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VARIOUS ENERGIES OF COMMUTING GRAPHS OF FINITE NONABELIAN GROUPS

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Abstract. The commuting graph of a finite nonabelian group G is a simple undirected graph, denoted by Γ_G, whose vertex set is the noncentral elements of G and two distinct vertices x and y are adjacent if and only if $xy = yx$. In this paper, we compute energy, Laplacian energy, and signless Laplacian energy of Γ_G for various families of finite nonabelian groups and analyze their values graphically. Our computations show that the conjecture posed in [*MATCH Commun. Math. Comput. Chem.* **59**, (2008) 343–354] holds for the commuting graph of some families of finite groups.

1. Introduction

Let $L(\mathcal{G})$ and $Q(\mathcal{G})$ be the Laplacian and signless Laplacian matrices of a graph $\mathcal{G}$, respectively. Then $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$ and $Q(\mathcal{G}) = D(\mathcal{G}) + A(\mathcal{G})$, where $A(\mathcal{G})$ and $D(\mathcal{G})$ are the adjacency and degree matrices of $\mathcal{G}$, respectively. The spectrum of $\mathcal{G}$ is a multiset given by $\text{spec}(\mathcal{G}) := \{\lambda_1^{p_1}, \lambda_2^{p_2}, \dots, \lambda_l^{p_l}\}$, where $\lambda_1, \lambda_2, \dots, \lambda_l$ are the eigenvalues of $A(\mathcal{G})$ with multiplicities $p_1, p_2, \dots, p_l$, respectively. Similarly, the Laplacian and signless Laplacian spectrums of $\mathcal{G}$ are defined by the multisets $\text{L-spec}(\mathcal{G}) := \{\mu_1^{q_1}, \mu_2^{q_2}, \dots, \mu_m^{q_m}\}$ and $\text{Q-spec}(\mathcal{G}) := \{\nu_1^{r_1}, \nu_2^{r_2}, \dots, \nu_n^{r_n}\}$, respectively, where $\mu_1, \mu_2, \dots, \mu_m$ are the eigenvalues of $L(\mathcal{G})$ with multiplicities $q_1, q_2, \dots, q_m$ and $\nu_1, \nu_2, \dots, \nu_n$ are the eigenvalues of $Q(\mathcal{G})$ with multiplicities $r_1, r_2, \dots, r_n$, respectively. A graph $\mathcal{G}$ is called integral if all the elements of $\text{spec}(\mathcal{G})$ are integers. Harary and Schwenk [12] introduced the concept of integral graphs in 1974. Similarly, $\mathcal{G}$ is called L-integral and Q-integral, respectively, if

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L-spec($\mathcal{G}$) and Q-spec($\mathcal{G}$) contain only integers. One may refer to [1,2,4,14,16,21] for various results of these graphs.

Depending on various spectra of a graph, there are various energies called *energy*, *Laplacian energy*, and *signless Laplacian energy* denoted by $E(\mathcal{G}), LE(\mathcal{G})$ and $LE^+(\mathcal{G})$, respectively. These energies are defined as follows:

$$E(\mathcal{G}) = \sum_{\lambda \in \text{spec}(\mathcal{G})} |\lambda|, \tag{1.1}$$

$$LE(\mathcal{G}) = \sum_{\mu \in \text{L-spec}(\mathcal{G})} \left| \mu - \frac{2|e(\mathcal{G})|}{|v(\mathcal{G})|} \right|, \tag{1.2}$$

and

$$LE^+(\mathcal{G}) = \sum_{\nu \in \text{Q-spec}(\mathcal{G})} \left| \nu - \frac{2|e(\mathcal{G})|}{|v(\mathcal{G})|} \right|, \tag{1.3}$$

where $v(\mathcal{G})$ and $e(\mathcal{G})$ denote the set of vertices and edges of $\mathcal{G}$, respectively.

The commuting graph of a finite nonabelian group G with center $Z(G)$ is a simple undirected graph, denoted by Γ_G, whose vertex set is $G \setminus Z(G)$, and two distinct vertices x and y are adjacent if and only if $xy = yx$. Various aspects of commuting graphs of finite groups can be found in [3, 13, 17, 19]. In [6–8, 18], Dutta and Nath have computed various spectra of Γ_G for different families of finite groups.

In this paper, we compute various energies of the commuting graphs of those families of finite nonabelian groups and analyze their values graphically. It may be mentioned here that various energies of the commuting graphs of some super integral groups are computed in [10, 20]. It is also worth mentioning that the Laplacian spectrum and energy of noncommuting graphs of some finite nonabelian groups are computed in [5] and [9], respectively.

The motivation of this paper lies in [11], where Gutman et al. posed the following conjecture.

Conjecture 1.1. $E(\mathcal{G}) \leq LE(\mathcal{G})$ for any graph $\mathcal{G}$.

The above conjecture was disproved in [15, 22], providing some counterexamples. Here we pose the following question comparing Laplacian and signless Laplacian energies of graphs.

Question 1.2. Is $LE(\mathcal{G}) \leq LE^+(\mathcal{G})$ for all graphs $\mathcal{G}$?

In this paper, we show that Conjecture 1.1 holds for commuting graphs of some families of finite groups. In particular, we show that the conjecture holds for the commuting graphs of the family of dihedral groups, quasidihedral groups, generalized quaternion groups, projective special linear groups $PSL(2, 2^k)$, general linear groups, the groups $A(n, \vartheta)$, and the family of metacyclic groups $M_{12n} = \langle a, b : a^6 = b^{2n} = 1, bab^{-1} = a^{-1} \rangle$ and $U_{6n} = \langle x, y : x^{2n} = y^3 = 1, x^{-1}yx = y^{-1} \rangle$. We also show that the inequality in Question 1.2 does not hold for commuting graphs of finite nonabelian groups in general.

2. Some computations

In this section, we compute various energies of the commuting graphs of some families of finite nonabelian groups. We begin with the family of groups G such that $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for any prime p.

Theorem 2.1. *Let G be a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime integer. Then*

$$E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G) = 2(p^2-1)|Z(G)| - 2(p+1).$$

Proof. The expression for $E(\Gamma_G)$ follows from [20, Theorem 3(i)].

We have $|v(\Gamma_G)| = (p^2-1)|Z(G)|$ and $\Gamma_G = (p+1)K_{(p-1)|Z(G)|}$. Therefore, $2|e(\Gamma_G)| = (p^2-1)|Z(G)|((p-1)|Z(G)|-1)$ and so

$$\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|} = (p-1)|Z(G)| - 1.$$

By [8, Theorem 2.3], we have

$$\text{L-spec}(\Gamma_G) = \{0^{p+1}, ((p-1)|Z(G)|)^{(p^2-1)|Z(G)|-p-1}\}.$$

Now, $\left|0 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = (p-1)|Z(G)| - 1$ and $\left|(p-1)|Z(G)| - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = 1$. Hence, by (1.2), we have

$$LE(\Gamma_G) = (p+1)((p-1)|Z(G)|-1) + (p^2-1)|Z(G)| - p - 1,$$

and the result follows.

By [8, Theorem 2.3], we also have

$$\text{Q-spec}(\Gamma_G) = \{(2(p-1)|Z(G)|-2)^{p+1}, ((p-1)|Z(G)|-2)^{(p^2-1)|Z(G)|-p-1}\}.$$

Now

$$\left|2(p-1)|Z(G)| - 2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = (p-1)|Z(G)| - 1, \text{and}$$
$$\left|(p-1)|Z(G)| - 2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = 1.$$

Hence, by (1.3), we have

$$LE^+(\Gamma_G) = (p+1)((p-1)|Z(G)|-1) + (p^2-1)|Z(G)| - p - 1,$$

and the result follows. □

As a consequence, we have the following result.

