Khayyam J. Math. 6 (2020), no. 1, 119–128
DOI: 10.22034/kjm.2019.97178

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ON JENSEN'S MULTIPLICATIVE INEQUALITY FOR POSITIVE CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1]

Communicated by H.R. Ebrahimi Vishki

ABSTRACT. We obtain some multiplicative refinements and reverses of Jensen's inequality for positive convex/concave functions of selfadjoint operators in Hilbert spaces. Natural applications for power and exponential functions are provided.

1. INTRODUCTION

The famous *Young inequality* for scalars says that if a, $b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)\,a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. Inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality*.

We recall that *Specht's ratio* is defined by [12]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality is due to Tominaga [13] and provides a multiplicative reverse for Young's inequality

Date: Received: 1 May 2019; Revised: 8 November 2019; Accepted: 27 November 2019.
2010 *Mathematics Subject Classification.* Primary 47A63; Secondary 26D15.
Key words and phrases. Young's Inequality, convex functions, Jensen's inequality, Selfadjoint operator, functions of selfadjoint operators.

$$(1-\nu)\,a+\nu b \le S\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{1.3}$$

where $a,\, b>0$, $\nu\in[0,1]$.

We consider *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ \ h>0. \tag{1.4}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\ge 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative reverse of Young inequality in terms of Kantorovich's constant has been obtained by Liao, Wu, and Zhao [8]

$$(1-\nu)\,a+\nu b \le K^{R}\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{1.5}$$

where $a,\, b>0$, $\nu\in[0,1]$, and $R=\max\{1-\nu,\nu\}$.

The following result that provides a vector operator version for the Jensen inequality is well known, see, for instance, [10] or [11, p. 5]:

Theorem 1.1. *Let A be a selfadjoint operator on the Hilbert space H, and assume that $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars m and M with $m<M$. If f is a convex function on $[m,M]$, then*

$$f(\langle Ax,x\rangle)\le\langle f(A)\,x,x\rangle \tag{1.6}$$

for each $x\in H$ with $\|x\|=1$.

As a special case of Theorem 1.1, we have the *Hölder-McCarthy inequality* [9]: Let A be a selfadjoint positive operator on a Hilbert space H; then

(i) $\langle A^r x,x\rangle\ge\langle Ax,x\rangle^r$ for all $r>1$ and $x\in H$ with $\|x\|=1$;

(ii) $\langle A^r x,x\rangle\le\langle Ax,x\rangle^r$ for all $0<r<1$ and $x\in H$ with $\|x\|=1$;

(iii) if A is invertible, then $\langle A^r x,x\rangle\ge\langle Ax,x\rangle^r$ for all $r<0$ and $x\in H$ with $\|x\|=1$.

In [2] (see also [3, p. 16]) we obtained the following additive reverse of (1.6):

Theorem 1.2. *Let I be an interval and let $f:I\to\mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A)\subset\mathring{I}$, then*

$$(0\le)\,\langle f(A)\,x,x\rangle-f(\langle Ax,x\rangle)\le\langle f'(A)\,Ax,x\rangle-\langle Ax,x\rangle\cdot\langle f'(A)\,x,x\rangle \tag{1.7}$$

for any $x\in H$ with $\|x\|=1$.

This is a generalization of the scalar discrete inequality obtained in [7]. For other reverse inequalities of this type see [3, p. 16].

Motivated by the above results, in this paper, we obtain some multiplicative refinements and reverses of Jensen's inequality for positive convex or concave functions of selfadjoint operators in Hilbert spaces. Natural applications for power and exponential functions are provided.

2. Refinements

We have the following theorem.

Theorem 2.1. *Let $f : I \to [0, \infty)$ be continuous on the interval I, and assume that for some $\nu \in (0, 1)$ the function f^{ν} is convex on I. Then f is convex on I and we have the inequality*

$$f(\langle Ax, x\rangle) \le f^{1-\nu}(\langle Ax, x\rangle)\langle f^{\nu}(A)x, x\rangle \le \langle f(A)x, x\rangle, \tag{2.1}$$

where A is a selfadjoint operator with $\mathrm{Sp}(A) \subset I$ *and* $x \in H$ *with* $\|x\| = 1$.

