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ON PAIR OF GENERALIZED DERIVATIONS IN RINGS

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Abstract. Let R be an associative ring with extended centroid C, let G and F be generalized derivations of R associated with nonzero derivations δ and d, respectively, and let $m, k, n \geq 1$ be fixed integers. In the present paper, we study the situations: (i)$F(x) \circ_m G(y) = (x \circ_n y)^k$, (ii) $[F(x), y]_m + [x, d(y)]_n = 0$ for all y, x in some appropriate subset of R.

1. Introduction

Throughout the present paper, R is always an associative ring with centre $Z(R)$, C is the extended centroid of R, and the Utumi quotient ring is denoted by U. For further information related to these concepts, we refer the reader to [2]. For any elements $x, y \in R$, $[x, y]$ and $x \circ y$ stand for the Lie commutator $xy - yx$ and the Jordan commutator $xy + yx$, respectively. Let $x, y \in R$, then we set $x \circ_0 y = x$, $x \circ_1 y = x \circ y = xy + yx$, and $x \circ_m y = (x \circ_{m-1} y)y + y(x \circ_{m-1} y)$ for $m \geqslant 2$. We also set $[x, y]_0 = x$ and $[x, y]_1 = xy - yx$. The Engel condition is a polynomial $[x, y]_m = [x, y]_{m-1}y - y[x, y]_{m-1}, m \geq 2$ in non-commuting indeterminates x and y. A ring R is said to satisfy the Engel condition if $[x, y]_m = 0$ for some integer $m \geq 1$. Recall that a ring R is a prime ring if for each $y, x \in R$, $yRx = \{0\}$ implies that either $y = 0$ or $x = 0$ and R is a semiprime ring if for each $z \in R$, $zRz = \{0\}$ implies that $z = 0$. Prime rings are always semiprime but the converse is not true in general.

In the present paper, we establish a relation within the structure of rings and the nature of suitable mappings that satisfy some certain identities. In particular, we discuss generalized derivations defined on a ring R. An additive map $d : R \to$

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R is called a derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, if d can be written as $d(x) = [b, x]$ for some element $b \in R$, then d is called an inner derivation (determined by b). We use the notation I_b to denote the inner derivation determined by the element b. By a generalized inner derivation on R, we mean a self mapping F on R which is additive and for each $x \in R$ satisfies $F(x) = bx + xc$, where b, c are fixed elements in R. We can see that such a mapping F satisfies $F(xy) = x[c, y] + F(x)y = xI_c(y) + F(x)y$, where I_c denotes the inner derivation determined by the element c. This observation gives the following definition, which is given in [4]: An additive mapping $F : R \to R$ is said to be a generalized derivation if $F(zw) = F(z)w + zd(w)$ for all $w, z \in R$, where d stands for some derivation on R. Some homely instances of generalized derivations are generalized inner derivations, derivations, and left multipliers. We recall that a self additive mapping F of R is said to be a left multiplier if $F(ab) = F(a)b$ for all $b, a \in R$.

Argaç and Inceboz [1] showed that if a nonzero derivation d of a prime ring R satisfies $(d(x \circ y))^k = x \circ y$ for all $y, x \in I$, where I is a nonzero ideal of R and k is a fixed positive integer, then the ring is commutative. Further, Huang [9] proved that if U is a square closed Lie ideal of a prime ring R with the characteristic different from 2 and a generalized derivation F with associated derivation d on R satisfying $F(y) \circ d(x) = y \circ x$ for any $y, x \in U$, then either R is commutative or $d = 0$.

Influence by the mentioned above results, we prove the following result.