Corollary 2.2. *Let G be a nonabelian group of order p^3, for any prime p. Then*

$$E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G) = 2p^3 - 4p - 2.$$

Proof. The result follows from Theorem 2.1, since $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. □

Theorem 2.3. *Let G be a finite group such that $\frac{G}{Z(G)} \cong D_{2m}$ for $m \geq 2$. Then*

(1) $E(\Gamma_G) = (4m-2)|Z(G)| - 2(m+1)$.

(2) *If* $m = 2$; $m = 3$ *and* $|Z(G)| = 1, 2$; *or* $m = 4$ *and* $|Z(G)| = 1$, *then*

$$LE(\Gamma_G) = \frac{(2m^3+2)|Z(G)| - 4m^2 - 2m + 2}{2m-1}.$$

(3) *If* $m = 3$ *and* $|Z(G)| \geq 3$; $m = 4$ *and* $|Z(G)| \geq 2$; *or* $m \geq 5$, *then*

$$LE(\Gamma_G) = \frac{(2m^3 - 6m^2 + 4m)|Z(G)|^2 + (2m^2 - 2m + 2)|Z(G)| - 4m + 2}{2m-1}.$$

(4) *If* $m = 2$, *then* $LE^+(\Gamma_G) = 6|Z(G)| - 6$.
(5) *If* $m = 3$ *and* $|Z(G)| = 1$, *then* $LE^+(\Gamma_G) = \frac{16}{5}$.
(6) *If* $m = 3$ *and* $|Z(G)| \geq 2$, *then* $LE^+(\Gamma_G) = \frac{12|Z(G)|^2+18|Z(G)|-30}{5}$.
(7) *If* $m = 4$ *and* $|Z(G)| \leq 6$, *then* $LE^+(\Gamma_G) = \frac{48|Z(G)|^2}{7}$.
(8) *If* $m = 4$ *and* $|Z(G)| > 6$, *then* $LE^+(\Gamma_G) = \frac{48|Z(G)|^2+8|Z(G)|-56}{7}$.
(9) *If* $m \geq 5$, *then* $LE^+(\Gamma_G) = \frac{(2m^3-6m^2+4m)|Z(G)|^2}{2m-1}$.

Proof. The expression for $E(\Gamma_G)$ follows from [20, Theorem 3(ii)].

Since $\Gamma_G = K_{(m-1)|Z(G)|} \sqcup mK_{|Z(G)|}$, we have $|v(\Gamma_G)| = (2m-1)|Z(G)|$ and $2|e(\Gamma_G)| = (m-1)|Z(G)|((m-1)|Z(G)| - 1) + m|Z(G)|(|Z(G)| - 1)$. Therefore,

$$\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|} = \frac{(m^2 - m + 1)|Z(G)| - 2m + 1}{2m-1}.$$

Note that for any two integers r, s, we have

$$r|Z(G)| + s - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|} = \frac{((2r+1)m - m^2 - r - 1)|Z(G)| + 2m(s+1) - s - 1}{2m-1}. \tag{2.1}$$

By [8, Theorem 2.5], we have

$$\text{L-spec}(\Gamma_G) = \{0^{m+1}, ((m-1)|Z(G)|)^{(m-1)|Z(G)|-1}, (|Z(G)|)^{m(|Z(G)|-1)}\}.$$

Therefore, using (2.1), we have

$$\left|0 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{(m^2 - m + 1)|Z(G)| - 2m + 1}{2m-1},$$
$$\left|(m-1)|Z(G)| - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{(m^2 - 2m)|Z(G)| + 2m - 1}{2m-1},$$

and

$$\left||Z(G)| - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \begin{cases} \frac{(3m-m^2-2)|Z(G)|+2m-1}{2m-1} & \text{if } m = 2; \text{or } m = 3 \text{ and } |Z(G)| = 1, 2; \\ & \text{or } m = 4 \text{ and } |Z(G)| = 1, \\ \frac{(-3m+m^2+2)|Z(G)|-2m+1}{2m-1} & \text{if } m = 3 \text{ and } |Z(G)| \geq 3; \\ & \text{or } m = 4 \text{ and } |Z(G)| \geq 2; \\ & \text{or } m \geq 5. \end{cases}$$

Therefore, if $m = 2$; $m = 3$ and $|Z(G)| = 1, 2$; or $m = 4$ and $|Z(G)| = 1$, then by (1.2), we have

$$\begin{aligned} LE(\Gamma_G) =& \frac{(m+1)((m^2-m+1)|Z(G)|-2m+1)}{2m-1} \\ &+ \frac{((m-1)|Z(G)|-1)((m^2-2m)|Z(G)|+2m-1)}{2m-1} \\ &+ \frac{(m(|Z(G)|-1))((3m-m^2-2)|Z(G)|+2m-1)}{2m-1}, \end{aligned}$$

and hence the result follows on simplification.

If $m = 3$ and $|Z(G)| \geq 3$; or $m = 4$ and $|Z(G)| \geq 2$; or $m \geq 5$, then by (1.2), we have

$$\begin{aligned} LE(\Gamma_G) =& \frac{(m+1)((m^2-m+1)|Z(G)|-2m+1)}{2m-1} \\ &+ \frac{((m-1)|Z(G)|-1)((m^2-2m)|Z(G)|+2m-1)}{2m-1} \\ &+ \frac{(m(|Z(G)|-1))((-3m+m^2+2)|Z(G)|-2m+1)}{2m-1}, \end{aligned}$$

and hence the result follows on simplification.

By [8, Theorem 2.5], we also have

$$\begin{aligned} \text{Q-spec}(\Gamma_G) =& \{(2(m-1)|Z(G)|-2)^1, ((m-1)|Z(G)|-2)^{(m-1)|Z(G)|-1}, \\ & (2|Z(G)|-2)^m, (|Z(G)|-2)^{m(|Z(G)|-1)}\}. \end{aligned}$$

Now, using (2.1), we have

$$\left|2(m-1)|Z(G)|-2-\tfrac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \tfrac{(3m^2-5m+1)|Z(G)|-2m+1}{2m-1},$$

$$\left|(m-1)|Z(G)|-2-\tfrac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \begin{cases} \frac{(m^2-2m)|Z(G)|-2m+1}{2m-1} & \text{if } m=3 \text{ and } |Z(G)| \geq 2; \\ & \text{or } m \geq 4, \\ \frac{(-m^2+2m)|Z(G)|+2m-1}{2m-1} & \text{if } m=2; \text{ or } m=3 \\ & \text{and } |Z(G)|=1, \end{cases}$$

$$\left|2|Z(G)|-2-\tfrac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \begin{cases} \frac{(5m-m^2-3)|Z(G)|-2m+1}{2m-1} & \text{if } m=2; \text{ or } m=3 \text{ and} \\ & |Z(G)| \geq 2; \text{ or} \\ & m=4 \text{ and } |Z(G)|>6, \\ \frac{(-5m+m^2+3)|Z(G)|+2m-1}{2m-1} & \text{if } m=3 \text{ and } |Z(G)|=1; \\ & \text{or } m=4 \text{ and } |Z(G)| \leq 6; \\ & \text{or } m \geq 5, \end{cases}$$

and $\left||Z(G)|-2-\tfrac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \tfrac{(-3m+m^2+2)|Z(G)|+2m-1}{2m-1}$.

If $m = 2$, then by (1.3) and substitution, we have

$$LE^+(\Gamma_G) = \frac{3|Z(G)|-3}{3} + \frac{3|Z(G)|-3}{3} + \frac{6|Z(G)|-6}{3} + \frac{6|Z(G)|-6}{3},$$

and the result follows on simplification.

If $m = 3$ and $|Z(G)| = 1$, then by (1.3) and substitution, we get $LE^+(\Gamma_G) = \frac{16}{5}$.

