1]) \to C([0,1])$ is defined as

$$\mathfrak{B}_n(f;x) = \sum_{k=0}^{n} \mathcal{P}_{n,k}(x)\, f\left(\frac{k}{n}\right), \tag{1.5}$$

where $\mathcal{P}_{n,k}(x) = \dbinom{n}{k} \dfrac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)}$. Here, $\beta(a,b)$ is the Beta function defined by $\beta(a,b) = \displaystyle\int_0^1 t^{a-1}(1-t)^{b-1}\, dt$ $(a,b>0)$.

It can be seen that the operator defined in (1.5) is a positive linear operator.

2. Auxiliary results

Lemma 2.1. *For $a,b>0$ and nonnegative integers n and k, we have the following expressions.*

(i) *For* $0 \leqslant k \leqslant n$, $\displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{\beta(a+k, b+n-k)}{\beta(a,b)} = 1$;

(ii) *For* $1 \leqslant k \leqslant n$, $\displaystyle\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(a+k, b+n-k)}{\beta(a,b)} = \frac{a}{a+b}$;

(iii) *For* $1 \leqslant k \leqslant n$,

$$\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(a+k, b+n-k)}{\beta(a,b)} \cdot \frac{k}{n} = \frac{(n-1)(a+1)a}{n(a+b+1)(a+b)} + \frac{a}{n(a+b)};$$

(iv) *For* $1 \leqslant k \leqslant n$,

$$\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(a+k, b+n-k)}{\beta(a,b)} \cdot \frac{k^2}{n^2} = \frac{(n-1)(n-2)(a+2)(a+1)a}{n^2(a+b+2)(a+b+1)(a+b)} + \frac{3(n-1)(a+1)a}{n^2(a+b+1)(a+b)} + \frac{a}{n^2(a+b)};$$

(v) *For* $1 \leqslant k \leqslant n$,

$$\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(a+k, b+n-k)}{\beta(a,b)} \cdot \frac{k^3}{n^3} = \frac{(n-1)(n-2)(n-3)(a+3)(a+2)(a+1)a}{n^3(a+b+3)(a+b+2)(a+b+1)(a+b)} + \frac{6(n-1)(n-2)(a+2)(a+1)a}{n^3(a+b+2)(a+b+1)(a+b)} + \frac{7(n-1)(a+1)a}{n^3(a+b+1)(a+b)} + \frac{a}{n^3(a+b)}.$$

Now, we obtain the first four raw moments of the operator defined in (1.5). The following lemma gives the raw moments of the operator defined in (1.5).

Lemma 2.2. *For $x \in [0,1]$ and the operator given in (1.5), the following equalities hold true.*

(i) $\mathfrak{B}_n(1; x) = 1$;

(ii) $\mathfrak{B}_n(t; x) = \dfrac{nx+1}{n+2}$;

(iii) $\mathfrak{B}_n(t^2; x) = \dfrac{(n-1)(nx+1)(nx+2)}{n(n+2)(n+3)} + \dfrac{nx+1}{n(n+2)}$;

(iv) $$\mathfrak{B}_n(t^3; x) = \frac{(n-1)(n-2)(nx+3)(nx+2)(nx+1)}{n^2(n+4)(n+3)(n+2)} + \frac{3(n-1)(nx+2)(nx+1)}{n^2(n+3)(n+2)} + \frac{nx+1}{n^2(n+2)};$$

(v) $$\mathfrak{B}_n(t^4; x) = \frac{(n-1)(n-2)(n-3)}{n^3} \cdot \frac{nx+4}{n+5} \cdot \frac{nx+3}{n+4} \cdot \frac{nx+2}{n+3} \cdot \frac{nx+1}{n+2} + \frac{6(n-1)(n-2)}{n^3} \cdot \frac{nx+3}{n+4} \cdot \frac{nx+2}{n+3} \cdot \frac{nx+1}{n+2} + \frac{7(n-1)}{n^3} \cdot \frac{nx+2}{n+3} \cdot \frac{nx+1}{n+2} + \frac{1}{n^3} \cdot \frac{nx+1}{n+2}.$$

Proof. (i) $\mathfrak{B}_n(1;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)}$

By part (i) of Lemma 2.1, we get

$$\mathfrak{B}_n(1;x) = 1.$$

(ii)

$$\begin{aligned}\mathfrak{B}_n(t;x) &= \sum_{k=0}^{n} \binom{n}{k} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k}{n} \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)}\end{aligned}$$

