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ON APPROXIMATION OF FUNCTIONS BELONGING TO SOME CLASSES OF FUNCTIONS BY $(N, p_n, q_n)(E, \theta)$ MEANS OF CONJUGATE SERIES OF ITS FOURIER SERIES

XHEVAT ZAHIR KRASNIQI[1]*, DEEPMALA[2]

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Abstract. We obtain some new results on the approximation of functions belonging to some classes of functions by $(N, p_n, q_n)(E, \theta)$ means of conjugate series of its Fourier series. These results, under conditions assumed here, are better than those obtained previously by others. In addition, several particular results are derived from our results as corollaries.

1. Introduction and preliminaries

Given two sequences $p := (p_n)$ and $q := (q_n)$, the convolution $(p * q)_n$ is defined by

$$R_n := (p * q)_n := \sum_{m=0}^{n} p_m q_{n-m}.$$

We write $P_n := (p*1)_n = \sum_{m=0}^{n} p_m$ and $Q_n := (1*q)_n = \sum_{m=0}^{n} q_m = \sum_{m=0}^{n} q_{n-m}$.

Let (s_n) be the sequence of partial sums of the numerical series $\sum_{n=0}^{\infty} u_n$. The generalized Nörlund transform of the sequence (s_n) is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_n^{p,q} = \frac{1}{R_n} \sum_{m=0}^{n} p_{n-m} q_m s_m,$$

where R_n is a sequence of nonzero real numbers. If $s_n \to s$ as $n \to \infty$ implies that $t_n^{p,q} \to s$ as $n \to \infty$, then the method (N, p_n, q_n) is said to be regular. A necessary and sufficient condition for the method (N, p_n, q_n) to be regular is that

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* Corresponding author.
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$\sum_{m=0}^{n} |p_{n-m}q_m| = \mathcal{O}(|(p*q)_n|)$ and $p_{n-m} = o(|(p*q)_n|)$ as $n \to \infty$ for every fixed $m \geq 0$ (see Borwein [1]).

The method (N, p_n, q_n) reduces to the Nörlund method (N, p_n), if $q_n = 1$ for all n; and to the Riesz method $(\overline{N}, q_n)$, if $p_n = 1$ for all n. It is a well-known fact that the (N, p_n) mean (respectively, $(\overline{N}, q_n)$ mean), includes as a special case, Cesàro and harmonic means (respectively, logarithmic mean).

Let $E_n^\theta = \frac{1}{(1+\theta)^n} \sum_{k=0}^{n} \binom{n}{k} \theta^{n-k} s_k$, $\theta > 0$. If $E_n^\theta \longrightarrow s$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to s by the Euler method (E, θ) and this method is regular; see [2].

The product summability $(N, p_n, q_n)(E, \theta)$ is obtained if we superimpose (E, θ) summability on (N, p_n, q_n) summability. The (N, p_n, q_n) transform of the (E, θ) transform defines the $(N, p_n, q_n)(E, \theta)$ transform $t_n^{p,q,\theta}$ of the nth partial sums s_n of the series $\sum_{n=0}^{\infty} u_n$ which is defined by the equality

$$t_n^{p,q,\theta} = \frac{1}{R_n} \sum_{k=0}^{n} \frac{p_k q_{n-k}}{(1+\theta)^{n-k}} \sum_{v=0}^{n-k} \binom{n-k}{v} \theta^k s_v.$$

If $t_n^{p,q,\theta} \longrightarrow s$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or the sequence (s_n) is said to be summable $(N, p_n, q_n)(E, \theta)$ to the sum s if the limit $\lim_{n\to\infty} t_n^{p,q,\theta}$ exists and is equal to the same number s.

Let f be a 2π periodic signal (function) and Lebesgue integrable, that is, $f \in L[0, 2\pi]$. Then the Fourier series of the signal (function) f at the point x is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx), \tag{1.1}$$

with its partial sums $s_n(f; x)$ being a trigonometric polynomial of order n with $n + 1$ terms.

The conjugate series of the Fourier series (1.1) is given by

$$\sum_{m=1}^{\infty} (b_m \cos mx - a_m \sin mx). \tag{1.2}$$

A signal (function) $f \in \text{Lip}\ \alpha$ if $|f(x+t) - f(x)| = \mathcal{O}(|t|^\alpha)$ for $0 < \alpha \leq 1$.

A signal (function) $f \in \text{Lip}\ (\alpha, r)$ for $0 \leq x \leq 2\pi$ if

$$\left\{ \int_0^{2\pi} |f(x+t) - f(x)|^r \right\}^{1/r} \leq M(|t|^\alpha) \tag{1.3}$$

for $r \geq 1$ and $0 < \alpha \leq 1$, where M is an absolute positive constant not necessarily the same at each occurrence (see McFadden [5]).

Moreover, it is said that $f \in \text{Lip}\ (\xi(t), r)$, $\xi(t) > 0$ if

$$f \in L^r[0, 2\pi] \quad \text{and} \quad \|f(x+t) - f(x)\|_r = \mathcal{O}\left(\xi(t)\right)$$

for $r \geq 1$, where the L_r-norm of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx \right)^{1/r}, \quad r \geq 1.$$

The L_∞-norm of a function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is defined by

$$\|f\|_\infty=\sup\{|f(x)|:x\in\mathbb{R}\}.$$

It should be noted here that if $\xi(t)=t^\alpha$, then Lip $(\xi(t),r)=$ Lip (α,r), and if $r\longrightarrow\infty$ in Lip(p,r) class, then this class reduces to the Lipα class.