If $m = 3$ and $|Z(G)| \geq 2$, then by (1.3) and substitution, we have

$$LE^+(\Gamma_G) = \frac{13|Z(G)| - 5}{5} + \frac{6|Z(G)|^2 - 13|Z(G)| + 5}{5} + \frac{9|Z(G)| - 15}{5} + \frac{6|Z(G)|^2 + 9|Z(G)| - 15}{5},$$

and the result follows on simplification.

If $m = 4$ and $|Z(G)| \leq 6$, then by (1.3) and substitution, we have

$$LE^+(\Gamma_G) = \frac{29|Z(G)| - 7}{7} + \frac{(3|Z(G)| - 1)(8|Z(G)| - 7)}{7} + \frac{4(-|Z(G)| + 7)}{7} + \frac{4(|Z(G)| - 1)(6|Z(G)| + 7)}{7},$$

and the result follows on simplification.

If $m = 4$ and $|Z(G)| > 6$, then by (1.3) and substitution, we have

$$LE^+(\Gamma_G) = \frac{29|Z(G)| - 7}{7} + \frac{24|Z(G)|^2 - 29|Z(G)| + 7}{7} + \frac{4|Z(G)| - 28}{7} + \frac{24|Z(G)|^2 + 4|Z(G)| - 28}{7},$$

and the result follows on simplification.

If $m \geq 5$, then by (1.3) and substitution, we have

$$\begin{aligned} LE^+(\Gamma_G) = &\frac{(3m^2 - 5m + 1)|Z(G)| - 2m + 1}{2m - 1} \\ &+ \frac{((m-1)|Z(G)| - 1)((m^2 - 2m)|Z(G)| - 2m + 1)}{2m - 1} \\ &+ \frac{m((-5m + m^2 + 3)|Z(G)| + 2m - 1)}{2m - 1} \\ &+ \frac{m(|Z(G)| - 1)((-3m + m^2 + 2)|Z(G)| + 2m - 1)}{2m - 1}, \end{aligned}$$

and hence the result follows on simplification. □

Using Theorem 2.3, we now compute the energy, Laplacian energy, and signless Laplacian energy of the commuting graphs of the groups M_{2mn}, D_{2m} and Q_{4n}, respectively.

Corollary 2.4. *Let $M_{2mn} = \langle a, b : a^m = b^{2n} = 1, bab^{-1} = a^{-1}\rangle$ be a metacyclic group, where $m > 2$.*

If m is odd, then

$$E(\Gamma_{M_{2mn}}) = (4m - 2)n - 2(m + 1),$$

$$LE(\Gamma_{M_{2mn}}) = \begin{cases} \frac{56n-40}{5} & \text{if } m = 3 \text{ and } n = 1, 2, \\ \frac{12n^2+14n-10}{5} & \text{if } m = 3 \text{ and } n \geq 3, \\ \frac{(2m^3-6m^2+4m)n^2+(2m^2-2m+2)n-4m+2}{2m-1} & \text{otherwise}, \end{cases}$$

and

$$LE^+(\Gamma_{M_{2mn}}) = \begin{cases} \frac{16}{5} & \text{if } m = 3 \text{ and } n = 1, \\ \frac{12n^2+18n-30}{5} & \text{if } m = 3 \text{ and } n \geq 2, \\ \frac{(2m^3-6m^2+4m)n^2}{2m-1} & \text{otherwise.} \end{cases}$$

If m is even, then

$$E(\Gamma_{M_{2mn}}) = (4m-4)n - (m+2),$$

$$LE(\Gamma_{M_{2mn}}) = \begin{cases} 12n-6 & \text{if } m = 4, \\ \frac{112n-40}{5} & \text{if } m = 6 \text{ and } n = 1, 2, \\ \frac{48n^2+28n-10}{5} & \text{if } m = 6 \text{ and } n > 2, \\ \frac{192n^2+52n-14}{7} & \text{if } m = 8, \\ \frac{(m^3-6m^2+8m)n^2+(m^2-2m+4)n-2m+2}{m-1} & \text{otherwise,} \end{cases}$$

and

$$LE^+(\Gamma_{M_{2mn}}) = \begin{cases} 12n-6 & \text{if } m = 4, \\ \frac{48n^2+36n-30}{5} & \text{if } m = 6, \\ \frac{192n^2}{7} & \text{if } m = 8 \text{ and } n \leq 3, \\ \frac{192n^2+16n-56}{7} & \text{if } m = 8 \text{ and } n > 3, \\ \frac{(m^3-6m^2+8m)n^2}{m-1} & \text{otherwise.} \end{cases}$$

Proof. The result follows from Theorem 2.3, using the facts

$$Z(M_{2mn}) = \begin{cases} \langle b^2 \rangle & \text{if } m \text{ is odd,} \\ \langle b^2 \rangle \cup a^{\frac{m}{2}} \langle b^2 \rangle & \text{if } m \text{ is even,} \end{cases} \text{ and } \frac{M_{2mn}}{Z(M_{2mn})} \cong \begin{cases} D_{2m} & \text{if } m \text{ is odd,} \\ D_m & \text{if } m \text{ is even.} \end{cases}$$

□

Putting $n = 1$ in Corollary 2.4, we get the following result.

Corollary 2.5. *Let $D_{2m} = \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1} \rangle$ be the dihedral group of order $2m$, where $m > 2$.*

If m is odd, then

$$E(\Gamma_{D_{2m}}) = 2m-4, \quad LE(\Gamma_{D_{2m}}) = \begin{cases} \frac{16}{5} & \text{if } m = 3, \\ \frac{2(m+1)(m-1)(m-2)}{2m-1} & \text{otherwise,} \end{cases}$$

$$\text{and} \quad LE^+(\Gamma_{D_{2m}}) = \begin{cases} \frac{16}{5} & \text{if } m = 3, \\ \frac{2m^3-6m^2+4m}{2m-1} & \text{otherwise.} \end{cases}$$

If m is even, then

$$E(\Gamma_{D_{2m}}) = 3m-6, \quad LE(\Gamma_{D_{2m}}) = \begin{cases} 6 & \text{if } m = 4, \\ \frac{72}{5} & \text{if } m = 6, \\ \frac{230}{7} & \text{if } m = 8, \\ \frac{m^3-5m^2+4m+6}{m-1} & \text{otherwise,} \end{cases}$$

$$\text{and} \quad LE^+(\Gamma_{D_{2m}}) = \begin{cases} 6 & \text{if } m = 4, \\ \frac{54}{5} & \text{if } m = 6, \\ \frac{192}{7} & \text{if } m = 8, \\ \frac{m^3-6m^2+8m}{m-1} & \text{otherwise.} \end{cases}$$

Corollary 2.6. *Let $Q_{4m} = \langle x, y : y^{2m} = 1, x^2 = y^m, xyx^{-1} = y^{-1} \rangle$, where $m \geq 2$, be the generalized quaternion group of order $4m$. Then*

$$E(\Gamma_{Q_{4m}}) = 6m - 6, \quad LE(\Gamma_{Q_{4m}}) = \begin{cases} 6 & \text{if } m = 2, \\ \frac{72}{5} & \text{if } m = 3, \\ \frac{230}{7} & \text{if } m = 4, \\ \frac{8m^3-20m^2+8m+6}{2m-1} & \text{otherwise,} \end{cases}$$

$$\text{and} \quad LE^+(\Gamma_{Q_{4m}}) = \begin{cases} 6 & \text{if } m = 2, \\ \frac{54}{5} & \text{if } m = 3, \\ \frac{192}{7} & \text{if } m = 4, \\ \frac{2m^3-6m^2+4m}{2m-1} & \text{otherwise.} \end{cases}$$

Proof. We have $Z(Q_{4m}) = \{1, a^m\}$ and $\frac{Q_{4m}}{Z(Q_{4m})} \cong D_{2m}$. Therefore, the result follows from Theorem 2.3. □

It may be mentioned here that Corollaries 2.5 and 2.6 are also obtained, by direct calculations, in [10, Theorems 2.2 and 2.1] along with [20, Theorem 1(ii), (iii)]. We also have the following result as a corollary of Theorem 2.3, noting that $|Z(U_{6n})| = n$ and the central quotient of U_{6n} is isomorphic to D_6.