Proof. Let t, $s \in I$ and let $\lambda \in [0, 1]$. Since f^{ν} is convex on I, then

$$f^{\nu}((1-\lambda)t + \lambda s) \le (1-\lambda) f^{\nu}(t) + \lambda f^{\nu}(s)$$

and by taking the power $\frac{1}{\nu} > 1$ and using the convexity of power function $g(t) = t^r$ with exponent $r = \frac{1}{\nu} > 1$, we have

$$\begin{aligned} f((1-\lambda)t + \lambda s) &\le [(1-\lambda) f^{\nu}(t) + \lambda f^{\nu}(s)]^{\frac{1}{\nu}} \\ &\le (1-\lambda)(f^{\nu}(t))^{\frac{1}{\nu}} + \lambda (f^{\nu}(s))^{\frac{1}{\nu}} = (1-\lambda) f(t) + \lambda f(s), \end{aligned}$$

which proves the convexity of f on I.

Now by Hölder–McCarthy inequality, we have for any $x \in H$ with $\|x\| = 1$

$$\langle f^{\nu}(A)x, x\rangle \le \langle f(A)x, x\rangle^{\nu}$$

and by Young's inequality we have

$$\begin{aligned} f^{1-\nu}(\langle Ax, x\rangle)\langle f^{\nu}(A)x, x\rangle &\le f^{1-\nu}(\langle Ax, x\rangle)\langle f(A)x, x\rangle^{\nu} \\ &\le (1-\nu) f(\langle Ax, x\rangle) + \nu\langle f(A)x, x\rangle \end{aligned} \tag{2.2}$$

for any $x \in H$ with $\|x\| = 1$.

The last inequality follows now by Jensen's inequality for the convex function f.

Applying Jensen's inequality for the convex function f^{ν}, we also have

$$f^{\nu}(\langle Ax, x\rangle) \le \langle f^{\nu}(A)x, x\rangle,$$

which implies that

$$f(\langle Ax, x\rangle) = f^{1-\nu}(\langle Ax, x\rangle) f^{\nu}(\langle Ax, x\rangle) \le f^{1-\nu}(\langle Ax, x\rangle)\langle f^{\nu}(A)x, x\rangle$$

that proves the first inequality in (2.1). □

The case of concave functions is as follows.

Theorem 2.2. *Let $f : I \to [0, \infty)$ be continuous and concave on the interval I. Then for any $\nu \in (0, 1)$, the function f^{ν} is concave on I and we have the inequality*

$$\begin{aligned} \left\langle f^{1/2}(A)x, x\right\rangle^2 &\le \left\langle f^{1-\nu}(A)x, x\right\rangle\langle f^{\nu}(A)x, x\rangle \\ &\le f^{1-\nu}(\langle Ax, x\rangle)\langle f^{\nu}(A)x, x\rangle \le f(\langle Ax, x\rangle), \end{aligned} \tag{2.3}$$

where A is a selfadjoint operator with $\mathrm{Sp}(A) \subset I$ *and* $x \in H$ *with* $\|x\| = 1$.

Proof. From (2.2) and the concavity of f ,we have

$$\begin{aligned} f^{1-\nu}\left(\langle Ax,x\rangle\right)\langle f^{\nu}\left(A\right)x,x\rangle &\leq f^{1-\nu}\left(\langle Ax,x\rangle\right)\langle f\left(A\right)x,x\rangle^{\nu} \\ &\leq \left(1-\nu\right)f\left(\langle Ax,x\rangle\right)+\nu\langle f\left(A\right)x,x\rangle \\ &\leq f\left(\langle Ax,x\rangle\right) \end{aligned}$$

for any $\nu\in(0,1)$ and $x\in H$ with $\|x\|=1$, that proves the last part of (2.3).

Now, let t, $s\in I$ and $\lambda\in[0,1]$. Since f is concave on I, then

$$f\left(\left(1-\lambda\right)t+\lambda s\right)\geq\left(1-\lambda\right)f\left(t\right)+\lambda f\left(s\right).$$

By taking the power $\nu\in(0,1)$ and using the concavity of power function $g(t)=t^r$ for the exponent $r=\nu\in(0,1)$, we have

$$f^{\nu}\left(\left(1-\lambda\right)t+\lambda s\right)\geq\left(\left(1-\lambda\right)f\left(t\right)+\lambda f\left(s\right)\right)^{\nu}\geq\left(1-\lambda\right)f^{\nu}\left(t\right)+\lambda f^{\nu}\left(s\right)$$

that shows that f^{ν} is concave on I.