A signal (function) f is approximated by the trigonometric polynomial $\tau_n(f;x)$ of order n, and the degree of approximation $E_n(f)$ of a function $f\in L_r$ is given by

$$E_n(f)=\min_n\|f(x)-\tau_n(f;x)\|_r,$$

in terms of n.

The degree of approximation of a function $f:\mathbb{R}\longrightarrow\mathbb{R}$ by a trigonometric polynomial $\tau_n(f;x)$ of order n under sup norm $\|\cdot\|_\infty$ is defined by

$$\|f(x)-\tau_n(f;x)\|_\infty=\sup\{|f(x)-\tau_n(f;x)|:x\in\mathbb{R}\}.$$

Throughout this paper, we will write

$$\psi_x(t):=\psi(t)=f(x+t)-f(x-t),\quad \triangle c_n=c_n-c_{n+1},\ n\geq 0,$$

and

$$\widetilde{K_n^{p,q,\theta}}(t)=\frac{1}{2\pi R_n}\sum_{k=0}^{n}p_kq_{n-k}\left(\frac{R(t)}{1+\theta}\right)^k\frac{\cos\left[\frac{(k+1)t}{2}-k\tan^{-1}\left(\frac{\theta-1}{\theta+1}\tan\frac{t}{2}\right)\right]}{\sin\frac{t}{2}},$$

where

$$R(t)=\sqrt{(\theta+1)^2-4\theta\sin^2\frac{t}{2}}.$$

The theory of approximation, which is originated from a well-known theorem of Weierstrass, has been an excitatory interdisciplinary field of study till nowadays. The approximations of the functions have a wide applications in signal analysis, digital communications, theory of machines in mechanical engineering, and in particular in digital signal processing; see [12] and [13] (also the interested reader could find several new results on these approximations and their applications into references given in [11], see also [3] and [6–8]).

Very recently, Mishra and Sonavane [11] determined the degree of approximation of a conjugate function $\tilde{f}$ of a function $f\in$ Lip(α,r) $(r\geq 1)$, by an $(N,q_n)(E,1)$ transform of partial sums of the conjugate series of a Fourier series.

Their results are the following statements.

Theorem 1.1 ([11]). *Let (N,p_n) be a regular Nörlund method of summability defined by a positive generating sequence (p_n). Let $f:[0,2\pi]\longrightarrow\mathbb{R}$ be a 2π periodic function, Lebesgue integrable, and belonging to the* Lip$(\alpha,r),(r\geq 1)$ *class. If either*

$$\text{(i)}\quad (n+1)p_n=\mathcal{O}(P_n),\qquad \text{(ii)}\quad \sum_{k=0}^{n-1}|\triangle p_k|=\mathcal{O}\left(\frac{P_n}{n+1}\right)\tag{1.4}$$

or

$$\text{(i)}' \quad (n+1)p_n = \mathcal{O}(P_n), \qquad \text{(ii)}' \quad \sum_{k=0}^{n-1}\left|\triangle\left(\frac{P_k}{k+1}\right)\right| = \mathcal{O}\left(\frac{P_n}{n+1}\right). \tag{1.5}$$

Then the degree of approximation of the function $\widetilde{f}$ *by the* $(N,p_n)(E,1)$ *transform*

$$\widetilde{t_n^{NE}} = \frac{1}{P_n}\sum_{k=0}^{n}\frac{p_k}{2^{n-k}}\sum_{v=0}^{n-k}\binom{n-k}{v}\widetilde{s}_v$$

of the partial sums $\widetilde{s}_n(f,x)$ *of the series* (1.2) *is given by the following estimations:*

$$\|\widetilde{t_n^{NE}} - \widetilde{f}\|_r = \begin{cases} \mathcal{O}\left((n+1)^{-\alpha}\right) & \text{for } 0<\alpha<1 \\ \mathcal{O}\left(\frac{\log(n+1)}{n+1}\right) & \text{for } \alpha = 1, \end{cases}$$

for all $n \in \{0,1,2,\dots\}$.

Theorem 1.2 ([11]). *Let* (N,p_n) *be a regular Nörlund method of summability defined by a positive generating sequence* (p_n). *Assume that* $\xi(t)$ *is a modulus of continuity such that*

$$\int_0^v \frac{\xi(t)}{t}dt = \mathcal{O}\left(\xi(v)\right), \quad 0<v<\pi. \tag{1.6}$$

Let $f:[0,2\pi]\longrightarrow\mathbb{R}$ *be a* 2π *periodic function, Lebesgue integrable, and belonging to the* $\mathrm{Lip}(\xi(t),r),(r\geq 1)$ *class. Then the degree of approximation of the function* $\widetilde{f}$ *by* $\widetilde{t_n^{NE}}$ *means of the partial sums* $\widetilde{s}_n(f,x)$ *of the series* (1.2) *is given by*

$$\|\widetilde{t_n^{NE}} - \widetilde{f}\|_r = \mathcal{O}\left(\frac{1}{n+1}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\xi(t)}{t^2}dt\right)$$

for all $n \in \{0,1,2,\dots\}$.