Corollary 2.7. *Let $U_{6n} = \langle x, y : x^{2n} = y^3 = 1, x^{-1}yx = y^{-1} \rangle$. Then*

$$E(\Gamma_{U_{6n}}) = 10n - 8, \quad LE(\Gamma_{U_{6n}}) = \begin{cases} \frac{56n-40}{5} & \text{if } n = 1, 2, \\ \frac{12n^2+14n-10}{5} & \text{if } n \geq 3, \end{cases}$$

$$\text{and} \quad LE^+(\Gamma_{U_{6n}}) = \begin{cases} \frac{16}{5} & \text{if } n = 1, \\ \frac{12n^2+18n-30}{5} & \text{if } n \geq 2. \end{cases}$$

Theorem 2.8. *If G is a finite group such that $\frac{G}{Z(G)}$ is isomorphic to $Sz(2) = \langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2 \rangle$, known as the Suzuki group, then*

$$E(\Gamma_G) = 38|Z(G)| - 12, \quad LE(\Gamma_G) = \begin{cases} \frac{732|Z(G)|-228}{19} & \text{if } |Z(G)| \leq 4, \\ \frac{120|Z(G)|^2+122|Z(G)|-38}{19} & \text{if } |Z(G)| > 4, \end{cases}$$

$$\text{and} \quad LE^+(\Gamma_G) = \begin{cases} \frac{484}{19}, & \text{if } |Z(G)| = 1, \\ \frac{120|Z(G)|^2+530|Z(G)|-190}{19}, & \text{if } |Z(G)| > 1. \end{cases}$$

Proof. The expression for $E(\Gamma_G)$ follows from [20, Theorem 2(iv)].

We have $|v(\Gamma_G)| = 19|Z(G)|$ and $|e(\Gamma_G)| = \frac{4|Z(G)|(4|Z(G)|-1)+15|Z(G)|(3|Z(G)|-1)}{2}$ as $\Gamma_G = K_{4|Z(G)|} \sqcup 5K_{3|Z(G)|}$. Therefore,

$$\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|} = \frac{4|Z(G)|(4|Z(G)|-1)+15|Z(G)|(3|Z(G)|-1)}{19|Z(G)|} = \frac{61|Z(G)|-19}{19}.$$

Note that for any two integers r, s, we have

$$r|Z(G)| + s - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|} = \frac{(19r-61)|Z(G)| + 19(s+1)}{19}. \tag{2.2}$$

By [8, Theorem 2.2], we have

$$\text{L-spec}(\Gamma_G) = \{0^6, (4|Z(G)|)^{4|Z(G)|-1}, (3|Z(G)|)^{15|Z(G)|-5}\}.$$

Using (2.2), we have $\left|0 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{61|Z(G)|-19}{19}$, $\left|4|Z(G)| - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{15|Z(G)|+19}{19}$, and

$$\left|3|Z(G)| - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{-4|Z(G)|+19}{19}\right| = \begin{cases} \frac{-4|Z(G)|+19}{19} & \text{if } |Z(G)| \leq 4, \\ \frac{4|Z(G)|-19}{19} & \text{if } |Z(G)| > 4. \end{cases}$$

Therefore, if $|Z(G)| \leq 4$, then by (1.2) and substitution, we have

$$\begin{aligned} LE(\Gamma_G) =& \frac{366|Z(G)|-114}{19} + \frac{60|Z(G)|^2 + 61|Z(G)| - 19}{19} \\ &+ \frac{-60|Z(G)|^2 + 305|Z(G)| - 95}{19}, \end{aligned}$$

and the result follows on simplification.

If $|Z(G)| > 4$, then by (1.2) and substitution, we have

$$\begin{aligned} LE(\Gamma_G) =& \frac{366|Z(G)|-114}{19} + \frac{60|Z(G)|^2 + 61|Z(G)| - 19}{19} \\ &+ \frac{60|Z(G)|^2 - 305|Z(G)| + 95}{19}, \end{aligned}$$

and the result follows on simplification.

By [8, Theorem 2.2], we also have

$$\begin{aligned} \text{Q-spec}(\Gamma_G) = \{&(8|Z(G)|-2)^1, (4|Z(G)|-2)^{4|Z(G)|-1}, \\ &(6|Z(G)|-2)^5, (3|Z(G)|-2)^{15|Z(G)|-5}\}. \end{aligned}$$

Now, using (2.2), we have

$\left|8|Z(G)| - 2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{91|Z(G)|-19}{19}$,

$\left|4|Z(G)| - 2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \begin{cases} \frac{-15|Z(G)|+19}{19} & \text{if } |Z(G)| = 1, \\ \frac{15|Z(G)|-19}{19} & \text{if } |Z(G)| > 1, \end{cases}$

$\left|6|Z(G)| - 2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{53|Z(G)|-19}{19}$ and $\left|3|Z(G)| - 2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{4|Z(G)|+19}{19}$.

Hence, if $|Z(G)| = 1$, then by (1.3) and substitution, we have

$$LE^+(\Gamma_G) = \frac{91|Z(G)| - 19}{19} + \frac{(4|Z(G)| - 1)(-15|Z(G)| + 19)}{19} + \frac{265|Z(G)| - 95}{19} + \frac{(15|Z(G)| - 5)(4|Z(G)| + 19)}{19} = \frac{484}{19}.$$

If $|Z(G)| > 1$, then by (1.3) and substitution, we have

$$LE^+(\Gamma_G) = \frac{91|Z(G)| - 19}{19} + \frac{(4|Z(G)| - 1)(15|Z(G)| - 19)}{19} + \frac{5(53|Z(G)| - 19)}{19} + \frac{(15|Z(G)| - 5)(4|Z(G)| + 19)}{19},$$

and hence the result follows on simplification. □

Theorem 2.9. *Let $QD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1}\rangle$ be the quasidihedral group, where $n \geq 4$. Then*

$$E(\Gamma_{QD_{2^n}}) = 3(2^{n-1} - 2),\ LE(\Gamma_{QD_{2^n}}) = \frac{2^{3n-3} - 5.2^{2n-2} + 2^{n+1} + 6}{2^{n-1} - 1},$$

$$\text{and} \quad LE^+(\Gamma_{QD_{2^n}}) = \frac{5.2^{3n-4} - 15.2^{2n-2} + 5.2^{n+2}}{2^{n-1} - 1}.$$

Proof. The expression for $E(\Gamma_{QD_{2^n}})$ follows from [20, Theorem 2(i)].