Theorem 1.1. *Let m, n, k be the fixed positive integers, and let I be a nonzero ideal of a prime ring R with characteristic different from 2. If R admits generalized derivations F and G with associated nonzero derivations d and δ, respectively, such that $F(x) \circ_m G(y) = (x \circ_n y)^k$ for all $x, y \in I$, then R is commutative.*

Bell and Daif [3] initiated the concept of the term strong commutativity preserving (SCP) maps and showed the following: Let I be a nonzero right ideal of a semiprime ring R. If a derivation d of R satisfies $[d(x), d(y)] = [x, y]$ for all $y, x \in I$, then I is central. Inspired by the work of Bell and Daif [3], Huang [10] proved the following: If I is a nonzero ideal of R, a prime ring having characteristic different from 2, which admits a nonzero derivation d satisfying $[d(x), d(y)]_m = [x, y]_n$ for any $y, x \in I$, for some fixed positive integers m, n, then R is commutative. Influence by these results, Dhara, Ali, and Pattanayak [6] showed the following: Let I a nonzero ideal of a 2-torsion free semiprime ring R that admits a generalized derivation F associated with derivation d such that $d(I) \neq \{0\}$. If $[d(y), F(x)] = \pm[y, x]$ holds for all $y, x \in I$, then R contains a central ideal that is nonzero.

Tendentious by the above results, we study the following condition: $[F(x), y]_m + [x, d(y)]_n = 0$ for any $y, x \in I$, where I is a nonzero ideal of R and F is a generalized derivation associated with the derivation d of R. Bluntly, we prove the following.

Theorem 1.2. *Let m and n be fixed positive integers and let I be a nonzero ideal of a prime ring R with characteristic different from 2. If a generalized derivation*

F with associated nonzero derivation d of R satisfies $[F(x), y]_m + [x, d(y)]_n = 0$ for all $x, y \in I$, then R is commutative.

Theorem 1.3. *Let m and n be fixed positive integers and let R be a semiprime ring with characteristic different from 2. If a generalized derivation F with associated nonzero derivation d of R satisfies $[F(x), y]_m = [x, d(y)]_n$ for all $x, y \in R$, then there exists an idempotent element e in U that is central such that the ring $(1-e)U$ is commutative and the derivation d vanishes identically on eU.*

2. Main results

We will use frequently the following important result due to Kharchenko [11]:

Let d be a nonzero derivation of a prime ring R and let I be a nonzero ideal of R. Let $g(z_1, \ldots, z_n, d(z_1), \ldots, d(z_n))$ be a differential identity in I, that is,

$$g(w_1, \ldots, w_n, d(w_1), \ldots, d(w_n)) = 0 \text{ for all } w_1, w_2, \ldots, w_n \in I.$$

Then one of the following holds:

(i) d is an inner in Q, where Q is a martingale ring of quotient of R, that is, d can be written as $d(x) = [p, x]$ for any $x \in R$ and for some $p \in Q$. Also we have

$$g(w_1, \ldots, w_n, [p, w_1], \ldots, [p, w_n]) = 0 \quad \text{for any } w_1, \ldots, w_n \in I.$$

(ii) d is Q-outer and the following GPI is satisfied by I:

$$g(w_1, \ldots, w_n, y_1, \ldots, y_n) = 0.$$

Remark 2.1. Let I be an ideal of R. Then

(i) U, R, and I satisfy the same differential identities; see [13, Theorem 2].

(ii) U, R, and I satisfy the same GPI with coefficients in U; see [5, Theorem 2].

Remark 2.2. Let F be a generalized derivation defined on a dense right ideal of a semiprime ring R. Then F can be uniquely extended to U that takes the form $F(x) = ax + d(x)$, where d is a derivation on U and for some $a \in U$. Moreover, a and d are uniquely determined by the generalized derivation F; see [14, Theorem 4].