By part (ii) of Lemma 2.1, we get

$$\mathfrak{B}_n(t;x) = \frac{nx+1}{n+2}.$$

(iii)

$$\begin{aligned}\mathfrak{B}_n(t^2;x) &= \sum_{k=0}^{n} \binom{n}{k} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k^2}{n^2} \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k^2}{n^2} \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k}{n}\end{aligned}$$

By part (iii) of Lemma 2.1, we get

$$\mathfrak{B}_n(t^2;x) = \frac{1}{n}\left[(n-1)\cdot\frac{nx+2}{n+3}\cdot\frac{nx+1}{n+2} + \frac{nx+1}{n+2}\right].$$

$$\therefore \mathfrak{B}_n(t^2;x) = \frac{(n-1)(nx+1)(nx+2)}{n(n+2)(n+3)} + \frac{nx+1}{n(n+2)}.$$

(iv)

$$\begin{aligned}\mathfrak{B}_n(t^3;x) &= \sum_{k=0}^{n} \binom{n}{k} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k^3}{n^3} \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k^2}{n^2}\end{aligned}$$

By part (iv) of Lemma 2.1, we have

$$\begin{aligned}\mathfrak{B}_n(t^3;x) =& \frac{(n-1)(n-2)(nx+3)(nx+2)(nx+1)}{n^2(n+4)(n+3)(n+2)} \\ &+ \frac{3(n-1)(nx+2)(nx+1)}{n^2(n+3)(n+2)} + \frac{nx+1}{n^2(n+2)}.\end{aligned}$$

(v)

$$\mathfrak{B}_n(t^4;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k^4}{n^4}$$
$$= \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)} \cdot \frac{k^3}{n^3}$$

From the part (iv) of Lemma 2.1, we have

$$\mathfrak{B}_n(t^4;x) = \frac{(n-1)(n-2)(n-3)}{n^3} \cdot \frac{nx+4}{n+5} \cdot \frac{nx+3}{n+4} \cdot \frac{nx+2}{n+3} \cdot \frac{nx+1}{n+2}$$
$$+ \frac{6(n-1)(n-2)}{n^3} \cdot \frac{nx+3}{n+4} \cdot \frac{nx+2}{n+3} \cdot \frac{nx+1}{n+2}$$
$$+ \frac{7(n-1)}{n^3} \cdot \frac{nx+2}{n+3} \cdot \frac{nx+1}{n+2} + \frac{1}{n^3} \cdot \frac{nx+1}{n+2}.$$

□

The following lemma gives the moment estimation of the operator defined in (1.5) about x.

Lemma 2.3. *For $x \in [0,1]$, the following equalities give p^{th} ($p = 0,1,2,3,4$) moments for the operator defined in (1.5) about x.*

(i) $\mathfrak{B}_n((t-x)^0;x) = 1$;

(ii) $\mathfrak{B}_n((t-x);x) = \dfrac{1-2x}{n+2}$;

(iii) $\mathfrak{B}_n((t-x)^2;x) = \dfrac{3(2x^2-2x+1)}{(n+2)(n+3)} - \dfrac{2nx(x-1)}{(n+2)(n+3)} + \dfrac{1}{n(n+2)(n+3)}$;

(iv)
$$\mathfrak{B}_n((t-x)^3;x) = \frac{6(-4x^3+8x^2-12x+1)}{(n+2)(n+3)(n+4)} + \frac{nx(24x^2-45x+2)}{(n+2)(n+3)(n+4)}$$
$$+ \frac{2(5x-9)}{n(n+2)(n+3)(n+4)} - \frac{3n^2x^2}{(n+2)(n+3)(n+4)}$$
$$+ \frac{12}{(n+2)(n+3)(n+4)};$$

(v)
$$\mathfrak{B}_n((t-x)^4;x) = \frac{12n^2x^2(x-1)^2 - 2nx(126x^3-252x^2+169x-43)}{(n+2)(n+3)(n+4)(n+5)}$$
$$+ \frac{3n(40x^4-80x^3+132x^2-92x+25)+2(85x^2-85x+26)}{n(n+2)(n+3)(n+4)(n+5)}$$
$$- \frac{3n+4}{n^3(n+2)(n+3)(n+4)(n+5)}.$$