We have $|v(\Gamma_{QD_{2^n}})| = 2(2^{n-1} - 1)$ and $|e(\Gamma_{QD_{2^n}})| = \frac{2^{2n-2} - 2^{n+1} + 6}{2}$, since $\Gamma_{QD_{2^n}} = 2^{n-2}K_2 \sqcup K_{2^{n-1}-2}$. Therefore,

$$\frac{2|e(\Gamma_{QD_{2^n}})|}{|v(\Gamma_{QD_{2^n}})|} = \frac{2^{2n-2} - 2^{n+1} + 6}{2(2^{n-1} - 1)}.$$

By [8, Proposition 2.10], we have

$$\text{L-spec}(\Gamma_{QD_{2^n}}) = \{0^{2^{n-2}+1}, (2^{n-1} - 2)^{2^{n-1}-3}, 2^{2^{n-2}}\}.$$

Therefore, $\left|0 - \frac{2|e(\Gamma_{QD_{2^n}})|}{|v(\Gamma_{QD_{2^n}})|}\right| = \frac{2^{2n-2} - 2^{n+1} + 6}{2(2^{n-1} - 1)}$, $\left|2^{n-1} - 2 - \frac{2|e(\Gamma_{QD_{2^n}})|}{|v(\Gamma_{QD_{2^n}})|}\right| = \frac{2^{2n-2} - 2^n - 2}{2(2^{n-1} - 1)}$ and $\left|2 - \frac{2|e(\Gamma_{QD_{2^n}})|}{|v(\Gamma_{QD_{2^n}})|}\right| = \frac{2^{2n-2} - 2^{n+2} + 10}{2(2^{n-1} - 1)}$. Hence, by (1.2) and substitution, we have

$$LE(\Gamma_{QD_{2^n}}) = \frac{(2^{n-2} + 1)(2^{2n-2} - 2^{n+1} + 6)}{2(2^{n-1} - 1)} + \frac{(2^{n-1} - 3)(2^{2n-2} - 2^n - 2)}{2(2^{n-1} - 1)} + \frac{2^{n-2}(2^{2n-2} - 2^{n+2} + 10)}{2(2^{n-1} - 1)},$$

and the result follows on simplification.

By [8, Proposition 2.10], we also have

$$\text{Q-spec}(\Gamma_{QD_{2^n}}) = \{(2^n - 6)^1, (2^{n-1} - 4)^{2^{n-1}-3}, 2^{2^{n-2}}, 0^{2^{n-2}}\}.$$

Now

$\left|2^n - 6 - \frac{2|e(\Gamma_{QD_{2^n}})|}{|v(\Gamma_{QD_{2^n}})|}\right| = \frac{3.2^{2n-2} - 3.2^{n+1} + 6}{2(2^{n-1} - 1)}$, $\left|2^{n-1} - 4 - \frac{2|e(\Gamma_{QD_{2^n}})|}{|v(\Gamma_{QD_{2^n}})|}\right| = \frac{2^{2n-2} - 3.2^n + 2}{2(2^{n-1} - 1)}$, $\left|2 - \frac{2|e(\Gamma_{QD_{2^n}})|}{|v(\Gamma_{QD_{2^n}})|}\right| = \frac{2^{2n-2} - 2^{n+2} + 10}{2(2^{n-1} - 1)}$, and $\left|0 - \frac{2|e(\Gamma_{QD_{2^n}})|}{|v(\Gamma_{QD_{2^n}})|}\right| = \frac{2^{2n-2} - 2^{n+1} + 6}{2(2^{n-1} - 1)}$.

Therefore, by (1.3) and substitution, we have

$$\begin{aligned} LE^+(\Gamma_{QD_{2^n}}) =& \frac{3.2^{2n-2} - 3.2^{n+1} + 6}{2(2^{n-1} - 1)} + \frac{(2^{n-1} - 3)(2^{2n-2} - 3.2^n + 2)}{2(2^{n-1} - 1)} \\ &+ \frac{2^{n-2}(2^{2n-2} - 2^{n+2} + 10)}{2(2^{n-1} - 1)} + \frac{2^{n-2}(2^{2n-2} - 2^{n+1} + 6)}{2(2^{n-1} - 1)}, \end{aligned}$$

and the result follows on simplification. □

Theorem 2.10. *Let G denote the projective special linear group $PSL(2, 2^k)$, where $k \geq 2$. Then*

$$E(\Gamma_G) = 2^{3k+1} - 2^{2k+1} - 2^{k+2} - 4,$$

$$LE(\Gamma_G) = \frac{2^{6k+1} - 2^{5k+1} - 3 \cdot 2^{4k} - 2^{3k+2} + 3 \cdot 2^{2k} + 3 \cdot 2^{k+1} + 4}{2^{3k} - 2^k - 1},$$

and

$$LE^+(\Gamma_G) = \begin{cases} \frac{3916}{59} & \text{if } k = 2, \\ \frac{2^{6k+1} - 2^{5k+1} - 2^{4k+3} - 3 \cdot 2^{3k+1} + 3 \cdot 2^{2k+1} + 2^{k+3} + 4}{2^{3k} - 2^k - 1} & \text{otherwise.} \end{cases}$$

Proof. The expression for $E(\Gamma_G)$ follows from [20, Theorem 2(ii)].

We have $|v(\Gamma_G)| = 2^{3k} - 2^k - 1$ and $|e(\Gamma_G)| = \frac{2^{4k} - 2^{3k+1} - 2^{2k} + 2^{k+1} + 2}{2}$, since $\Gamma_G = (2^k + 1)K_{2^k - 1} \sqcup 2^{k-1}(2^k + 1)K_{2^k - 2} \sqcup 2^{k-1}(2^k - 1)K_{2^k}$. Therefore,

$$\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|} = \frac{2^{4k} - 2^{3k+1} - 2^{2k} + 2^{k+1} + 2}{2^{3k} - 2^k - 1}.$$

By [8, Proposition 2.11], we have

$$\begin{aligned} \text{L-spec}(\Gamma_G) = \{&0^{2^{2k} + 2^k + 1}, (2^k - 1)^{2^{2k} - 2^k - 2}, (2^k - 2)^{2^{k-1}(2^{2k} - 2^{k+1} - 3)}, \\ &(2^k)^{2^{k-1}(2^{2k} - 2^{k+1} + 1)}\}. \end{aligned}$$

Therefore, $\left|0 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{2^{4k} - 2^{3k+1} - 2^{2k} + 2^{k+1} + 2}{2^{3k} - 2^k - 1}$, $\left|2^k - 1 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{2^{3k} - 2^{k+1} - 1}{2^{3k} - 2^k - 1}$, $\left|2^k - 2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{2^k}{2^{3k} - 2^k - 1}$, and $\left|2^k - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{2^{3k+1} - 3.2^k - 2}{2^{3k} - 2^k - 1}$. Hence, by (1.2) and substitution, we have

$$\begin{aligned} LE(\Gamma_G) =& \frac{(2^{2k} + 2^k + 1)(2^{4k} - 2^{3k+1} - 2^{2k} + 2^{k+1} + 2)}{2^{3k} - 2^k - 1} \\ &+ \frac{(2^{2k} - 2^k - 2)(2^{3k} - 2^{k+1} - 1)}{2^{3k} - 2^k - 1} + \frac{2^{k-1}(2^{2k} - 2^{k+1} - 3)2^k}{2^{3k} - 2^k - 1} \\ &+ \frac{2^{k-1}(2^{2k} - 2^{k+1} + 1)(2^{3k+1} - 3.2^k - 2)}{2^{3k} - 2^k - 1}, \end{aligned}$$

and the result follows on simplification.

By [8, Proposition 2.11], we also have

$$\begin{aligned} \text{Q-spec}(\Gamma_G) = \{&(2^{k+1} - 4)^{2^k + 1}, (2^k - 3)^{2^{2k} - 2^k - 2}, (2^{k+1} - 6)^{2^{k-1}(2^k + 1)}, \\ &(2^k - 4)^{2^{k-1}(2^{2k} - 2^{k+1} - 3)}, (2^{k+1} - 2)^{2^{k-1}(2^k - 1)}, (2^k - 2)^{2^{k-1}(2^{2k} - 2^{k+1} + 1)}\}. \end{aligned}$$

Therefore, $\left|2^{k+1}-4-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{2^{4k}-2^{3k+1}-2^{2k}+2}{2^{3k}-2^k-1}$, $\left|2^{k}-3-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{2^{3k}-1}{2^{3k}-2^k-1}$,

$$\left|2^{k+1}-6-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \begin{cases} \frac{-2^{4k}+2^{3k+2}+2^{2k}-2^{k+1}-4}{2^{3k}-2^k-1} & \text{if } k=2, \\ \frac{2^{4k}-2^{3k+2}-2^{2k}+2^{k+1}+4}{2^{3k}-2^k-1} & \text{otherwise,} \end{cases}$$