Applying Jensen's inequality for the concave function $f^{1-\nu}$, we have

$$\left\langle f^{1-\nu}\left(A\right)x,x\right\rangle\leq f^{1-\nu}\left(\langle Ax,x\rangle\right)$$

for $x\in H$ with $\|x\|=1$, that proves the second inequality in (2.3).

Now, by using the Schwarz type inequality for continuous functions of selfadjoint operators

$$\langle g\left(A\right)h\left(A\right)x,x\rangle^{2}\leq\left\langle g^{2}\left(A\right)x,x\right\rangle\left\langle h^{2}\left(A\right)x,x\right\rangle$$

for $x\in H$ with $\|x\|=1$, then by choosing $g=f^{\frac{1-\nu}{2}}$ and $h=f^{\frac{\nu}{2}}$, we get the first inequality in (2.3). □

3. Upper bounds

The following reverse inequalities also hold.

Theorem 3.1. *Let $f:[m,M]\to[0,\infty)$ be a continuous function, and assume that*

$$0<\gamma=\min_{t\in[m,M]}f\left(t\right)<\max_{t\in[m,M]}f\left(t\right)=\Gamma<\infty. \tag{3.1}$$

Then for any A, a selfadjoint operator with

$$m1_{H}\leq A\leq M1_{H}, \tag{3.2}$$

we have the inequality

$$\left(1-\nu\right)f\left(\langle Ax,x\rangle\right)+\nu\langle f\left(A\right)x,x\rangle\leq S\left(\frac{\Gamma}{\gamma}\right)\langle f^{\nu}\left(A\right)x,x\rangle f^{1-\nu}\left(\langle Ax,x\rangle\right) \tag{3.3}$$

for any $x\in H$ with $\|x\|=1$, where $\nu\in[0,1]$.

Moreover, if f is convex on $[m,M]$, then

$$f^{\nu}\left(\langle Ax,x\rangle\right)\leq S\left(\frac{\Gamma}{\gamma}\right)\langle f^{\nu}\left(A\right)x,x\rangle \tag{3.4}$$

while, if f is concave on $[m,M]$, then

$$\langle f\left(A\right)x,x\rangle\leq S\left(\frac{\Gamma}{\gamma}\right)\langle f^{\nu}\left(A\right)x,x\rangle f^{1-\nu}\left(\langle Ax,x\rangle\right) \tag{3.5}$$

for any $x \in H$ *with* $\|x\| = 1$.

Proof. By Tominaga's inequality, we have

$$(1-\nu) + \nu s \le S(s) s^{\nu}$$

for any $s > 0$ and $\nu \in [0,1]$.

If $s \in \left[\frac{\gamma}{\Gamma}, \frac{\Gamma}{\gamma}\right]$, then

$$(1-\nu) + \nu s \le S(s) s^{\nu} \le s^{\nu} \max_{s \in \left[\frac{\gamma}{\Gamma}, \frac{\Gamma}{\gamma}\right]} S(s) = S\left(\frac{\Gamma}{\gamma}\right) s^{\nu}, \tag{3.6}$$

and since for any $t \in [m, M]$ and $x \in H$ with $\|x\| = 1$, we have

$$\frac{f(t)}{f(\langle Ax, x\rangle)} \in \left[\frac{\gamma}{\Gamma}, \frac{\Gamma}{\gamma}\right].$$

Hence by (3.6), we get

$$(1-\nu) f(\langle Ax, x\rangle) + \nu f(t) \le S\left(\frac{\Gamma}{\gamma}\right) f^{\nu}(t) f^{1-\nu}(\langle Ax, x\rangle) \tag{3.7}$$

for any $t \in [m, M]$ and $x \in H$ with $\|x\| = 1$.

If we use the functional calculus for the operator A, we have by (3.7) that

$$(1-\nu) f(\langle Ax, x\rangle) + \nu f(A) \le S\left(\frac{\Gamma}{\gamma}\right) f^{\nu}(A) f^{1-\nu}(\langle Ax, x\rangle) \tag{3.8}$$

for any $x \in H$ with $\|x\| = 1$.