Theorem 1.3 ([11]). *Let* (N,p_n) *be a regular Nörlund method of summability defined by a positive generating sequence* (p_n) *satisfying* (1.4) *or* (1.5). *Let* $f:[0,2\pi]\longrightarrow\mathbb{R}$ *be a* 2π*-periodic function, Lebesgue integrable, and belonging to the* $\mathrm{Lip}(\xi(t),r),(r\geq 1)$ *class and let* $\frac{\xi(t)}{t}$ *be monotone decreasing in* $(\pi/(n+1),\pi)$. *Then the degree of approximation of the function* $\widetilde{f}$ *by* $\widetilde{t_n^{NE}}$ *means of the partial sums* $\widetilde{s}_n(f,x)$ *of the series* (1.2) *is given by*

$$\|\widetilde{t_n^{NE}} - \widetilde{f}\|_r = \mathcal{O}\left(\xi\left(\frac{1}{n+1}\right)\log(n+1)\right),$$

for all $n \in \{0,1,2,\dots\}$.

The purpose of this paper is to determine the degree of approximation of the function $\widetilde{f}$ by $\widetilde{t_n^{p,q,\theta}}$ means of the partial sums $\widetilde{s}_n(f,x)$ of the series (1.2), in the cases when $f\in\mathrm{Lip}(\alpha,r)$ and $f\in\mathrm{Lip}(\xi(t),r),(r\geq 1)$. Our results will significantly extend the above mentioned results.

2. Helpful lemmas

The following auxiliary statements are needed for the proofs of our main results.

Lemma 2.1. *If* $0 \le t \le \frac{\pi}{n+1}$, *then* $\left|\widetilde{K_n^{p,q,\theta}}(t)\right| = \mathcal{O}(t^{-1})$, $n \in \{0,1,2,\dots\}$.

Proof. If $0 \le t \le \frac{\pi}{n+1}$ and $n \in \{0,1,2,\dots\}$, then according to Young's inequality $\sin\frac{t}{2} \ge \frac{t}{\pi}$, we get

$$\begin{aligned}
\left|\widetilde{K_n^{p,q,\theta}}(t)\right| &\le \frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_{n-k}\left|\frac{R(t)}{1+\theta}\right|^k \frac{\left|\cos\left[\frac{(k+1)t}{2} - k\tan^{-1}\left(\frac{\theta-1}{\theta+1}\tan\frac{t}{2}\right)\right]\right|}{\left|\sin\frac{t}{2}\right|}\\
&\le \frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{|R(t)|^{2\cdot\frac{k}{2}}}{(1+\theta)^k}\frac{1}{\frac{t}{\pi}}\\
&\le \frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{\left|(\theta+1)^2 - 4\theta\sin^2\frac{t}{2}\right|^{\frac{k}{2}}}{(1+\theta)^k}\frac{\pi}{t}\\
&\le \frac{1}{2R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{[(\theta+1)^2]^{\frac{k}{2}}}{(1+\theta)^k}\frac{1}{t}\\
&= \mathcal{O}\left(\frac{1}{t}\right).
\end{aligned}$$

□

Lemma 2.2. *If* $\frac{\pi}{n+1} < t \le \pi$, $0 \le k \le n$, $\theta \ne 3$, *and either*

$$\text{(i)}\quad (k+1)^2 p_k q_{n-k} = \mathcal{O}(R_k), \qquad \text{(ii)}\quad \sum_{k=0}^{n-1} |\triangle\, p_k q_{n-k}| = \mathcal{O}\left(\frac{R_n}{(n+1)^2}\right) \tag{2.1}$$

or

$$\text{(i)}'\quad (k+1)^2 p_k q_{n-k} = \mathcal{O}(R_k), \qquad \text{(ii)}'\quad \sum_{k=0}^{n-1}\left|\triangle\left(\frac{R_k}{k+1}\right)\right| = \mathcal{O}\left(\frac{R_n}{(n+1)^2}\right), \tag{2.2}$$

then

$$\left|\widetilde{t_n^{p,q,\theta}}(t)\right| = \mathcal{O}_\theta\left(\frac{1}{(n+1)^2 t^3}\right), \quad n \in \{0,1,2,\dots\}.$$

Proof. Let the conditions (2.1) be satisfied. For the sake of brevity, we denote

$$\varphi(t;\theta) := \varphi := \tan^{-1}\left(\frac{\theta-1}{\theta+1}\tan\frac{t}{2}\right)$$

and

$$\begin{aligned} P_\ell(t) &= \cos\frac{t}{2}-\cos\varphi-2\cos\frac{t}{2}\cos(\ell+1)\left(\frac{t}{2}-\varphi\right)\cos^2\left(\frac{t}{4}-\frac{\varphi}{2}\right)\\ &\quad+\sin\left(\frac{t}{2}-\varphi\right)\sin\left[\frac{t}{2}+(\ell+1)\left(\frac{t}{2}-\varphi\right)\right]\\ &\quad+2\sin\frac{t}{2}\sin(\ell+1)\left(\frac{t}{2}-\varphi\right)\sin^2\left(\frac{t}{4}-\frac{\varphi}{2}\right). \end{aligned}$$