Proof. (i) Using the part (i) of Lemma 2.2 and linearity of $\mathfrak{B}_n$, we have

$$\mathfrak{B}_n((t-x)^0;x) = \mathfrak{B}_n(1;x) = 1.$$

(ii) Using the parts (i) and (ii) of Lemma 2.2 and linearity of $\mathfrak{B}_n$, we have

$$\mathfrak{B}_n((t-x);x) = \frac{nx+1}{n+2} - x.$$
$$\therefore \mathfrak{B}_n((t-x);x) = \frac{1-2x}{n+2}.$$

(iii) Proceeding in similar manner as above with parts (i), (ii), and (iii) of Lemma 2.2, we have

$$\begin{aligned}\mathfrak{B}_n((t-x)^2;x) =& \frac{(n-1)(nx+1)(nx+2)}{n(n+2)(n+3)} + \frac{nx+1}{n(n+2)} \\ &- 2x\left(\frac{nx+1}{n+2}\right) - x^2. \\ \therefore \mathfrak{B}_n((t-x)^2;x) =& \frac{3(2x^2-2x+1)}{(n+2)(n+3)} - \frac{2nx(x-1)}{(n+2)(n+3)} \\ &+ \frac{1}{n(n+2)(n+3)}.\end{aligned}$$

(iv) Using the parts (i) to (iv) of Lemma 2.2 and linearity of $\mathfrak{B}_n$, we obtain

$$\begin{aligned}\mathfrak{B}_n((t-x)^3;x) =& \left(\frac{(n-1)(n-2)(nx+3)(nx+2)(nx+1)}{n^2(n+4)(n+3)(n+2)}\right. \\ &\left.+\frac{3(n-1)(nx+2)(nx+1)}{n^2(n+3)(n+2)} + \frac{nx+1}{n^2(n+2)}\right) \\ &- 3x\left(\frac{(n-1)(nx+1)(nx+2)}{n(n+2)(n+3)} + \frac{nx+1}{n(n+2)}\right) \\ &+ 3x^2\left(\frac{nx+1}{n+2}\right) - x^3. \\ \therefore \mathfrak{B}_n((t-x)^3;x) =& \frac{6(-4x^3+8x^2-12x+1)}{(n+2)(n+3)(n+4)} + \frac{nx(24x^2-45x+2)}{(n+2)(n+3)(n+4)} \\ &+ \frac{2(5x-9)}{n(n+2)(n+3)(n+4)} - \frac{3n^2x^2-12}{(n+2)(n+3)(n+4)}.\end{aligned}$$

(v) With the similar procedure as above, we obtain

$$\begin{aligned}\mathfrak{B}_n((t-x)^4;x) =& \left(\frac{(n-1)(n-2)(n-3)(nx+4)(nx+3)(nx+2)(nx+1)}{n^3(n+2)(n+3)(n+4)(n+5)}\right. \\ &+\frac{6(n-1)(n-2)(nx+3)(nx+2)(nx+1)}{n^3(n+2)(n+3)(n+4)} \\ &\left.+\frac{7(n-1)(nx+2)(nx+1)}{n^3(n+2)(n+3)} + \frac{1}{n^3}\cdot\frac{nx+1}{n+2}\right) \\ &- 4x\left(\frac{(n-1)(n-2)(nx+3)(nx+2)(nx+1)}{n^2(n+4)(n+3)(n+2)}\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{3(n-1)(nx+2)(nx+1)}{n^2(n+3)(n+2)}+\frac{nx+1}{n^2(n+2)}\Bigg)\\
&+6x^2\left(\frac{(n-1)(nx+1)(nx+2)}{n(n+2)(n+3)}+\frac{nx+1}{n(n+2)}\right)\\
&-4x^3\left(\frac{nx+1}{n+2}\right)+x^4.
\end{aligned}
$$

$$
\begin{aligned}
\therefore \mathfrak{B}_n((t-x)^4;x) =&\frac{12n^2x^2(x-1)^2-2nx(126x^3-252x^2+169x-43)}{(n+2)(n+3)(n+4)(n+5)}\\
&+\frac{3(40x^4-80x^3+132x^2-92x+25)}{(n+2)(n+3)(n+4)(n+5)}\\
&+\frac{2(85x^2-85x+26)}{n(n+2)(n+3)(n+4)(n+5)}\\
&-\frac{3n+4}{n^3(n+2)(n+3)(n+4)(n+5)}.
\end{aligned}
$$

□

3. Main result

The following theorem shows the convergence of the sequence of operators (1.5) for a function $f \in C([0,1])$. Here, we consider $C([0,1])$ endowed with the norm $||f|| = \sup_{x\in[0,1]} |f(x)|$.