Proof of Theorem 1.1. By the hypotheses, we have

$$F(x) \circ_m G(y) = (x \circ_n y)^k \quad \text{for any } x, y \in I. \tag{2.1}$$

Now since R is a prime ring and F, G are generalized derivations of R, by Remark 2.2, $G(x) = bx + \delta(x)$ and $F(x) = ax + d(x)$ for some $b, a \in U$ and derivations δ, d on U. By Remark 2.1, we have

$$F(x) \circ_m G(y) = (x \circ_n y)^k \tag{2.2}$$

for any $y, x \in U$. Hence

$$(ax + d(x)) \circ_m (by + \delta(y)) = (x \circ_n y)^k \quad \text{for any } y, x \in U, \tag{2.3}$$

that is,

$$ax \circ_m by + d(x) \circ_m by + ax \circ_m \delta(y) + d(x) \circ_m \delta(y) = (x \circ_n y)^k. \tag{2.4}$$

Here the proof is divided into three cases:
Case 1 If both δ and d are inner derivations, then there exist elements q and $p \in U$, respectively, such that $d(x) = [q, x]$ and $\delta(x) = [p, x]$ for any $x \in U$. So, we have

$$\begin{aligned} H(x, y) =& ax \circ_m by + [q, x] \circ_m by + ax \circ_m [p, y] \\ &+ [q, x] \circ_m [p, y] - (x \circ_n y)^k = 0 \quad \text{for any } y, x \in U. \end{aligned} \tag{2.5}$$

If C is infinite, then $U \bigotimes_C \bar{C}$ satisfies (2.5), where $\bar{C}$ stands for the algebraic closure of C. By [7], U and $U \bigotimes_C \bar{C}$ are centrally closed and prime. Therefore, we may replace R by $U \bigotimes_C \bar{C}$ or U according to C is infinite or finite. Thus we may assume that R is centrally closed over C, which is either algebraically closed and $H(x, y) = 0$ for any $y, x \in R$ or finite. By the use of Martindale's theorem [7], R is a primitive ring with D as an associative division ring as well as R has nonzero $soc(R)$. Also by the use of Jacobson's theorem [8], R and the dense ring of linear transformations for some vector space V over C are isomorphic, that is, $R \cong M_k(D)$, where $k = dim_D V$. Assume that $dim_D V \geqslant 2$, otherwise we are done. Also assume that there exists $v \in V$ such that qv and v are linearly D-independent.

If pv does not belong to the span of $\{v, qv\}$, then $\{v, pv, qv\}$ is linearly independent. By the density of ring R, there exist $y, x \in R$ such that

$$xv = 0, \quad xqv = -v, \quad ypv = v, \quad xpv = 0, \quad yv = 0, \quad yqv = v. \tag{2.6}$$

Multiplying equation (2.5) by v from right and using conditions in equation (2.6), we get $(-1)^{m-1}2^{m-1}v = 0$, a contradiction.

If pv belongs to the span of $\{v, qv\}$, then $p = v\alpha + qv\beta$ for some $\alpha, 0 \neq \beta \in D$. Again by the density of ring R, there exist $y, x \in R$ such that

$$xv = 0, \quad xqv = -v, \quad yqv = v, \quad yv = 0. \tag{2.7}$$

Again multiplying equation (2.5) by v from right and using conditions in equations (2.7), we get $(-1)^{m-1}2^{m-1}v\beta = 0$, a contradiction.

Therefore, $\{v, qv\}$ is linearly dependent over D and hence $q \in Z(R)$, that is, $d = 0$ which is a contradiction to our hypotheses. Similarly, we can show that $\delta = 0$, which contradicts our hypotheses.
Case 2 Assume that both δ and d are not both inner derivations of U. let δ and d are C-linearly dependent modulo D_{int}. Let $\delta = ad(p) + \beta d$, for some $\beta \in C$, where $ad(p)$ is an inner derivation induced by the element $p \in U$. Observe that if either $\beta = 0$ or d is inner, then δ is also inner which contradicts. So, $\beta \neq 0$ and d is not inner. Then by (2.3), we have

$$(ax + d(x)) \circ_m (by + \beta d(y) + [p, y]) = (x \circ_n y)^k \quad \text{for any } y, x \in U,$$

that is,

$$ax \circ_m (by + \beta d(y) + [p, y]) + d(x) \circ_m (by + \beta d(y) + [p, y]) = (x \circ_n y)^k.$$