Then, applying the summation by parts, we obtain

$$\begin{aligned} \left|\widetilde{t_n^{p,q,\theta}}(t)\right| &= \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n}p_kq_{n-k}\left(\frac{R(t)}{1+\theta}\right)^k\frac{\cos\left[\frac{(k+1)t}{2}-k\varphi\right]}{\sin\frac{t}{2}}\right|\\ &= \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n-1}\triangle\left(p_kq_{n-k}\right)\sum_{j=0}^{k}\frac{\left[(\theta+1)^2-4\theta\sin^2\frac{t}{2}\right]^{\frac{j}{2}}\cos\left[\frac{(j+1)t}{2}-j\varphi\right]}{(1+\theta)^j\sin\frac{t}{2}}\right.\\ &\quad\left.+p_nq_0\sum_{j=0}^{n}\frac{\left[(\theta+1)^2-4\theta\sin^2\frac{t}{2}\right]^{\frac{j}{2}}\cos\left[\frac{(j+1)t}{2}-j\varphi\right]}{(1+\theta)^j\sin\frac{t}{2}}\right|\\ &\le \frac{1}{2\pi R_n}\sum_{k=0}^{n-1}\left|\triangle\left(p_kq_{n-k}\right)\right|\max_{0\le\ell\le k}\left|\sum_{j=0}^{\ell}\frac{\cos\left[\frac{(j+1)t}{2}-j\varphi\right]}{\sin\frac{t}{2}}\right|\\ &\quad+p_nq_0\max_{0\le\ell\le n}\left|\sum_{j=0}^{\ell}\frac{\cos\left[\frac{(j+1)t}{2}-j\varphi\right]}{\sin\frac{t}{2}}\right|\\ &\le \frac{1}{2tR_n}\left\{\sum_{k=0}^{n-1}\left|\triangle\left(p_kq_{n-k}\right)\right|\max_{0\le\ell\le k}\left|\Re\sum_{j=0}^{\ell}e^{i\left[\frac{(j+1)t}{2}-j\varphi\right]}\right|\right.\\ &\quad\left.+p_nq_0\max_{0\le\ell\le n}\left|\Re\sum_{j=0}^{\ell}e^{i\left[\frac{(j+1)t}{2}-j\varphi\right]}\right|\right\}\\ &= \frac{1}{2tR_n}\left\{\sum_{k=0}^{n-1}\left|\triangle\left(p_kq_{n-k}\right)\right|\max_{0\le\ell\le k}\left|\Re e^{\frac{it}{2}}\frac{1-e^{i(\ell+1)\left(\frac{t}{2}-\varphi\right)}}{1-e^{i\left(\frac{t}{2}-\varphi\right)}}\right|\right.\\ &\quad\left.+p_nq_0\max_{0\le\ell\le n}\left|\Re e^{\frac{it}{2}}\frac{1-e^{i(\ell+1)\left(\frac{t}{2}-\varphi\right)}}{1-e^{i\left(\frac{t}{2}-\varphi\right)}}\right|\right\}\\ &= \frac{1}{2tR_n}\left\{\sum_{k=0}^{n-1}\left|\triangle\left(p_kq_{n-k}\right)\right|\max_{0\le\ell\le k}\left|\frac{P_\ell(t)}{4\sin^2\left(\frac{t}{4}-\frac{\varphi}{2}\right)}\right|\right.\\ &\quad\left.+p_nq_0\max_{0\le\ell\le n}\left|\frac{P_\ell(t)}{4\sin^2\left(\frac{t}{4}-\frac{\varphi}{2}\right)}\right|\right\} \end{aligned}$$

$$= \frac{7}{8tR_n}\left\{\sum_{k=0}^{n-1}\frac{|\triangle(p_kq_{n-k})|}{\sin^2\left(\frac{t}{4}-\frac{\varphi}{2}\right)}+\frac{p_nq_0}{\sin^2\left(\frac{t}{4}-\frac{\varphi}{2}\right)}\right\}$$

$$\leq \frac{7\pi^2}{32tR_n}\left\{\sum_{k=0}^{n-1}\frac{|\triangle(p_kq_{n-k})|}{\left|\frac{t}{4}-\frac{\varphi}{2}\right|^2}+\frac{p_nq_0}{\left|\frac{t}{4}-\frac{\varphi}{2}\right|^2}\right\}.$$

Using the inequality of [4, pp. 288–289]

$$\tan^{-1}u < u \quad \text{for} \quad u > 0,$$

we have

$$\begin{aligned}\left|\widetilde{t_n^{p,q,\theta}}(t)\right| &\leq \frac{7\pi^2}{32tR_n\left[\frac{t}{4}-\frac{\theta-1}{2(\theta+1)}\tan\frac{t}{2}\right]^2}\left\{\sum_{k=0}^{n-1}|\triangle(p_kq_{n-k})|+p_nq_0\right\}\\ &\leq \frac{7\pi^2}{32tR_n\left[\frac{t}{4}-\frac{\theta-1}{2(\theta+1)}\tan\frac{t}{2}\right]^2}\mathcal{O}\left(\frac{R_n}{(n+1)^2}+\frac{R_n}{(n+1)^2}\right)\\ &= \mathcal{O}\left(\frac{1}{(n+1)^2t\left(t-\frac{2(\theta-1)}{\theta+1}\tan\frac{t}{2}\right)^2}\right)\\ &= \mathcal{O}\left(\frac{1}{(n+1)^2t^3\left(\frac{3-\theta}{\theta+1}\right)^2}\right)=\mathcal{O}_\theta\left(\frac{1}{(n+1)^2t^3}\right).\end{aligned}$$

The proof of this lemma, under conditions (2.2), can be done in a similar way. We will skip it, and the proof of the lemma is completed. □

Lemma 2.3 ([11])**.** *Let $f \in \mathrm{Lip}(\alpha, r)$, let $0 < \alpha \leq 1$, and let $r \geq 1$. Then*

$$\left[\int_0^{2\pi}|\psi(x,t)|^r dx\right]^{\frac{1}{r}} = \mathcal{O}(|t^\alpha|).$$

Lemma 2.4 ([11])**.** *Let $f \in \mathrm{Lip}(\xi, r)$, and let $r \geq 1$. Then*

$$\left[\int_0^{2\pi}|\psi(x,t)|^r dx\right]^{\frac{1}{r}} = \mathcal{O}(\xi(t)).$$

Now we pass to the main results of this paper.