Theorem 3.1. *For every function $f \in C([0,1])$, it follows that*

$$||\mathfrak{B}_n(f;x) - f(x)|| \to 0 \text{ uniformly as } n \to \infty.$$

Proof. From Lemma 2.2, we have

$$
\begin{aligned}
\mathfrak{B}_n(1;x) &= 1,\\
\mathfrak{B}_n(t;x) &= \frac{nx+1}{n+2},\\
\text{and } \mathfrak{B}_n(t^2;x) &= \frac{(n-1)(nx+1)(nx+2)}{n(n+2)(n+3)}+\frac{nx+1}{n(n+2)}.
\end{aligned}
$$

Now, it follows that $\mathfrak{B}_n(t^m;x)$ converges uniformly to x^m $(m = 0,1,2)$ on $[0,1]$. Hence, the result follows by Korovkin's theorem [17]. □

4. Rate of convergence

For a function $f \in C([a,b])$, the modulus of continuity is defined as

$$\omega_f(\delta) \equiv \omega(f,\delta) = \sup_{\substack{x-\delta\leqslant t\leqslant x+\delta\\ a\leqslant x\leqslant b}} \{|f(t)-f(x)|\} \;\; ; \text{ where } \delta > 0.$$

Now, we estimate the rate of convergence of the sequence of operators (1.5). The following theorem gives the rate of convergence of the sequence of operators (1.5) in terms of modulus of continuity of a function $f \in C([0,1])$.

Theorem 4.1. *For every function* $f \in C([0,1])$,

$$|\mathfrak{B}_n(f;x) - f(x)| \leqslant 2\ \omega(f, \sqrt{\delta_n}),$$

where $\delta_n = \mathfrak{B}_n((t-x)^2;x)$.

Proof. Since the operator (1.5) is monotonic, we have

$$\begin{aligned}
|\mathfrak{B}_n(f;x) - f(x)| &\leqslant \sum_{k=0}^{n} \mathcal{P}_{n,k}(x)\ \left| f\left(\frac{k}{n}\right) - f(x) \right| \\
&\leqslant \sum_{k=0}^{n} \mathcal{P}_{n,k}(x)\ \omega_f\left(\frac{k}{n} - x\right) \\
&\leqslant \sum_{k=0}^{n} \mathcal{P}_{n,k}(x)\ \left(1 + \frac{1}{\delta^2}\left(\frac{k}{n} - x\right)^2\right) \omega_f(\delta) \\
&= \left[1 + \frac{1}{\delta^2} \sum_{k=0}^{n} \mathcal{P}_{n,k}(x)\ \left(\frac{k}{n} - x\right)^2\right] \omega_f(\delta) \\
&= \left[1 + \frac{1}{\delta^2}\ \mathfrak{B}_n((t-x)^2;x)\right] \omega_f(\delta).
\end{aligned}$$

If we take $\delta = \delta_n = (\mathfrak{B}_n((t-x)^2;x))$, then

$$\mathfrak{B}_n(f;x) - f(x)| \leqslant 2\ \omega_f(\sqrt{\delta_n})\ .$$

□

Let $f \in C([0,1])$, $C > 0$ and let $0 < r \leqslant 1$. A function f is said to be in the (Lipschitz) class $\mathrm{Lip}_C(r)$ if the inequality

$$|f(t) - f(x)| \leqslant C\ |t-x|^r$$

holds for $t, x \in [0,1]$.

Theorem 4.2. *Let* $f \in \mathrm{Lip}_C(r)$. *Then*

$$|\mathfrak{B}_n(f;x) - f(x)| \leqslant C\ \delta_n^r(x)$$

where $\delta_n(x) = (\mathfrak{B}_n((t-x)^2;x))^{\frac{1}{2}}$.

Proof. Using the monotonicity of the operator (1.5), we obtain

$$\begin{aligned}
|\mathfrak{B}_n(f;x) - f(x)| &\leqslant \sum_{k=0}^{n} \mathcal{P}_{n,k}(x)\ \left| f\left(\frac{k}{n}\right) - f(x) \right| \\
&\leqslant C\ \sum_{k=0}^{n} \mathcal{P}_{n,k}(x)\ \left|\frac{k}{n} - x\right|^r \\
&= C\ \sum_{k=0}^{n} \left(\mathcal{P}_{n,k}(x)\ \left(\frac{k}{n} - x\right)^2\right)^{\frac{r}{2}} (\mathcal{P}_{n,k}(x))^{\frac{2-r}{2}}\ .
\end{aligned}$$

