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ON THE NORM OF JORDAN $*$-DERIVATIONS

ABOLFAZL NIAZI MOTLAGH

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Abstract. Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Let $T \in B(\mathcal{H})$. In this paper, we determine the norm of the inner Jordan $*$-derivation $\Delta_T : X \mapsto TX - X^*T$ acting on the Banach algebra $B(\mathcal{H})$. More precisely, we show that

$$\left\|\Delta_T\right\| \geq 2 \sup_{\lambda \in W_0(T)} |\mathrm{Im}(\lambda)|$$

in which $W_0(T)$ is the maximal numerical range of operator T.

1. Introduction and preliminaries

Let $\mathfrak{A}$ be a $*$-algebra. A Jordan $*$-derivation on $\mathfrak{A}$ is a linear mapping $E : \mathfrak{A} \longrightarrow \mathfrak{A}$ which satisfies

$$E(a^2) = aE(a) + E(a)a^*$$

for all $a \in \mathfrak{A}$. Note that for a fixed $a \in \mathfrak{A}$ the mapping $\Delta_a(x) = ax - x^*a$ is a Jordan $*$-derivation; such a Jordan $*$-derivation is said to be inner.

In [1, 4], we can see the following results:

(1) Every Jordan $*$-derivation on complex $*$-algebra with identity is inner.
(2) Every Jordan $*$-derivation on the algebra of all bounded linear operators on a real Hilbert space $\mathcal{H}$ with $\dim\mathcal{H} > 1$, is inner.
(3) Every Jordan $*$-derivation on the quaternion algebra is inner.

Let $\mathcal{H}$ be a complex infinite dimensional Hilbert space and let $B(\mathcal{H})$ be the Banach algebra of all bounded linear operators on $\mathcal{H}$. For operators $\mathrm{A}, \mathrm{B} \in B(\mathcal{H})$, we define the generalized Jordan $*$-derivation $\Delta_{\mathrm{A},\mathrm{B}}$ by

$$\Delta_{\mathrm{A},\mathrm{B}}(X) = \mathrm{A}X - X^*\mathrm{B}$$

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for all $X \in B(\mathcal{H})$. Note that if A = B, then $\Delta_{\mathrm{A,A}} = \Delta_{\mathrm{A}}$ is a Jordan $*$-derivation.

The notion of numerical range (also called field of values) was firstly introduced by O. Toeplitz [6] in 1918 for matrices, but his definition applies equally well to operators on infinite dimensional Hilbert spaces.

The numerical range of $\mathrm{A} \in B(\mathcal{H})$ is defined by $W(\mathrm{A}) = \{\langle \mathrm{A}x, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$ and the numerical radius of A is defined by $w(\mathrm{A}) = \sup\{|\lambda| : \lambda \in W(\mathrm{A})\}$ where $\langle .,.\rangle$ and $\|.\|$ stand, respectively, for the scalar product on $\mathcal{H}$ and the norm associated with it. In [3], it was shown that $\overline{W(\mathrm{A})}$ is a compact convex subset of $\mathbb{C}$ and that $\sigma(\mathrm{A}) \subseteq \overline{W(\mathrm{A})}$, where $\sigma(\mathrm{A})$, the spectrum of A, consists of those complex numbers λ such that $\mathrm{A} - \lambda I$ is not invertible. The spectral radius is given by $r(\mathrm{A}) = \sup\{|\lambda| : \lambda \in \sigma(\mathrm{A})\}$. The relation between the numerical range and the spectrum of an operator has been studied by several mathematicians; see, for instance, [2, 3].

The concept of maximal numerical range was introduced by Stampfli [5] for proving the norm of a derivation.

Definition 1.1. Let A be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. The maximal numerical range of A is defined to be the set

$$W_0(\mathrm{A}) = \{\lambda \in \mathbb{C} : \langle \mathrm{A}x_n, x_n\rangle \to \lambda, \text{ where } \|x_n\| = 1 \text{ and } \|\mathrm{A}x_n\| \to \|\mathrm{A}\|\}.$$

It was shown in [5, Lemma 2] that $W_0(\mathrm{A})$ is convex and is contained in the closure of $W(\mathrm{A})$.

In the next section, we investigate the norm of an inner Jordan $*$-derivation $\Delta_T : X \mapsto TX - X^*T$ acting on $B(\mathcal{H})$. We show that

If $\lambda \in W_0(T)$, then $\|\Delta_T\| \geq 2(\|T\|^2 - |\lambda|^2)^{\frac{1}{2}}$.
$\|\Delta_T\| \geq 2\sup_{\lambda \in W_0(T)} |\mathrm{Im}(\lambda)|$.
$\|\Delta_T\| = 2\|T\|$ if and only if $0 \in W_0(T)$.
If $i\|T\| \in W_0(T)$, then $\|\Delta_T\| = 2\|T\|$.