3. Main results

We prove firstly the following main result.

Theorem 3.1. *Let (N, p_n, q_n) be a generalized regular Nörlund method of summability defined by two positive generating sequences (p_n) and (q_n). Let $f : [0, 2\pi] \longrightarrow \mathbb{R}$ be a 2π periodic function, Lebesgue integrable, and belonging to the* $\mathrm{Lip}(\alpha, r)$*,*

$(r\geq 1)$ class. If either (2.1) or (2.2) holds true, then the degree of approximation of the function $\widetilde{f}$ by $(N,p_n,q_n)(E,\theta)$ transform

$$\widetilde{t_n^{p,q,\theta}}(x)=\frac{1}{R_n}\sum_{k=0}^{n}\frac{p_kq_{n-k}}{(1+\theta)^k}\sum_{v=0}^{k}\binom{k}{v}\theta^{k-v}\widetilde{s}_v(x)$$

of the partial sums $\widetilde{s}_n(f,x)$ of the series (1.2) is given by the following estimation:

$$\left\|\widetilde{t_n^{p,q,\theta}}-\widetilde{f}\right\|_{L^r}=\mathcal{O}_\theta\left((n+1)^{-\alpha}\right),\quad 0<\alpha\leq 1,$$

for all $n\in\{0,1,2,\dots\}$ and $\theta\neq 3$.

Proof. For the partial sums $\widetilde{s}_v(f,x)$ of the series (1.2), the following equality holds true

$$\widetilde{s}_v(f,x)-\widetilde{f}(x)=\frac{1}{2\pi}\int_0^\pi\psi(x,t)\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt.$$

Therefore, the (E,θ) means of $\widetilde{s}_v(f,x)-\widetilde{f}(x)$, have this form

$$\widetilde{E_k^\theta}(x)-\widetilde{f}(x)=\frac{1}{2\pi(1+\theta)^k}\int_0^\pi\frac{\psi(x,t)}{\sin\frac{t}{2}}\sum_{v=0}^{k}\binom{k}{v}\theta^{k-v}\cos\left(v+\frac{1}{2}\right)tdt,$$

which can be rewritten as follows:

$$\begin{aligned}\widetilde{E_k^\theta}(x)-\widetilde{f}(x)&=\frac{1}{2\pi(1+\theta)^k}\int_0^\pi\frac{\psi(x,t)}{\sin\frac{t}{2}}\Re\left\{\sum_{v=0}^{k}\binom{k}{v}\theta^{k-v}e^{i\left(v+\frac{1}{2}\right)t}\right\}dt\\&=\frac{1}{2\pi(1+\theta)^k}\int_0^\pi\frac{\psi(x,t)}{\sin\frac{t}{2}}\Re\left\{e^{\frac{it}{2}}\sum_{v=0}^{k}\binom{k}{v}\theta^{k-v}e^{ivt}\right\}dt\\&=\frac{1}{2\pi(1+\theta)^k}\int_0^\pi\frac{\psi(x,t)}{\sin\frac{t}{2}}\Re\left\{e^{\frac{it}{2}}\left(\theta+e^{it}\right)^k\right\}dt\\&=\frac{1}{2\pi(1+\theta)^k}\int_0^\pi\frac{\psi(x,t)}{\sin\frac{t}{2}}\Re\left\{e^{\frac{i(k+1)t}{2}}\left(\theta e^{-\frac{it}{2}}+e^{\frac{it}{2}}\right)^k\right\}dt\\&=\frac{1}{2\pi(1+\theta)^k}\int_0^\pi\frac{\psi(x,t)}{\sin\frac{t}{2}}\Re\left\{e^{\frac{i(k+1)t}{2}}\left[(\theta+1)\cos\frac{t}{2}-i(\theta-1)\sin\frac{t}{2}\right]^k\right\}dt.\end{aligned}$$

Denoting

$$R(t)\cos\varphi(t;\theta)=(\theta+1)\cos\frac{t}{2}\quad\text{and}\quad R(t)\sin\varphi(t;\theta)=(\theta-1)\sin\frac{t}{2},$$

we easily obtain that

$$R(t)=\sqrt{(\theta+1)^2-4\theta\sin^2\frac{t}{2}},$$

and

$$\varphi(t;\theta)=\tan^{-1}\left(\frac{\theta-1}{\theta+1}\tan\frac{t}{2}\right).$$

So, we have

$$\begin{aligned}\widetilde{E_k^\theta}(x)-\widetilde{f}(x) &= \frac{1}{2\pi(1+\theta)^k}\int_0^\pi \frac{\psi(x,t)}{\sin\frac{t}{2}}\Re\left\{e^{\frac{i(k+1)t}{2}}R^k(t)e^{-i\varphi(t;\theta)k}\right\}dt\\ &= \frac{1}{2\pi(1+\theta)^k}\int_0^\pi \frac{\psi(x,t)}{\sin\frac{t}{2}}\Re\left\{e^{\frac{i(k+1)t}{2}}R^k(t)e^{-ik\varphi(t;\theta)}\right\}dt\\ &= \frac{1}{2\pi(1+\theta)^k}\int_0^\pi \frac{\psi(x,t)}{\sin\frac{t}{2}}R^k(t)\cos\left[\frac{(k+1)t}{2}-k\tan^{-1}\left(\frac{\theta-1}{\theta+1}\tan\frac{t}{2}\right)\right]dt.\end{aligned}$$