Applying Holder's inequality for the sum, we obtain

$$\begin{aligned}|\mathfrak{B}_n(f;x)-f(x)| &\leqslant C\left(\sum_{k=0}^{n}\mathcal{P}_{n,k}(x)\ \left(\frac{k}{n}-x\right)^2\right)^{\frac{r}{2}}\left(\sum_{k=0}^{n}\mathcal{P}_{n,k}(x)\right)^{\frac{2-r}{2}}\\ &= C\left(\sum_{k=0}^{n}\mathcal{P}_{n,k}(x)\ \left(\frac{k}{n}-x\right)^2\right)^{\frac{r}{2}}\\ &= C\left(\mathfrak{B}_n((t-x)^2;x)\right)^{\frac{r}{2}}\\ &= C\ \delta_n^r(x),\end{aligned}$$

where $\delta_n(x)=\left(\mathfrak{B}_n\left((t-x)^2;x\right)\right)^{\frac{1}{2}}$.

□

Now, we establish a direct result for the operator (1.5) using the Peetre's K-functional and the second order modulus of continuity.

Let $f\in C([a,b])$. Let $\delta>0$ and let $\mathcal{W}^2=\{g\in C\left([a,b]\right):g',g''\in C\left([a,b]\right)\}$. The Peetre's K-functional is defined by

$$K_2(f,\delta)=\inf_{g\in\mathcal{W}^2}\{||f-g||+\delta||g''||\}.$$

For a function $f\in C([a,b])$, the second order modulus of continuity is defined by

$$\omega_2(f,\delta)=\sup_{0<h<\delta}\ \sup_{a\leqslant x\leqslant b}\ \{|f(x+2h)-2f(x+h)+f(x)|\}\ \ ;\text{ where }\delta>0.$$

The relation between the Peetre's K-functional and the second order modulus of continuity is given as follows.

There is a constant $M>0$ such that

$$K_2(f,\delta)\leqslant M\cdot\omega_2(f,\delta). \tag{4.1}$$

Theorem 4.3. *Let $f\in C\left([0,1]\right)$ and let $g\in C\left([0,1]\right)$ be such that $g',g''\in C\left([0,1]\right)$. Then for all $n\in\mathbb{N}$, there is a constant $M>0$ such that*

$$\left|\mathfrak{B}_n(f;x)-f(x)-g'(x)\left(\frac{1-2x}{n+2}\right)\right|\leqslant M\cdot\omega_2(f,\delta_n(x)),$$

where $\delta_n(x)=\left(\mathfrak{B}_n\left((t-x)^2;x\right)\right)^{\frac{1}{2}}$.

Proof. Set $\mathcal{W}^2$ for $[0,1]$. For $g\in\mathcal{W}^2$, employ the Taylor's expansion to obtain

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-u)g''(u)\ du.$$

By Lemma (2.3), we have

$$\mathfrak{B}_n(g;x)=g(x)+g'(x)\left(\frac{1-2x}{n+2}\right)+\mathfrak{B}_n\left(\int_x^t(t-u)g''(u)\ du;x\right).$$

$$\therefore \left|\mathfrak{B}_n(g;x) - g(x) - g'(x)\left(\frac{1-2x}{n+2}\right)\right| \leqslant \mathfrak{B}_n\left(\int_x^t |t-u|\ |g''(u)|\ du; x\right).$$

$$\therefore \left|\mathfrak{B}_n(g;x) - g(x) - g'(x)\left(\frac{1-2x}{n+2}\right)\right| \leqslant \mathfrak{B}_n\left((t-x)^2;x\right)||g''||.$$

Hence, we get

$$\left|\mathfrak{B}_n(g;x) - g(x) - g'(x)\left(\frac{1-2x}{n+2}\right)\right| \leqslant \left(\frac{3(2x^2-2x+1)}{(n+2)(n+3)} - \frac{2nx(x-1)}{(n+2)(n+3)} + \frac{1}{n(n+2)(n+3)}\right)||g''||.$$

Now,

$$\left|\mathfrak{B}_n(f;x) - f(x) - g'(x)\left(\frac{1-2x}{n+2}\right)\right| \leqslant |\mathfrak{B}_n(f-g;x) - (f-g)(x)| + \left|\mathfrak{B}_n(g;x) - g(x) - g'(x)\left(\frac{1-2x}{n+2}\right)\right|.$$