$\left|2^{k}-4-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{2^{3k+1}-2^k-2}{2^{3k}-2^k-1}$, $\left|2^{k+1}-2-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{2^{4k}-2^{2k}-2^{k+1}}{2^{3k}-2^k-1}$, and $\left|2^{k}-2-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{2^k}{2^{3k}-2^k-1}$. Therefore, by (1.3) we have, if $k=2$, then

$$\begin{aligned} LE^+(\Gamma_G) =& \frac{(2^k+1)(2^{4k}-2^{3k+1}-2^{2k}+2)}{2^{3k}-2^k-1} + \frac{(2^{2k}-2^k-2)(2^{3k}-1)}{2^{3k}-2^k-1} \\ &+ \frac{2^{k-1}(2^k+1)(-2^{4k}+2^{3k+2}+2^{2k}-2^{k+1}-4)}{2^{3k}-2^k-1} \\ &+ \frac{2^{k-1}(2^{2k}-2^{k+1}-3)(2.2^{3k}-2^k-2)}{2^{3k}-2^k-1} \\ &+ \frac{2^{k-1}(2^k-1)(2^{4k}-2^{2k}-2^{k+1})}{2^{3k}-2^k-1} + \frac{2^{k-1}(2^{2k}-2^{k+1}+1)2^k}{2^{3k}-2^k-1} \\ =& \frac{3916}{59}. \end{aligned}$$

Otherwise,

$$\begin{aligned} LE^+(\Gamma_G) =& \frac{(2^k+1)(2^{4k}-2^{3k+1}-2^{2k}+2)}{2^{3k}-2^k-1} + \frac{(2^{2k}-2^k-2)(2^{3k}-1)}{2^{3k}-2^k-1} \\ &+ \frac{2^{k-1}(2^k+1)(2^{4k}-2^{3k+2}-2^{2k}+2^{k+1}+4)}{2^{3k}-2^k-1} \\ &+ \frac{2^{k-1}(2^{2k}-2^{k+1}-3)(2^{3k+1}-2^k-2)}{2^{3k}-2^k-1} \\ &+ \frac{2^{k-1}(2^k-1)(2^{4k}-2^{2k}-2^{k+1})}{2^{3k}-2^k-1} + \frac{2^{k-1}(2^{2k}-2^{k+1}+1)2^k}{2^{3k}-2^k-1}. \end{aligned}$$

Hence, the result follows on simplification. □

Theorem 2.11. *Let G denote the general linear group $GL(2,q)$, where $q = p^n > 2$ and p is a prime. Then*

$$\begin{aligned} E(\Gamma_G) =& 2q^4 - 2q^3 - 4q^2 - 2q, \\ LE(\Gamma_G) =& \frac{q^9 - 2q^8 - 4q^7 + 10q^6 + q^5 - 11q^4 + 2q^3 + 5q^2 - 2q}{(q-1)(q^3-q-1)}, \quad \textit{and} \\ LE^+(\Gamma_G) =& \frac{2q^9 - 10q^7 - 22q^6 - 18q^5 + 51q^4 - 16q^3 - 30q^2 + 3q}{2(q-1)(q^3-q-1)}. \end{aligned}$$

Proof. The expression for $E(\Gamma_G)$ follows from [20, Theorem 2(iii)].

We have $|v(\Gamma_G)| = (q-1)(q^3-q-1)$ and $|e(\Gamma_G)| = \frac{q^6-2q^5-2q^4+4q^3+2q^2-3q}{2}$ as $\Gamma_G = \frac{q(q+1)}{2}K_{q^2-3q+2} \sqcup \frac{q(q-1)}{2}K_{q^2-q} \sqcup (q+1)K_{q^2-2q+1}$. Therefore,

$$\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|} = \frac{q^6-2q^5-2q^4+4q^3+2q^2-3q}{(q-1)(q^3-q-1)}.$$

By [8, Proposition 2.12], we have

$$\text{L-spec}(\Gamma_G) = \{0^{q^2+q+1}, (q^2-3q+2)^{\frac{q(q+1)(q^2-3q+1)}{2}}, (q^2-q)^{\frac{q(q-1)(q^2-q-1)}{2}}, (q^2-2q+1)^{q(q+1)(q-2)}\}.$$

Therefore,
$\left|0 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{q^6-2q^5-2q^4+4q^3+2q^2-3q}{(q-1)(q^3-q-1)}$, $\left|q^2-3q+2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{2q^5-6q^4+3q^3+3q^2-2}{(q-1)(q^3-q-1)}$, $\left|q^2-q - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{2q^4-3q^3-q^2+2q}{(q-1)(q^3-q-1)}$, and $\left|q^2-2q+1 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{q^5-4q^4+3q^3+2q^2-q-1}{(q-1)(q^3-q-1)}$.

Hence, by (1.2) and substitution, we have

$$\begin{aligned} LE(\Gamma_G) =& \frac{(q^2+q+1)(q^6-2q^5-2q^4+4q^3+2q^2-3q)}{(q-1)(q^3-q-1)} \\ &+ \frac{q(q+1)(q^2-3q+1)(2q^5-6q^4+3q^3+3q^2-2)}{2(q-1)(q^3-q-1)} \\ &+ \frac{q(q-1)(q^2-q-1)(2q^4-3q^3-q^2+2q)}{2(q-1)(q^3-q-1)} \\ &+ \frac{q(q+1)(q-2)(q^5-4q^4+3q^3+2q^2-q-1)}{(q-1)(q^3-q-1)}, \end{aligned}$$

and the result follows on simplification.

By [8, Proposition 2.12], we also have

$$\begin{aligned} \text{Q-spec}(\Gamma_G) =& \{(2q^2-6q-2)^{\frac{q(q+1)}{2}}, (q^2-3q)^{\frac{q(q+1)(q^2-3q+1)}{2}}, (2q^2-2q-2)^{\frac{q(q-1)}{2}}, \\ & (q^2-q-2)^{\frac{q(q-1)(q^2-q-1)}{2}}, (2q^2-4q)^{q+1}, (q^2+2q-1)^{q(q+1)(q-2)}\}. \end{aligned}$$

Therefore, $\left|2q^2-6q-2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{q^6-6q^5+4q^4+4q^3+2q^2-3q-2}{(q-1)(q^3-q-1)}$,
$\left|q^2-3q - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{q^5-4q^4+q^3+q^2}{(q-1)(q^3-q-1)}$, $\left|2q^2-2q-2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{q^6-2q^5+2q^2+q-2}{(q-1)(q^3-q-1)}$,
$\left|q^2-q-2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{q^3-q^2-2q+2}{(q-1)(q^3-q-1)}$, $\left|2q^2-4q - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{q^6-4q^5+4q^4-q}{(q-1)(q^3-q-1)}$, and

$\left|q^2+2q-1-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{3q^5-2q^4-5q^3+5q-1}{(q-1)(q^3-q-1)}$. Therefore, by (1.3) and substitution, we have

$$\begin{aligned} LE^+(\Gamma_G) =& \frac{q(q+1)(q^6-6q^5+4q^4+4q^3+2q^2-3q-2)}{2(q-1)(q^3-q-1)} \\ &+ \frac{q(q+1)(q^2-3q+1)(q^5-4q^4+q^3+q^2)}{2(q-1)(q^3-q-1)} \\ &+ \frac{q(q-1)(q^6-2q^5+2q^2+q-2)}{2(q-1)(q^3-q-1)} \\ &+ \frac{q(q-1)(q^2-q-1)(q^3-q^2-2q+2)}{2(q-1)(q^3-q-1)} \\ &+ \frac{(q+1)(q^6-4q^5+4q^4-q)}{(q-1)(q^3-q-1)} \\ &+ \frac{q(q+1)(q-2)(3q^5-2q^4-5q^3+5q-1)}{(q-1)(q^3-q-1)}. \end{aligned}$$

Hence, the result follows on simplification. □

Theorem 2.12. *Let $F = GF(2^n), n \geq 2$ and let ϑ be the Frobenius automorphism of F, that is, $\vartheta(x) = x^2$ for all $x \in F$. If G denotes the group*

$$A(n,\vartheta) := \left\{ U(a,b) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \vartheta(a) & 1 \end{bmatrix} : a, b \in F \right\}$$

under the operation $U(a,b)U(a',b') := U(a+a', b+b'+a'\vartheta(a))$, then

$$E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G) = 2(2^n-1)^2.$$

Proof. The expression for $E(\Gamma_G)$ follows from [20, Theorem 2(v)].