2. Main results

Using some techniques from [5], we have the following result.

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert space and let $T \in B(\mathcal{H})$. If $\lambda \in W_0(T)$, then $\|\Delta_T\| \geq 2(\|T\|^2 - |\lambda|^2)^{\frac{1}{2}}$.*

Proof. Suppose that $\lambda \in W_0(T)$. Then there exists a sequence $\{x_n\} \subseteq \mathcal{H}$ such that $\|x_n\| = 1$, $\lim_n \|Tx_n\| = \|T\|$ and $\lim_n \langle Tx_n, x_n\rangle = \lambda$. Set $Tx_n = \alpha_n x_n + \beta_n y_n$ in which $\langle x_n, y_n\rangle = 0$ and $\|y_n\| = 1$ for all $n \in \mathbb{N}$. Hence

$$|\langle Tx_n, x_n\rangle| = |\langle \alpha_n x_n + \beta_n y_n, x_n\rangle| = |\alpha_n| \longrightarrow \lambda.$$

Now for all $n \in \mathbb{N}$ define $V_n : \mathcal{H} \mapsto \mathcal{H}$ by $V_n(x_n) = x_n, V_n(y_n) = -y_n$ and $V_n\Big|_{span\{x_n,y_n\}^\perp} = 0$. It is easy to see that $V_n^* = V_n$ and

$$\begin{aligned}\big\|\Delta_T(V_n)(x_n)\big\| &= \big\|(TV_n - V_n^*T)(x_n)\big\| \\ &= \big\|Tx_n - V_n^*(\alpha_n x_n + \beta_n y_n)\big\| \\ &= \big\|\alpha_n x_n + \beta_n y_n - \alpha_n x_n + \beta_n y_n\big\| \\ &= 2|\beta_n|.\end{aligned}$$

Since $0 \leq \lim_n \big(\|T\|^2 - \|Tx_n\|^2\big) = \lim_n \big(\|T\|^2 - |\alpha_n|^2 - |\beta_n|^2\big) = 0$, there exists a sequence $\{\varepsilon_n\} \subseteq \mathbb{R}^+$ such that $\varepsilon_n \to 0$ and $0 \leq (\|T\|^2 - |\alpha_n|^2)^{\frac{1}{2}} - |\beta_n| \leq \frac{1}{2}\varepsilon_n$. Using $|\alpha_n| \to \lambda$ and $\big\|\Delta_T(V_n)(x_n)\big\| = 2|\beta_n| \geq 2(\|T\|^2 - |\alpha_n|^2)^{\frac{1}{2}} - \varepsilon_n$ implies that $\big\|\Delta_T\big\| \geq 2(\|T\|^2 - |\lambda|^2)^{\frac{1}{2}}$. □

Now, The following corollary shows the relation between the norm of Δ_T and the maximal numerical range.

Corollary 2.2. *Let $\mathcal{H}$ be a Hilbert space and let $T \in B(\mathcal{H})$. Then $\big\|\Delta_T\big\| = 2\|T\|$ if and only if $0 \in W_0(T)$.*

Proof. Let $0 \in W_0(T)$. By Theorem (2.1), we can conclude that $\big\|\Delta_T\big\| \geq 2\|T\|$ and that the upper estimate $\big\|\Delta_T\big\| \leq 2\|T\|$ is trivial, so $\big\|\Delta_T\big\| = 2\|T\|$.

Now, let $\big\|\Delta_T\big\| = 2\|T\|$. Then there exist sequences $\{x_n\} \subseteq \mathcal{H}$ and $\mathrm{A}_n \subseteq B(\mathcal{H})$ such that $\|x_n\| = \|\mathrm{A}_n\| = 1$ and $\big\|\Delta_T(\mathrm{A}_n)(x_n)\big\| \to 2\|T\|$. Since $\lim_n \|\mathrm{A}_n x_n\| = 1$, we have $\lim_n \|Tx_n\| = \|T\|$ and $\lim_n \|T\mathrm{A}_n x_n\| = \|T\|$, $\lim_n \|\mathrm{A}_n^* x_n\| = 1$.