Consequently, the (N,p_n,q_n) means of $\widetilde{E_k^\theta}(x)-\widetilde{f}(x)$ are

$$\begin{aligned}&\frac{1}{R_n}\sum_{k=0}^n p_kq_{n-k}\left\{\widetilde{E_k^\theta}(x)-\widetilde{f}(x)\right\}\\ &= \frac{1}{2\pi R_n}\sum_{k=0}^n \frac{p_kq_{n-k}}{(1+\theta)^k}\int_0^\pi \frac{\psi(x,t)}{\sin\frac{t}{2}}R^k(t)\cos\left[\frac{(k+1)t}{2}-k\tan^{-1}\left(\frac{\theta-1}{\theta+1}\tan\frac{t}{2}\right)\right]dt\end{aligned}$$

or

$$\widetilde{t_n^{p,q,\theta}}(x)-\widetilde{f}(x)=\int_0^\pi \psi(x,t)\widetilde{K_n^{p,q,\theta}}(t)dt.$$

Whence, using generalized Minkowski's inequality (see [15, p. 18]) and Lemma 2.3, we have

$$\begin{aligned}\left\|\widetilde{t_n^{p,q,\theta}}(x)-\widetilde{f}(x)\right\|_{L^r} &= \left[\int_0^{2\pi}\left|\widetilde{t_n^{p,q,\theta}}(x)-\widetilde{f}(x)\right|^r dx\right]^{1/r}\\ &= \left[\int_0^{2\pi}\left|\int_0^\pi \psi(x,t)\widetilde{K_n^{p,q,\theta}}(t)dt\right|^r dx\right]^{1/r}\\ &= \int_0^\pi\left[\int_0^{2\pi}\left|\psi(x,t)\right|^r dx\right]^{1/r}\left|\widetilde{K_n^{p,q,\theta}}(t)\right|dt\\ &= \int_0^{\frac{\pi}{n+1}}\mathcal{O}(t^\alpha)\left|\widetilde{K_n^{p,q,\theta}}(t)\right|dt+\int_{\frac{\pi}{n+1}}^\pi \mathcal{O}(t^\alpha)\left|\widetilde{K_n^{p,q,\theta}}(t)\right|dt\\ &:= J_1+J_2,\end{aligned}$$

since by Lemma 2.3 the implication $f\in \mathrm{Lip}\alpha \Longrightarrow \psi\in \mathrm{Lip}\alpha$ holds true.

Based on Lemma 2.1, we obtain

$$J_1=\int_0^{\frac{\pi}{n+1}}\mathcal{O}(t^\alpha)\mathcal{O}(t^{-1})dt=\mathcal{O}((n+1)^{-\alpha}).$$

Now applying Lemma 2.2, we have

$$\begin{aligned} J_2 &= \int_{\frac{\pi}{n+1}}^{\pi} \mathcal{O}(t^\alpha)\left|\widetilde{K_n^{p,q,\theta}}(t)\right|dt \\ &= \int_{\frac{\pi}{n+1}}^{\pi} \mathcal{O}(t^\alpha)\mathcal{O}_\theta\left(\frac{1}{(n+1)^2t^3}\right)dt \\ &= \mathcal{O}_\theta\left(\frac{1}{(n+1)^2}\right)\frac{t^{\alpha-2}}{\alpha-2}\Bigg|_{\frac{\pi}{n+1}}^{\pi} = \mathcal{O}_\theta\left(\frac{1}{(n+1)^\alpha}\right). \end{aligned}$$

Therefore, we have

$$\left\|\widetilde{t_n^{p,q,\theta}} - \widetilde{f}\right\|_{L^r} = \mathcal{O}_\theta\left(\frac{1}{(n+1)^\alpha}\right), \quad \theta \neq 3, \quad 0 < \alpha \leq 1.$$

□

Note that $\widetilde{t_n^{p,q,\theta}}$ means generalize $\widetilde{t_n^{NE}}$ means considered in [9, 11], in the case when $q_n = 1$ for all n, and therefore this theorem has a wider range of applications. Another advantage of this theorem is the fact that, for $\alpha = 1$, it gives a degree of approximation of Jackson's order, even in this case, while in the previous results proved by others do not.

Example 3.2. In order to support our main results, we take $p_n = n+1$, $q_n = 1$, and $\theta = 1$, then $E_n^1(f;x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}s_k(f;x)$ means as well as $\widetilde{t_n^N}(f;x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)s_k(f;x)$ means, converge to $f(x)$ faster than $s_n(x)$ in the interval $[-\pi,\pi]$, where $s_n(x)$ denotes the partial sums of the series

$$\sum_{n=1}^{\infty}\frac{2[1-(-1)^n]}{\pi n}\sin nx, \quad -\pi \leq x \leq \pi,$$

which is the Fourier series of the function

$$f(x) = \begin{cases} -1 & \text{if } -\pi \leq x < 0, \\ 1 & \text{if } \ \ 0 \leq x \leq \pi. \end{cases}$$

This nice example given in [11] (see its details at the end of it) shows that the product summability means $\widetilde{t_n^{NE}}(f;x)$ (these are special cases of the means considered in this paper) of the Fourier series of $f(x)$ overshoots the Gibbs Phenomenon and presents the smoothing effect of this method providing better approximates than partial sums $s_n(f;x)$.