We have $|v(\Gamma_G)| = 2^n(2^n-1)$ and $|e(\Gamma_G)| = \frac{2^{3n}-2^{2n+1}+2^n}{2}$, since $\Gamma_G = (2^n-1)K_{2^n}$. Therefore,

$$\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|} = 2^n - 1.$$

By [8, Proposition 2.13], we have

$$\text{L-spec}(\Gamma_G) = \{0^{2^n-1}, (2^n)^{2^{2n}-2^{n+1}+1}\}.$$

Therefore, $\left|0-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = 2^n-1$ and $\left|2^n-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = 1$. Hence, by (1.2), we have

$$LE(\Gamma_G) = (2^n-1)(2^n-1) + (2^{2n}-2^{n+1}+1) = 2(2^n-1)^2.$$

By [8, Proposition 2.13], we also have

$$\text{Q-spec}(\Gamma_G) = \{(2^{n+1}-2)^{2^n-1}, (2^n-2)^{2^{2n}-2^{n+1}+1}\}.$$

Therefore, $\left|2^{n+1}-2-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = 2^n-1$ and $\left|2^n-2-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = 1$. Therefore, by (1.3) , have

$$LE^+(\Gamma_G) = (2^n-1)(2^n-1) + (2^{2n}-2^{n+1}+1) = 2(2^n-1)^2.$$ □

Theorem 2.13. *Let $F = GF(p^n)$, where p is prime. If G denotes the group*

$$\left\{ V(a,b,c) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} : a,b,c \in F \right\}$$

under the operation $V(a,b,c)V(a',b',c') = V(a+a', b+b'+ca', c+c')$, then

$$E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G) = 2(p^{3n} - 2p^n - 1).$$

Proof. The expression for $E(\Gamma_G)$ follows from [20, Theorem 2(vi)].

We have $|v(\Gamma_G)| = p^n(p^{2n} - 1)$ and $|e(\Gamma_G)| = \frac{p^{5n} - p^{4n} - 2p^{3n} + p^{2n} + p^n}{2}$, since $\Gamma_G = (p^n + 1)K_{p^{2n} - p^n}$. Therefore,

$$\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|} = p^{2n} - p^n - 1.$$

By [8, Proposition 2.14], we have

$$\text{L-spec}(\Gamma_G) = \{0^{p^n+1}, (p^{2n} - p^n)^{p^{3n} - 2p^n - 1}\}.$$

Therefore, $\left|0 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = p^{2n} - p^n - 1$ and $\left|p^{2n} - p^n - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = 1$. Hence, by (1.2), we have

$$LE(\Gamma_G) = (p^n + 1)(p^{2n} - p^n - 1) + (p^{3n} - 2p^n - 1) = 2(p^{3n} - 2p^n - 1).$$

By [8, Proposition 2.14], we also have

$$\text{Q-spec}(\Gamma_G) = \{(2p^{2n} - 2p^n - 2)^{p^n+1}, (p^{2n} - p^n - 2)^{p^{3n} - 2p^n - 1}\}.$$

Therefore, $\left|2p^{2n} - 2p^n - 2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = p^{2n} - p^n - 1$ and $\left|p^{2n} - p^n - 2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = 1$. Therefore, by (1.3), we have

$$LE^+(\Gamma_G) = (p^n + 1)(p^{2n} - p^n - 1) + (p^{3n} - 2p^n - 1) = 2(p^{3n} - 2p^n - 1). \qquad \square$$

Proposition 2.14. *Let G be a nonabelian group of order pq, where p and q are primes with $p \mid (q-1)$. Then*

$$E(\Gamma_G) = 2q(p-1) - 3, \quad LE(\Gamma_G) = \begin{cases} \frac{16}{5} & \text{if } p = 2, \\ & \text{and } q = 3, \\ \frac{2pq^3 - 2p^2q^2 + 4p^2q - 8pq - 2q^3 + 4q^2 - 2q + 4}{pq-1} & \text{otherwise,} \end{cases}$$

$$\textit{and} \quad LE^+(\Gamma_G) = \begin{cases} \frac{16}{5} & \text{if } p = 2 \text{ and } q = 3, \\ \frac{2pq^3 - 2p^2q^2 + 2p^2q - 2pq - 2q^3 + 2q^2}{pq-1} & \text{otherwise.} \end{cases}$$

Proof. The expression for $E(\Gamma_G)$ follows from [20, Theorem 2(vii)].

We have $|v(\Gamma_G)| = pq - 1$ and $|e(\Gamma_G)| = \frac{p^2q - 3pq + q^2 - q + 2}{2}$, since $\Gamma_G = qK_{p-1} \sqcup K_{q-1}$. Therefore,

$$\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|} = \frac{p^2q - 3pq + q^2 - q + 2}{pq - 1}.$$

By [8, Proposition 2.9], we have

$$\text{L-spec}(\Gamma_G) = \{0^{q+1}, (q-1)^{q-2}, (p-1)^{pq-2q}\}.$$

Therefore, $\left|0 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{p^2q-3pq+q^2-q+2}{pq-1}$, $\left|q-1 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{pq^2-p^2q+2pq-q^2-1}{pq-1}$, and

$$\left|p-1-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \begin{cases} \frac{-q^2+q+2pq-p-1}{pq-1} & \text{if } p=2 \text{ and } q=3, \\ \frac{q^2-q-2pq+p+1}{pq-1} & \text{otherwise.} \end{cases}$$

Hence, by (1.2) we have, if $p = 2$ and $q = 3$, then

$$\begin{aligned} LE(\Gamma_G) =& \frac{(q+1)(p^2q-3pq+q^2-q+2)}{pq-1} + \frac{(q-2)(pq^2-p^2q+2pq-q^2-1)}{pq-1} \\ &+ \frac{(pq-2q)(-q^2+q+2pq-p-1)}{pq-1}. \end{aligned}$$

Otherwise,

$$\begin{aligned} LE(\Gamma_G) =& \frac{(q+1)(p^2q-3pq+q^2-q+2)}{pq-1} + \frac{(q-2)(pq^2-p^2q+2pq-q^2-1)}{pq-1}. \\ &+ \frac{(pq-2q)(q^2-q-2pq+p+1)}{pq-1}. \end{aligned}$$

Hence, the result follows on simplification.

By [8, Proposition 2.9], we also have

$$\text{Q-spec}(\Gamma_G) = \{(2q-4)^1, (q-3)^{q-2}, (2p-4)^q, (p-3)^{pq-2q}\}.$$

Therefore, $\left|2q-4 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{2pq^2-p^2q-pq-q^2-q+2}{pq-1}$,

$$\left|q-3-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \begin{cases} \frac{-pq^2+p^2q+q^2-1}{pq-1} & \text{if } p=2 \text{ and } q=3\ , \\ \frac{pq^2-p^2q-q^2+1}{pq-1} & \text{otherwise,} \end{cases}$$

$\left|2p-4 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{-p^2q+pq+q^2-q+2p-2}{pq-1}$, and $\left|p-3 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \frac{q^2-q+p-1}{pq-1}$.