Then by the use of Kharchenko's theorem [11], we have

$$ax \circ_m (by + \beta y_1 + [p, y]) + x_1 \circ_m (by + \beta y_1 + [p, y]) = (x \circ_n y)^k$$

for all $y, x, y_1, x_1 \in I$. Setting $y = 0 = x$, we obtain

$$x_1 \circ_m y_1 = 0 \tag{2.8}$$

for all $y_1, x_1 \in I$. By [5, Theorem 2], Q as well as R satisfies the polynomial identity $x_1 \circ_m y_1 = 0$. By [12, Lemma 1], we have $R \subseteq M_n(F)$, the ring of $n \times n$ matrices over some field F, where $n \geq 1$. Also, $M_n(F)$ and R satisfy the same polynomial identity, that is, $x_1 \circ_m y_1 = 0$, for any $y_1, x_1 \in M_n(F)$. We use e_{ij} to denote matrix unit with 1 in (i, j)th-entry and zero elsewhere. Taking $y_1 = e_{11}$ and $x_1 = e_{12}$, we see that $x_1 \circ_m y_1 = e_{12} \neq 0$, a contradiction.

The case when $d = ad(q) + \gamma\delta$ for some $\gamma \in C$ and $ad(q)$, an inner derivation induced by an element $q \in U$, is similar.

Case 3 Now assume that δ and d are C-linearly independent modulo D_{int}. Therefore, from (2.4), we have

$$ax \circ_m by + d(x) \circ_m by + ax \circ_m \delta(y) + d(x) \circ_m \delta(y) = (x \circ_n y)^k$$

for any $y, x \in U$. Then by the use of Kharchenko's theorem [11], we have

$$ax \circ_m by + z \circ_m by + ax \circ_m w + z \circ_m w = (x \circ_n y)^k$$

for any $w, z, y, x \in I$. Particularly, for $y = x = 0$, we have

$$z \circ_m w = 0, \tag{2.9}$$

which is the same as equation (2.8). Therefore, by a similar argument as above, this leads that R is commutative. This finishes the proof of the theorem. □

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By hypotheses, we have

$$[F(x), y]_m + [x, d(y)]_n = 0 \quad \text{for any } y, x \in I. \tag{2.10}$$

By Remark 2.1, we have

$$[F(x), y]_m + [x, d(y)]_n = 0 \quad \text{for any } y, x \in U. \tag{2.11}$$

By Remark 2.2, it follows that $F(x) = ax + d(x)$ for some $a \in U$ and derivation d on U. Then we have

$$[ax + d(x), y]_m + [x, d(y)]_n = 0 \quad \text{for any } y, x \in U. \tag{2.12}$$

That is,

$$[ax, y]_m + [d(x), y]_m + [x, d(y)]_n = 0 \quad \text{for any } y, x \in U. \tag{2.13}$$

In the light of Kharchenko's theorem [11, Theorem 2], the proof is divided into two cases:

Case I Let d be an inner derivation of U, that is, $d(x) = [q, x]$ for any $x \in U$ and for some $q \in U$. Then

$$H(x, y) = [ax, y]_m + [[q, x], y]_m + [x, [q, y]]_n = 0 \quad \text{for any } y, x \in U. \tag{2.14}$$