If we take the inner product over $y \in H$ with $\|y\| = 1$ in (3.8), then we get

$$(1-\nu) f(\langle Ax, x\rangle) + \nu \langle f(A) y, y\rangle \le S\left(\frac{\Gamma}{\gamma}\right) \langle f^{\nu}(A) y, y\rangle f^{1-\nu}(\langle Ax, x\rangle)$$

that for $y = x$ we get the desired inequality.

If f is convex, then

$$f(\langle Ax, x\rangle) \le \langle f(A) x, x\rangle$$

for any $x \in H$ with $\|x\| = 1$, then by (3.3) we get

$$f(\langle Ax, x\rangle) \le S\left(\frac{\Gamma}{\gamma}\right) \langle f^{\nu}(A) x, x\rangle f^{1-\nu}(\langle Ax, x\rangle)$$

for any $x \in H$ with $\|x\| = 1$, which is equivalent to (3.4).

If f is concave, then

$$\langle f(A) x, x\rangle \le f(\langle Ax, x\rangle)$$

and by (3.3), we get (3.5). □

Remark 3.2. If for some $\nu \in (0,1)$ the function f^{ν} is convex on $[m, M]$, then according to Theorem 2.1, f is convex on I and inequality (3.4) is trivially satisfied since $S\left(\frac{\Gamma}{\gamma}\right) \ge 1$.

If f is convex on I and for some $\nu \in (0,1)$ the function f^{ν} is concave on $[m,M]$, then from (3.4), we have the meaningful inequality

$$1 \leq \frac{f^{\nu}(\langle Ax,x\rangle)}{\langle f^{\nu}(A)x,x\rangle} \leq S\left(\frac{\Gamma}{\gamma}\right) \tag{3.9}$$

for any $x \in H$ with $\|x\| = 1$.

Inequality (3.5) can be written in equivalent form as

$$\frac{\langle f(A)x,x\rangle}{f(\langle Ax,x\rangle)} \leq S\left(\frac{\Gamma}{\gamma}\right)\frac{\langle f^{\nu}(A)x,x\rangle}{f^{\nu}(\langle Ax,x\rangle)} \tag{3.10}$$

for any $x \in H$ with $\|x\| = 1$.

We also have the following result.

Theorem 3.3. *With the assumptions of Theorem 3.1, we have the inequality*

$$(1-\nu) f(\langle Ax,x\rangle) + \nu \langle f(A)x,x\rangle \leq K^{R}\left(\frac{\Gamma}{\gamma}\right)\langle f^{\nu}(A)x,x\rangle f^{1-\nu}(\langle Ax,x\rangle) \tag{3.11}$$

for any $x \in H$ with $\|x\| = 1$, where $\nu \in [0,1]$ and $R = \max\{\nu, 1-\nu\}$.

Moreover, if f is convex on $[m,M]$, then

$$\frac{f^{\nu}(\langle Ax,x\rangle)}{\langle f^{\nu}(A)x,x\rangle} \leq K^{R}\left(\frac{\Gamma}{\gamma}\right) \tag{3.12}$$

while, if f is concave on $[m,M]$, then

$$\frac{\langle f(A)x,x\rangle}{f(\langle Ax,x\rangle)} \leq K^{R}\left(\frac{\Gamma}{\gamma}\right)\frac{\langle f^{\nu}(A)x,x\rangle}{f^{\nu}(\langle Ax,x\rangle)} \tag{3.13}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. By inequality (1.5), we have

$$(1-\nu) + \nu s \leq K^{R}(s)\, s^{\nu}$$

for any $s > 0$ and $\nu \in [0,1]$.

If $s \in \left[\frac{\gamma}{\Gamma}, \frac{\Gamma}{\gamma}\right]$, then

$$(1-\nu) + \nu s \leq S(s)\, s^{\nu} \leq s^{\nu} \max_{s \in \left[\frac{\gamma}{\Gamma}, \frac{\Gamma}{\gamma}\right]} K^{R}(s) = K^{R}\left(\frac{\Gamma}{\gamma}\right) s^{\nu} \tag{3.14}$$

and since, for any $t \in [m,M]$ and $x \in H$ with $\|x\| = 1$, we have

$$\frac{f(t)}{f(\langle Ax,x\rangle)} \in \left[\frac{\gamma}{\Gamma}, \frac{\Gamma}{\gamma}\right].$$

Hence by (3.14), we get

$$(1-\nu) f(\langle Ax,x\rangle) + \nu f(t) \leq K^{R}\left(\frac{\Gamma}{\gamma}\right) f^{\nu}(t)\, f^{1-\nu}(\langle Ax,x\rangle)$$

for any $t \in [m,M]$ and $x \in H$ with $\|x\| = 1$.