Therefore, by (1.3), we have, if $p = 2$ and $q = 3$, then

$$\begin{aligned} LE^+(\Gamma_G) =& \frac{2pq^2-p^2q-pq-q^2-q+2}{pq-1} + \frac{(q-2)(-pq^2+p^2q+q^2-1)}{pq-1} \\ &+ \frac{q(-p^2q+pq+q^2-q+2p-2)}{pq-1} + \frac{(pq-2q)(q^2-q+p-1)}{pq-1}. \end{aligned}$$

Otherwise,

$$\begin{aligned} LE^+(\Gamma_G) =& \frac{2pq^2-p^2q-pq-q^2-q+2}{pq-1} + \frac{(q-2)(pq^2-p^2q-q^2+1)}{pq-1} \\ &+ \frac{q(-p^2q+pq+q^2-q+2p-2)}{pq-1} + \frac{(pq-2q)(q^2-q+p-1)}{pq-1}. \end{aligned}$$

Hence, the result follows on simplification. □

3. Graphical representation

In this section, we analyze the graphical representations of various energies of commuting graphs of the groups D_{2m}, Q_{4m}, QD_{2^n}, U_{6n}, $GL(2,q)$, M_{12n}, and $A(n,\vartheta)$.

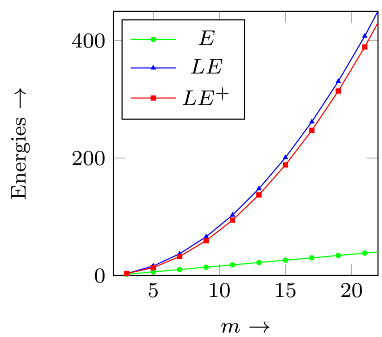

Figure 1. Energies of $\Gamma_{D_{2m}}$, m is odd

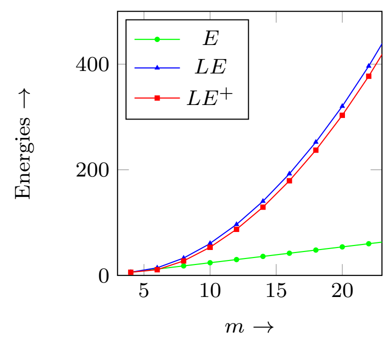

Figure 2. Energies of $\Gamma_{D_{2m}}$, m is even

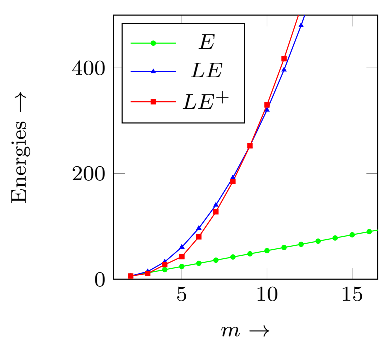

Figure 3. Energies of $\Gamma_{Q_{4m}}$

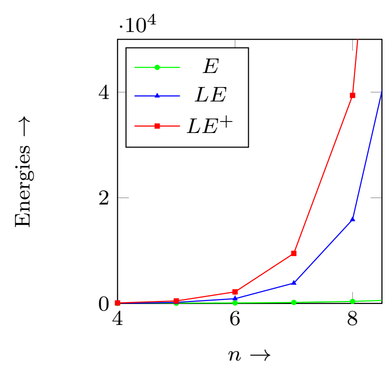

Figure 4. Energies of $\Gamma_{QD_{2^n}}$

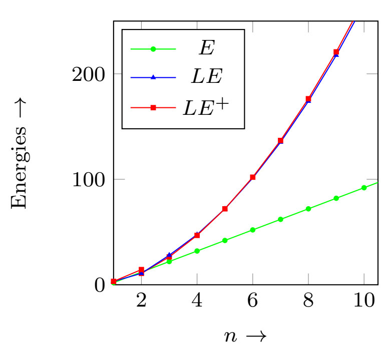

Figure 5. Energies of $\Gamma_{U_{6n}}$

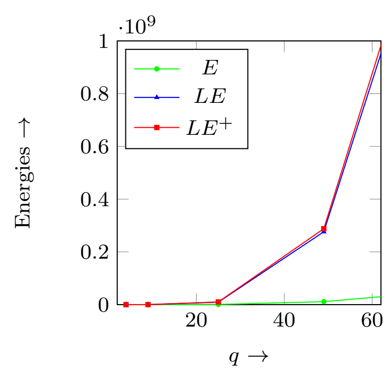

Figure 6. Energies of $\Gamma_{GL(2,q)}$

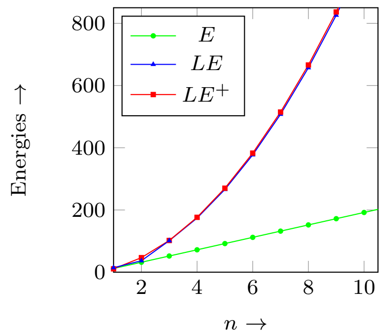

FIGURE 7. Energies of $\Gamma_{M_{12n}}$, m is odd

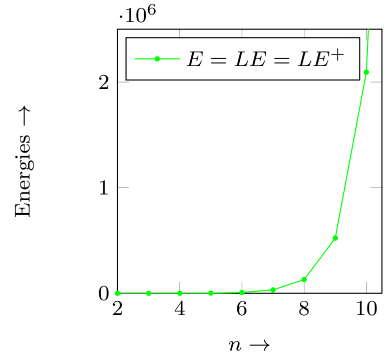

FIGURE 8. Energies of $\Gamma_{A(n,\vartheta)}$

From the above figures, one can conclude that Conjecture 1.1 holds for the commuting graphs of the family of dihedral groups, quasidihedral groups, generalized quaternion groups, general linear groups, the groups $A(n, \vartheta)$, and the family of metacyclic groups $M_{12n} = \langle a, b : a^6 = b^{2n} = 1, bab^{-1} = a^{-1}\rangle$ and $U_{6n} = \langle x, y : x^{2n} = y^3 = 1, x^{-1}yx = y^{-1}\rangle$. We also have

$$LE(\Gamma_{PSL(2,2^k)}) - E(\Gamma_{PSL(2,2^k)}) = 3 \cdot 2^k + \frac{2^k}{2^{3k} - 2^k - 1} > 0.$$

Therefore, Conjecture 1.1 also holds for the commuting graphs of projective special linear groups $PSL(2, 2^k)$. In the light of the above discussion and [10, Theorems 3.1 and 3.2], it follows that Conjecture 1.1 holds for Γ_G if Γ_G is planar or toroidal.

It is also observed that Laplacian energy and signless Laplacian energy of the commuting graph of a finite nonabelian group are not comparable in general. For example, in Figures 1 and 2, $LE(\Gamma_{D_{2m}}) > LE^+(\Gamma_{D_{2m}})$; however in Figure 4, $LE(\Gamma_{QD_{2^n}}) < LE^+(\Gamma_{QD_{2^n}})$. Also, in Figure 3, $LE(\Gamma_{Q_{4m}}) > LE^+(\Gamma_{Q_{4m}})$ for $3 \leq m \leq 8$ whereas $LE(\Gamma_{Q_{4m}}) < LE^+(\Gamma_{Q_{4m}})$ for $m > 9$. In most of the cases

$$E(\Gamma_G) \leq \min\{LE(\Gamma_G), LE^+(\Gamma_G)\}.$$

However, in Figure 8, $E(\Gamma_{A(n,\vartheta)}) = LE(\Gamma_{A(n,\vartheta)}) = LE^+(\Gamma_{A(n,\vartheta)})$. We conclude this paper with the following natural questions.

Question 3.1. Is Conjecture 1.1 true for commuting graphs of finite nonabelian groups?

Question 3.2. Can we determine all finite nonabelian groups G such that

(a) $LE(\Gamma_G) < LE^+(\Gamma_G)$
(b) $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G)$?

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