Using the relation $|\mathfrak{B}_n(f;x)| \leqslant ||f||$, we obtain

$$\left|\mathfrak{B}_n(f;x) - f(x) - g'(x)\left(\frac{1-2x}{n+2}\right)\right| \leqslant ||f-g|| + \left(\frac{3(2x^2-2x+1)}{(n+2)(n+3)} - \frac{2nx(x-1)}{(n+2)(n+3)} + \frac{1}{n(n+2)(n+3)}\right)||g''||.$$

Now taking infimum on the right side of the above inequality over all $g \in \mathcal{W}^2$, we get

$$\left|\mathfrak{B}_n(f;x) - f(x) - g'(x)\left(\frac{1-2x}{n+2}\right)\right| \leqslant K_2(f, \delta_n^2(x)),$$

where

$$\delta_n^2(x) = \frac{3(2x^2-2x+1)}{(n+2)(n+3)} - \frac{2nx(x-1)}{(n+2)(n+3)} + \frac{1}{n(n+2)(n+3)} = \mathfrak{B}_n\left((t-x)^2;x\right).$$

In the view of the property of Peetre's K-functional given in (4.1), we get

$$\left|\mathfrak{B}_n(f;x) - f(x) - g'(x)\left(\frac{1-2x}{n+2}\right)\right| \leqslant M \cdot \omega_2(f, \delta_n(x)).$$

□

5. Weighted approximation

Let $B_\rho([0,1])$ be the space of functions defined on $[0,1]$ satisfying the condition $|f(x)| \leqslant M_f \rho(x)$, where M_f is a constant depending only on f. Moreover $\rho(x)$ is a weight function defined on $[0,1]$ and bounded away from zero. By $C_\rho([0,1])$, we denote the subspace of $B_\rho([0,1])$, of all continuous functions belonging to $B_\rho([0,1])$. We define a norm $||\cdot||_\rho$ on $C_\rho([0,1])$ as $||f||_\rho = \sup\limits_{x\in[0,1]} \dfrac{|f(x)|}{\rho(x)}$.

Theorem 5.1. *For each $f \in C_\rho([0,1])$,*

$$\lim_{n\to\infty} ||\mathfrak{B}_n(f;x) - f(x)||_\rho = 0.$$

Proof. Using the subsequent lemma to [8, Theorem 1], we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} ||\mathfrak{B}_n(t^r;x) - x^r||_\rho = 0,\ \ r = 0,1,2. \tag{5.1}$$

Since, $\mathfrak{B}_n(1;x) = 1$, the first condition of (5.1) is satisfied for $r = 0$.

Now,

$$\begin{aligned} ||\mathfrak{B}_n(t;x) - x||_\rho &= \sup_{x\in[0,1]} \frac{|\mathfrak{B}_n(t;x) - x|}{\rho(x)} \\ &= \sup_{x\in[0,1]} \left|\frac{1-2x}{n+2}\right| \times \frac{1}{\rho(x)} \text{ (From Lemma 2.2)} \\ &= \frac{1}{n+2} \sup_{x\in[0,1]} \left|\frac{1-2x}{\rho(x)}\right|. \end{aligned}$$

$\therefore\ ||\mathfrak{B}_n(t;x) - x||_\rho \to 0$ as $n \to \infty$.

Therefore, the second condition of (5.1) is satisfied for $r = 1$.
Similarly, we have

$$\begin{aligned} ||\mathfrak{B}_n(t^2;x) - x||_\rho &= \sup_{x\in[0,1]} \frac{|\mathfrak{B}_n(t^2;x) - x|}{\rho(x)} \\ &= \sup_{x\in[0,1]} \left|\frac{(n-1)(nx+1)(nx+2)}{n(n+2)(n+3)} + \frac{nx+1}{n(n+2)} - x^2\right| \times \frac{1}{\rho(x)} \\ &\qquad \text{(From Lemma 2.2)} \\ &= \sup_{x\in[0,1]} \left|\frac{2nx(2-3x)}{(n+2)(n+3)} + \frac{3(1-2x^2)}{(n+2)(n+3)} + \frac{1}{n(n+2)(n+3)}\right| \times \frac{1}{\rho(x)} \\ &\leqslant \frac{2n}{(n+2)(n+3)} \sup_{x\in[0,1]} \left|\frac{x(2-3x)}{\rho(x)}\right| + \frac{3}{(n+2)(n+3)} \sup_{x\in[0,1]} \left|\frac{1-2x^2}{\rho(x)}\right| \\ &\quad + \frac{1}{n(n+2)(n+3)} \sup_{x\in[0,1]} \left|\frac{1}{\rho(x)}\right|. \end{aligned}$$

$\therefore\ ||\mathfrak{B}_n(t^2;x) - x^2||_\rho \to 0$ as $n \to \infty$.
Therefore, the third condition of (5.1) is satisfied for $r = 2$. Hence, we have

$$\lim_{n\to\infty} ||\mathfrak{B}_n(f;x) - f(x)||_\rho = 0\ .$$

□

6. Voronovskaya type theorem

In this section, we establish a Voronovskaya type asymptotic formula for the operator $\mathfrak{B}_n$.