Theorem 3.3. *Let (N, p_n, q_n) be a generalized regular Nörlund method of summability defined by two positive generating sequences (p_n) and (q_n) satisfying conditions (2.1) or (2.2). Assume that $\xi(t)$ is a modulus of continuity such that*

$$\int_0^v \frac{\xi(t)}{t}dt = \mathcal{O}\left(\xi(v)\right), \quad 0 < v < \pi. \tag{3.1}$$

Let $f : [0, 2\pi] \longrightarrow \mathbb{R}$ be a 2π periodic function, Lebesgue integrable, and belonging to the $\mathrm{Lip}(\xi(t), r), (r \geq 1)$ *class. Then the degree of approximation of the* *function* $\widetilde{f}$ *by* $\widetilde{t_n^{p,q,\theta}}$ *means of the partial sums* $\widetilde{s}_n(f,x)$ *of the series* (1.2) *is given by*

$$\left\|\widetilde{t_n^{p,q,\theta}} - \widetilde{f}\right\|_{L^r} = \mathcal{O}_\theta\left(\frac{1}{(n+1)^2}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\xi(t)}{t^3}dt\right)$$

for all $n \in \{0,1,2,\dots\}$.

Proof. Using the generalized Minkowski's inequality and Lemma 2.4, we have

$$\begin{aligned}
\left\|\widetilde{t_n^{p,q,\theta}}(x) - \widetilde{f}(x)\right\|_{L^r} &= \left[\int_0^{2\pi}\left|\widetilde{t_n^{p,q,\theta}}(x) - \widetilde{f}(x)\right|^r dx\right]^{1/r} \\
&= \left[\int_0^{2\pi}\left|\int_0^{\pi}\psi(x,t)\widetilde{K_n^{p,q,\theta}}(t)dt\right|^r dx\right]^{1/r} \\
&= \int_0^{\pi}\left[\int_0^{2\pi}\left|\psi(x,t)\right|^r dx\right]^{1/r}\left|\widetilde{K_n^{p,q,\theta}}(t)\right|dt \\
&= \int_0^{\frac{\pi}{n+1}}\mathcal{O}(\xi(t))\left|\widetilde{K_n^{p,q,\theta}}(t)\right|dt + \int_{\frac{\pi}{n+1}}^{\pi}\mathcal{O}(\xi(t))\left|\widetilde{K_n^{p,q,\theta}}(t)\right|dt \\
&:= J_1^{(1)} + J_2^{(1)}.
\end{aligned}$$

Based on Lemma 2.1 and condition (3.1), we obtain

$$\begin{aligned}
J_1^{(1)} &= \int_0^{\frac{\pi}{n+1}}\mathcal{O}(\xi(t))\left|\widetilde{K_n^{p,q,\theta}}(t)\right|dt = \mathcal{O}\left(\int_0^{\frac{\pi}{n+1}}\frac{\xi(t)}{t}dt\right) \\
&= \mathcal{O}\left(\xi\left(\frac{\pi}{n+1}\right)\right) = \mathcal{O}\left(\frac{1}{(n+1)^2}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\xi(t)}{t^3}dt\right),
\end{aligned}$$

since

$$\begin{aligned}
\frac{1}{(n+1)^2}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\xi(t)}{t^3}dt &\geq \frac{\xi\left(\frac{\pi}{n+1}\right)}{(n+1)^2}\int_{\frac{\pi}{n+1}}^{\pi}\frac{dt}{t^3} \\
&\geq \frac{\xi\left(\frac{\pi}{n+1}\right)}{(n+1)^2}\left(\frac{1}{-2t^2}\right)\Bigg|_{\frac{\pi}{n+1}}^{\pi} \geq \frac{1}{4\pi^2}\xi\left(\frac{\pi}{n+1}\right).
\end{aligned}$$

On the other hand, Lemma 2.2 gives

$$J_2^{(1)} = \int_{\frac{\pi}{n+1}}^{\pi}\mathcal{O}(\xi(t))\left|\widetilde{K_n^{p,q,\theta}}(t)\right|dt = \mathcal{O}_\theta\left(\frac{1}{(n+1)^2}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\xi(t)}{t^3}dt\right).$$

Consequently, we clearly have

$$\left\|\widetilde{t_n^{p,q,\theta}} - \widetilde{f}\right\|_{L^r} = \mathcal{O}_\theta\left(\frac{1}{(n+1)^2}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\xi(t)}{t^3}dt\right),$$

which proves the statement of the theorem. □

Theorem 3.4. *Let (N, p_n, q_n) be a generalized regular Nörlund method of summability defined by two positive generating sequences (p_n) and (q_n) satisfying (1.4) or (1.5) and (3.1). Let $f : [0, 2\pi] \longrightarrow \mathbb{R}$ be a 2π-periodic function, Lebesgue integrable, and belonging to the $\mathrm{Lip}(\xi(t), r), (r \geq 1)$ class, and let $\frac{\xi(t)}{t}$ be monotone decreasing in $(\pi/(n+1), \pi)$. Then the degree of approximation of the function $\widetilde{f}$ by $\widetilde{t_n^{p,q,\theta}}$ means of the partial sums $\widetilde{s}_n(f, x)$ of the series (1.2) is given by*

$$\left\|\widetilde{t_n^{p,q,\theta}} - \widetilde{f}\right\|_{L^r} = \mathcal{O}_\theta\left(\xi\left(\frac{\pi}{n+1}\right)\right),$$

for all $n \in \{0, 1, 2, \dots\}$ and $\theta \neq 3$.