If C is infinite, then $U \bigotimes_C \bar{C}$ satisfies (2.14), where $\bar{C}$ stands for the algebraic closure of C. By [7], U and $U \bigotimes_C \bar{C}$ are centrally closed and prime. Therefore, we may replace R by $U \bigotimes_C \bar{C}$ or U according to C is infinite or finite. Thus we may assume that R is centrally closed over C, which is either algebraically closed and $H(x,y)=0$ for any $y,x \in R$ or finite. By the use of Martindale's theorem [7], R is a primitive ring with D as associative division ring as well as R has nonzero $soc(R)$. Also by the use of Jacobson's theorem [8], R and the dense ring of linear transformations for some vector space V over C are isomorphic, that is, $R \cong M_k(D)$, where $k = dim_D V$. Assume that $dim_D V \geqslant 2$, otherwise we are done. Also assume that there exists $v \in V$ such that qv and v are linearly D-independent. Since $dim_D V \geqslant 2$, it is possible to find $w \in V$ such that $\{w, qv, v\}$ is linearly independent over D. By the density of the ring R, we can find $y, x \in R$ such that

$$xv = 0, \quad xqv = w, \quad yw = v, \quad xw = 0, \quad yv = 0, \quad yqv = v. \tag{2.15}$$

Multiplying equation (2.14) by v from right and using conditions in equation (2.15), we get $v = 0$, which is a contradiction to the linearly independent of the set $\{v, qv\}$. Therefore, $\{qv, v\}$ is linearly dependent and hence $q \in Z(R)$, that is, $d = 0$, which is a contradiction to our hypotheses. Hence our assumption $dim_D V \geqslant 2$ is wrong. Therefore, $dim_D V = 1$ and hence R is commutative.

Case II Let d be an outer derivation. Then

$$[ax, y]_m + [t, y]_m + [x, s]_n = 0 \quad \text{for any } y, x, t, s \in I. \tag{2.16}$$

In particular, choosing $y = 0$, we get $[x, s]_n = 0$ for any $s, x \in I$, that is, $[x, s]_m = 0 = [I_x(s)_{m-1}, s]$ for all $s, x \in I$. By [12, Theorem 1], either R is commutative or $I_x = \{0\}$, that is, $I \subseteq Z(R)$ that is R is commutative by Mayne [15]. □

Now we prove the last result.

Proof of Theorem 1.3. We know that any derivation defined on a semiprime ring R can be uniquely extended to a derivation on U, where U is a left Utumi ring of quotient of R, and hence every derivation of R can be defined on U; see [13, Lemma 2]. Also, U, R, and I satisfy the same generalized polynomial identity (GPI) and differential identities (see [5, 13]). By [14, Theorem 4], F can be expressed as $F(x) = d(x) + ax$ for some $a \in U$ and a derivation d defined on U. We have

$$[ax, y]_m + [d(x), y]_m + [x, d(y)]_n = 0 \quad \text{for any } y, x \in U. \tag{2.17}$$

Let $M(C) = \{A \mid A \text{ is a maximal ideal of } C\}$ and let $P \in M(C)$. Then PU is a prime ideal of U, which is invariant under all derivations of U by the theory of orthogonal completions of semiprime ring (see [13, pp. 31–32]). Also, $\bigcap\{PU \mid P \in M(C)\ \} = \{0\}$. Set $\overline{U} = U/PU$. Now any derivation d of R canonically induces a derivation $\overline{d}$ on $\overline{U}$ defined by $\overline{d}(\overline{x}) = \overline{d(x)}$ for any $x \in \bar{U}$. Then

$$[\bar{a}\bar{x}, \bar{y}]_m + [\overline{d(x)}, \bar{y}]_m + [\bar{x}, \overline{d(y)}]_n = 0$$

for all $\overline{y}, \overline{x} \in \overline{U}$. It is clear that $\overline{U}$ is a prime ring. So by the use of Theorem 1.2, we have, either $[U, U] \subseteq PU$ or $d(U) \subseteq PU$ for any $P \in M(C)$. This gives that $d(U)[U, U] \subseteq PU$ for any $P \in M(C)$. Since $\bigcap\{PU \mid P \in M(C)\ \} = \{0\}$, we have $d(U)[U, U] = \{0\}$. Again using the standard theory of orthogonal completion of semiprime ring [2], it is obvious that there exists an element e that is a central idempotent in U such that on the direct sum decomposition $U = eU \oplus (1-e)U$, such that d vanishes identically on eU and the ring $(1-e)U$ is commutative. □

The following examples demonstrate that R to be *prime* cannot be omitted in the hypotheses of Theorems 1.1 and 1.2.