Set $y_n = T\mathrm{A}_n x_n + \mathrm{A}_n^* T x_n$. Using the $\lim_n \big\|\Delta_T(\mathrm{A}_n)(x_n)\big\| = 2\|T\|$ implies that $\lim_n \|y_n\| = 0$. Since T is a bounded operator, there exists a subsequence $\{x_{n_k}\}$ such that $\big\langle Tx_{n_k}, x_{n_k}\big\rangle$ is a convergence sequence. Therefore, without loss of generality, one can assume that $\lim_n \big\langle Tx_n, x_n\big\rangle = \lambda$; hence $\lambda \in W_0(T)$. On the other hand, $\lim_n \big\langle T\mathrm{A}_n x_n, \mathrm{A}_n x_n\big\rangle = -\lambda$ because

$$\begin{aligned}\big\langle T\mathrm{A}_n x_n, \mathrm{A}_n x_n\big\rangle &= \big\langle -\mathrm{A}_n^* T x_n + y_n, \mathrm{A}_n x_n\big\rangle \\ &= -\big\langle \mathrm{A}_n^* T x_n, \mathrm{A}_n x_n\big\rangle + \big\langle y_n, \mathrm{A}_n x_n\big\rangle \\ &= -\big\langle Tx_n, \mathrm{A}_n^2 x_n\big\rangle + \big\langle y_n, \mathrm{A}_n x_n\big\rangle \\ &= -\big\langle Tx_n, x_n\big\rangle + \big\langle Tx_n, x_n - \mathrm{A}_n^2 x_n\big\rangle + \big\langle y_n, \mathrm{A}_n x_n\big\rangle.\end{aligned}$$

The equality $\lim_n \|\mathrm{A}_n x_n\| = 0$ implies $\lim_n \|x_n - \mathrm{A}_n^2 x_n\| = 0$, and so

$$\lim_n \big\langle Tx_n, x_n - \mathrm{A}_n^2 x_n\big\rangle = 0, \lim_n \big\langle y_n, \mathrm{A}_n x_n\big\rangle = 0.$$

Hence, $\lim_n \big\langle T\mathrm{A}_n x_n, \mathrm{A}_n x_n\big\rangle = \lim_n -\big\langle Tx_n, x_n\big\rangle = -\lambda$, and therefore $-\lambda \in W_0(T)$. Since we have $-\lambda, \lambda \in W_0(T)$ and $W_0(T)$ is convex, $0 = \frac{1}{2}\lambda + \frac{1}{2}(-\lambda) \in W_0(T)$. □

In the following theorem, we give a lower bound for $\big\|\Delta_T\big\|$.

Theorem 2.3. *Let $\mathcal{H}$ be a Hilbert space and let $T \in B(\mathcal{H})$. Then*

$$\big\|\Delta_T\big\| \geq 2 \sup_{\lambda \in W_0(T)} |\mathrm{Im}(\lambda)|.$$

Proof. Let $\lambda \in W_0(T)$. Then there exists a sequence $\{x_n\} \subseteq \mathcal{H}$ such that $\|x_n\| = 1, \lim_n \|Tx_n\| = \|T\|$ and $\lim_n \big\langle Tx_n, x_n \big\rangle = \lambda$. Hence,

$$\begin{aligned}\big\|\Delta_T\big(x_n \otimes Tx_n\big)(x_n)\big\| &= \|T(x_n \otimes Tx_n)(x_n) - (x_n \otimes Tx_n)^* T(x_n)\| \\ &= \|T(x_n \otimes Tx_n)(x_n) - (Tx_n \otimes x_n)T(x_n)\| \\ &= \|T(\langle x_n, Tx_n\rangle x_n) - \langle Tx_n, x_n\rangle Tx_n\| \\ &= \|\langle x_n, Tx_n\rangle Tx_n - \langle Tx_n, x_n\rangle Tx_n\| \\ &= |\langle x_n, Tx_n\rangle - \langle Tx_n, x_n\rangle| \|Tx_n\|.\end{aligned}$$

But since

$$\big\|\Delta_T\big(x_n \otimes Tx_n\big)(x_n)\big\| \leq \|T\| \big\|\Delta_T\big\|,$$

we have

$$|\langle x_n, Tx_n\rangle - \langle Tx_n, x_n\rangle| \|Tx_n\| \leq \|T\| \big\|\Delta_T\big\|.$$

Therefore

$$\begin{aligned}\lim_n |\langle x_n, Tx_n\rangle - \langle Tx_n, x_n\rangle| \|Tx_n\| &\leq \lim_n \|T\| \big\|\Delta_T\big\| \\ |\overline{\lambda} - \lambda| \|T\| &\leq \|T\| \big\|\Delta_T\big\| \\ |\overline{\lambda} - \lambda| &\leq \| \big\|\Delta_T\big\| \\ 2|\mathrm{Im}(\lambda)| &\leq \| \big\|\Delta_T\big\|.\end{aligned}$$

Then

$$\big\|\Delta_T\big\| \geq 2 \sup_{\lambda \in W_0(T)} |\mathrm{Im}(\lambda)|.$$

□

Corollary 2.4. *If $i\|T\| \in W_0(T)$, where $i^2 = -1$, then $\big\|\Delta_T\big\| = 2\|T\|$.*

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Department of Mathematics, Faculty of basic sciences, University of Bojnord, P.O. Box 1339, Bojnord, Iran.

E-mail address: a.niazi@ub.ac.ir; niazimotlagh@gmail.com