Proof. Under conditions of our theorem and from the proof of the previous theorem, we have

$$\begin{aligned}
\left\|\widetilde{t_n^{p,q,\theta}} - \widetilde{f}\right\|_{L^r} &= \mathcal{O}_\theta\left(\frac{1}{(n+1)^2}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\xi(t)}{t^3}dt\right) \\
&= \mathcal{O}_\theta\left(\frac{1}{(n+1)^2}\frac{\xi\left(\frac{\pi}{n+1}\right)}{\left(\frac{\pi}{n+1}\right)}\int_{\frac{\pi}{n+1}}^{\pi}\frac{dt}{t^2}\right) \\
&= \mathcal{O}_\theta\left(\xi\left(\frac{\pi}{n+1}\right)\right).
\end{aligned}$$

□

4. Corollaries

Several corollaries can be derived from our main results. Let us see some of them as follows. If we take $p_n = 1$ for all n and $\theta = 1$, then we obtain (note that conditions (2.1) or (2.2) become simpler) the following corollary.

Corollary 4.1. *Let (N, q_n) be a regular Nörlund method of summability defined by a positive sequence (q_n). Let $f : [0, 2\pi] \longrightarrow \mathbb{R}$ be a 2π periodic function, Lebesgue integrable, and belonging to the $\mathrm{Lip}(\alpha, r), (r \geq 1)$ class. If either (2.1) or (2.2) holds true for $p_n = 1$, then the degree of approximation of the function $\widetilde{f}$ by the $(N, q_n)(E, 1)$ transform*

$$\widetilde{t_n^{NE}}(x) = \frac{1}{Q_n}\sum_{k=0}^{n}\frac{q_{n-k}}{2^k}\sum_{v=0}^{k}\binom{k}{v}\widetilde{s}_v(x)$$

of the partial sums $\widetilde{s}_n(f, x)$ of the series (1.2) is given by the following estimation:

$$\left\|\widetilde{t_n^{NE}} - \widetilde{f}\right\|_{L^r} = \mathcal{O}\left((n+1)^{-\alpha}\right), \quad 0 < \alpha \leq 1,$$

for all $n \in \{0, 1, 2, \dots\}$.

Corollary 4.2. *Let (N, q_n) be a regular Nörlund method of summability defined by a positive generating sequence (q_n) satisfying conditions (2.1) or (2.2) for $p_n = 1$.*

Assume that $\xi(t)$ *is a modulus of continuity such that*

$$\int_0^v \frac{\xi(t)}{t} dt = \mathcal{O}\left(\xi(v)\right), \quad 0 < v < \pi.$$

Let $f : [0, 2\pi] \longrightarrow \mathbb{R}$ *be a* 2π *periodic function, Lebesgue integrable, and belonging to the* $\operatorname{Lip}(\xi(t), r), (r \geq 1)$ *class. Then the degree of approximation of the function* $\widetilde{f}$ *by* $\widetilde{t_n^{NE}}$ *means of the partial sums* $\widetilde{s}_n(f, x)$ *of the series* (1.2) *is given by*

$$\left\| \widetilde{t_n^{NE}} - \widetilde{f} \right\|_{L^r} = \mathcal{O}\left(\frac{1}{(n+1)^2} \int_{\frac{\pi}{n+1}}^{\pi} \frac{\xi(t)}{t^3} dt \right)$$

for all $n \in \{0, 1, 2, \dots\}$.

Corollary 4.3. *Let* (N, p_n, q_n) *be a regular Nörlund method of summability defined by a positive generating sequence* (p_n) *satisfying* (1.4) *or* (1.5) *and* (3.1). *Let* $f : [0, 2\pi] \longrightarrow \mathbb{R}$ *be a* 2π*-periodic function, Lebesgue integrable, and belonging to the* $\operatorname{Lip}(\xi(t), r), (r \geq 1)$ *class, and let* $\frac{\xi(t)}{t}$ *be monotone decreasing in* $(\pi/(n+1), \pi)$. *Then the degree of approximation of the function* $\widetilde{f}$ *by* $\widetilde{t_n^{NE}}$ *means of the partial sums* $\widetilde{s}_n(f, x)$ *of the series* (1.2) *is given by*

$$\left\| \widetilde{t_n^{NE}} - \widetilde{f} \right\|_{L^r} = \mathcal{O}\left(\xi\left(\frac{\pi}{n+1} \right) \right),$$

for all $n \in \{0, 1, 2, \dots\}$.

5. Conclusion

The theory of approximations is a very important field of study in many researchers. In particular, the theory of trigonometric approximation is of great mathematical interest and still receives considerable attention. In many problems, we encountered the determination of the degree of approximation of periodic functions belonging to generalized Lipschitz classes using the product of matrix operators. In this paper, we considered the product $(N, p_n, q_n)(E, \theta)$, which is obtained by superimposing (E, θ) summability, $(\theta > 0, \theta \neq 3)$, on (N, p_n, q_n) summability, to determine the degree of approximation (under some special conditions on the sequences $\{p_n\}$ and $\{q_n\}$) of periodic functions belonging to generalized Lipschitz classes (Theorems 3.1, 3.3, and 3.4). These results, presented here, are more general than those proved earlier by others, and from them, we derived several new corollaries.

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[1] University of Prishtina "Hasan Prishtina", Faculty of Education, Department of Mathematics and Informatics, Avenue "Mother Theresa" 5, 10000 Prishtina, Kosovo.
E-mail address: xhevat.krasniqi@uni-pr.edu

[2] Mathematics Discipline, PDPM Indian Institute of Information Technology, Design and Manufacturing, Jabalpur, Dumna Airport Road, P.O.: Khamaria, Jabalpur 482 005, Madhya Pradesh, India.
E-mail address: dmrai23@gmail.com