Example 2.3. For any ring K with characteristic different from two, let $R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x, y \in K \right\}$ and $I = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \in K \right\}$. Then R is a ring under the usual addition and multiplication of matrices and I is a nonzero ideal of R. Define maps $F, G, d, \delta : R \to R$ by $F\left(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x & 2y \\ 0 & 0 \end{pmatrix}$, $G\left(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$, $\delta\left(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -y \\ 0 & 0 \end{pmatrix}$, and $d\left(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. Then F and G are generalized derivations on R associated with the nonzero derivations d and δ, respectively, satisfying $F(x) \circ_m G(y) = (x \circ_n y)^k$ for all $x, y \in I$. However R is not commutative. Hence Theorem 1.1 is not true for arbitrary rings.

Example 2.4. Let $R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, y, z \in K \right\}$ and $I = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \in K \right\}$, where K is a ring with characteristic different from two. Then R is a ring under the usual addition and multiplication of matrices and I is a nonzero ideal of R. Define maps $F, d : R \to R$ by $F\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ and $d\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. Then F is a generalized derivation on R associated with the nonzero derivation d satisfying $[F(x), y]_m + [x, d(y)]_n = 0$ for all $x, y \in I$. However R is not commutative. Hence Theorem 1.2 does not hold for arbitrary rings.

References

1. N. Argaç and H.G. Inceboz, *Derivations of prime and semiprime rings*, J. Korean Math. Soc. **46** (2009) 997–1005.
2. K.I. Beidar, W.S. Martindale III and A.V. Mikhalev, *Rings with Generalized Identities*, Monographs and Textbooks in Pure and Applied Mathematics, 196. Marcel Dekker, New York, 1996.
3. H.E. Bell and M.N. Daif, *On commutativity and strong commutativity-preserving maps*, Cand. Math. Bull. **37** (1994) 443–447.
4. M. Brešar, *On the distance of the composition of two derivations to the generalized derivations*, Glasgow Math. J. **33** (1991) 89–93.
5. C.L. Chuang, *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc. **103** (1988) 723–728.
6. B. Dhara, S. Ali and A. Pattanayak, *Identities with generalized derivations in semiprime rings*, Demonstratio Math. **46** (2013) 453–460.
7. T. Erickson, W.S. Martindale III and J.M. Osborn, *Prime nonassociative algebras*, Pacific J. Math. **60** (1975) 49–63.

8. N. Jacobson, *Structure of Rings,* Colloq. Publ. 37, Amer. Math. Soc. Providence, RI, 1956.
9. S. Huang, *Generalized derivations of prime rings*, Int. J. Math. Math. Sci. **2007** (2007), Art. ID 85612, 6 pp.
10. S. Huang, *Derivations with Engel conditions in prime and semiprime rings*, Czechoslovak Math. J. **61** (2011) 1135–1140.
11. V.K. Kharchenko, *Differential identities of prime rings*, (Russian) Algebra i Logika **17** (1978), no. 2, 220–238, 242–243.
12. C. Lanski, *An Engel condition with derivation*, Proc. Amer. Math. Soc. **118** (1993) 731–734.
13. T.K. Lee, *Semiprime rings with differential identities*, Bull. Inst. Math. Acad. Sin. **8** (1992) 27–38.
14. T.K. Lee, *Generalized derivations of left faithful rings*, Comm. Algebra, **27** (1999) 4057–4073.
15. J.H. Mayne, *Centralizing mappings of prime rings*, Canad. Math. Bull. **27** (1984) 122–126.

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