Lemma 6.1. *For every $x \in [0,1]$, we have*

$$\lim_{n\to\infty} n \cdot \mathfrak{B}_n((t-x);x) = 1-2x$$
$$\text{and } \lim_{n\to\infty} n \cdot \mathfrak{B}_n((t-x)^2;x) = 2x(1-x).$$

The proof of the above lemma is clear from the expressions (ii) and (iii) of Lemma 2.3.

Theorem 6.2. *If $f \in C([0,1])$ such that $f', f'' \in C([0,1])$ and $x \in [0,1]$, then we have*

$$\lim_{n\to\infty} n(\mathfrak{B}_n(f;x) - f(x)) = (1-2x) \cdot f'(x) + x(1-x) \cdot f''(x) \ .$$

Proof. Let $f, f', f'' \in C([0,1])$ and let $x \in [0,1]$ be fixed.
By Taylor's expansion, we write

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2} f''(x) + h(t)(t-x)^2, \tag{6.1}$$

where $h(t) \in C([0,1])$ is the Peano type remainder and $\lim_{t\to x} h(t) = 0$.

Using the linearity of $\mathfrak{B}_n$ in (6.1), we get

$$\begin{aligned} n(\mathfrak{B}_n(f;x) - f(x)) =& n \cdot f'(x)\mathfrak{B}_n((t-x);x) + \frac{n}{2} \cdot f''(x)\mathfrak{B}_n((t-x)^2;x) \\ &+ n \cdot \mathfrak{B}_n(h(t)(t-x)^2;x). \end{aligned} \tag{6.2}$$

Using the Cauchy–Schwarz inequality, we have

$$\mathfrak{B}_n(h(t)(t-x)^2;x) \leqslant \sqrt{\mathfrak{B}_n(h^2(t);x)} \cdot \sqrt{\mathfrak{B}_n((t-x)^4;x)} \ .$$
$$\therefore n \cdot \mathfrak{B}_n(h(t)(t-x)^2;x) \leqslant \sqrt{\mathfrak{B}_n(h^2(t);x)} \cdot \sqrt{n^2 \cdot \mathfrak{B}_n((t-x)^4;x)}. \tag{6.3}$$

Now, for $x \in [0,1]$, From (v) of Lemma 2.3, $\lim_{n\to\infty} n^2 \cdot \mathfrak{B}_n((t-x)^4;x)$ is nonnegative and finite. As $h(t) \in C([0,1])$ and $\lim_{t\to x} h(t) = 0$, by using uniform convergence of the operators $\mathfrak{B}_n$ for $f(t) \in C([0,1])$, we have

$$\lim_{n\to\infty} \mathfrak{B}_n(h^2(t);x) = h^2(x) = 0$$

uniformly for $x \in [0,1]$. Hence, from (6.3), we get

$$\lim_{n\to\infty} n \cdot \mathfrak{B}_n(h(t)(t-x)^2;x) = 0.$$

Hence, from (6.2), we get

$$\lim_{n\to\infty} n(\mathfrak{B}_n(f;x) - f(x)) = (1-2x) \cdot f'(x) + x(1-x) \cdot f''(x) \ .$$

□

7. Graphical illustration and comparison

In this section, we show the approximation of some continuous functions by the operator $\mathfrak{B}_n$ graphically using Maple. Also, we show the graphical comparison of some operators and the operator $\mathfrak{B}_n$.

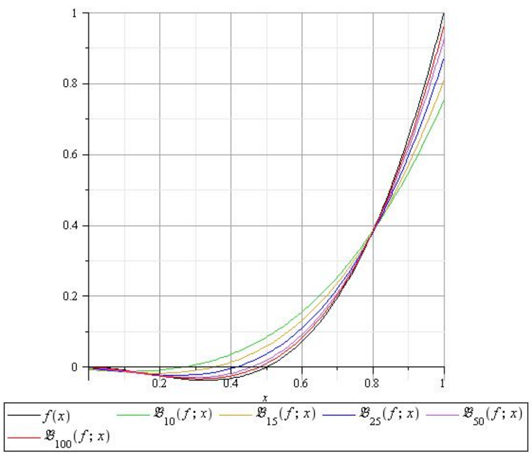

Figure 7-1

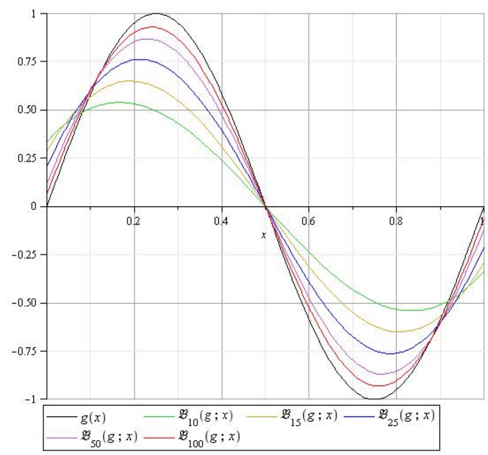

Figure 7-2

We consider two functions $f(x) = 2x^3 - x^2$ and $g(x) = \sin 2\pi x$ defined on $[0, 1]$. Figure 7-1 shows the approximation of the function f by the operator $\mathfrak{B}_n$ for $n = 10, 15, 25, 50$ and 100. Figure 7-2 shows the approximation of the function g by the operator $\mathfrak{B}_n$ for $n = 10, 15, 25, 50$ and 100.

Now, we show the comparison of few operators and the operator $\mathfrak{B}_n$ graphically. The comparison of the operators (1.2), (1.3), and (1.4) with the operator $\mathfrak{B}_n$ is shown in the following figures. The figures 7-3, 7-4, and 7-5 show the comparison of the operators (1.2), (1.3), (1.4), and $\mathfrak{B}_n$ approximating the function f for $n = 10, 25$ and 50, respectively.

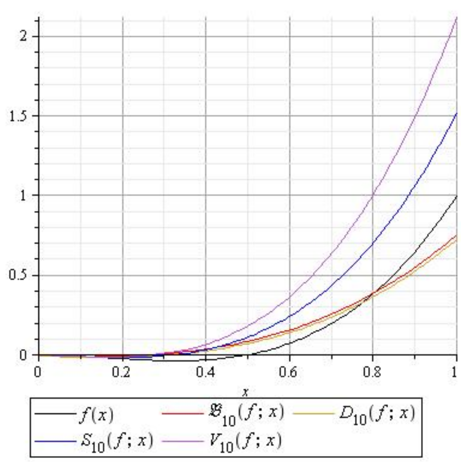

Figure 7-3

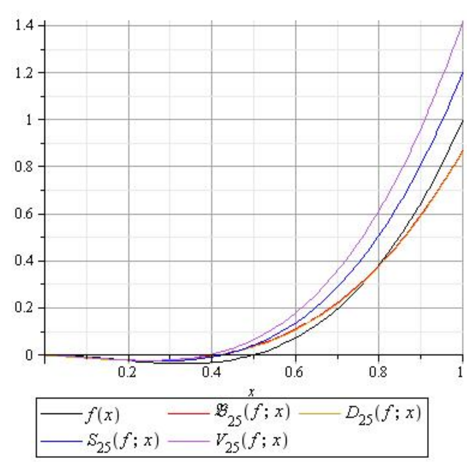

Figure 7-4

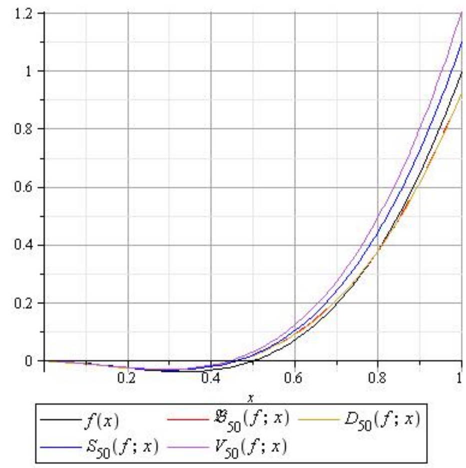

FIGURE 7-5

The figures 7-3, 7-4, and 7-5 show that approximations of the operator $\mathfrak{B}_n$ and the operator (1.2) are identical as n increases. The operator $\mathfrak{B}_n$ approximates the function f quite better than the operators (1.3) and (1.4) on $[0, 1]$.

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