Now the proof goes along the lines of the proof in Theorem 3.1 and we omit the details. □

In the recent paper [6], the author obtained the following reverse of Young's inequality:

$$\frac{(1-\nu)\,a+\nu b}{a^{1-\nu}b^{\nu}} \le \exp\left[4\nu\,(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{3.15}$$

for any $a,\ b>0$ and $\nu\in[0,1]$, where K is Kantorovich's constant defined in (1.4).

If $a,\ b\in[m,M]$, then by the properties of K, we have the upper bound:

$$\frac{(1-\nu)\,a+\nu b}{a^{1-\nu}b^{\nu}} \le \exp\left[4\nu\,(1-\nu)\left(K\left(\frac{M}{m}\right)-1\right)\right] \tag{3.16}$$

for any and $\nu\in[0,1]$.

Using a similar argument as in Theorem 2.1 and inequality (3.16) we can also state the following result.

Theorem 3.4. *With the assumptions of Theorem 3.1, we have the inequality*

$$\begin{aligned}&(1-\nu)\,f\left(\langle Ax,x\rangle\right)+\nu\left\langle f\left(A\right)x,x\right\rangle \qquad (3.17)\\ &\le \exp\left[4\nu\,(1-\nu)\left(K\left(\frac{\Gamma}{\gamma}\right)-1\right)\right]\left\langle f^{\nu}\left(A\right)x,x\right\rangle f^{1-\nu}\left(\langle Ax,x\rangle\right)\end{aligned}$$

for any $x\in H$ with $\|x\|=1$, where $\nu\in[0,1]$.

Moreover, if f is convex on $[m,M]$, then

$$\frac{f^{\nu}\left(\langle Ax,x\rangle\right)}{\left\langle f^{\nu}\left(A\right)x,x\right\rangle} \le \exp\left[4\nu\,(1-\nu)\left(K\left(\frac{\Gamma}{\gamma}\right)-1\right)\right] \tag{3.18}$$

while, if f is concave on $[m,M]$, then

$$\frac{\left\langle f\left(A\right)x,x\right\rangle}{f\left(\langle Ax,x\rangle\right)} \le \exp\left[4\nu\,(1-\nu)\left(K\left(\frac{\Gamma}{\gamma}\right)-1\right)\right]\frac{\left\langle f^{\nu}\left(A\right)x,x\right\rangle}{f^{\nu}\left(\langle Ax,x\rangle\right)} \tag{3.19}$$

for any $x\in H$ with $\|x\|=1$.

The proof follows as above, however the details are not presented here.

The interested reader may establish similar results by employing the following multiplicative reverses of Young inequalities.

Lemma 3.5 ([4])**.** *If $a,\ b\in[\gamma,\Gamma]\subset(0,\infty)$ and $\nu\in[0,1]$, then we have*

$$\frac{(1-\nu)\,a+\nu b}{a^{1-\nu}b^{\nu}} \le \max\left\{\kappa_{\gamma,\Gamma}\left(\nu\right),\kappa_{\gamma,\Gamma}\left(1-\nu\right)\right\}, \tag{3.20}$$

where

$$\kappa_{\gamma,\Gamma}\left(\nu\right) := \frac{(1-\nu)\,\gamma+\nu\Gamma}{\gamma^{1-\nu}\Gamma^{\nu}}. \tag{3.21}$$

Also the following result holds.

Lemma 3.6 ([5])**.** *If $a,\ b\in[\gamma,\Gamma]\subset(0,\infty)$ and $\nu\in[0,1]$, then we have*

$$\frac{(1-\nu)\,a+\nu b}{a^{1-\nu}b^{\nu}} \le \exp\left[\frac{1}{2}\nu\,(1-\nu)\left(\frac{\Gamma}{\gamma}-1\right)^{2}\right]. \tag{3.22}$$

Inequality (3.22) can also be obtained from inequality (2.9) from the paper [1].

4. Some examples

Now, let $\nu \in (0,1)$ and let $r \geq \frac{1}{\nu} > 1$. Consider the function $f : [0,\infty) \to [0,\infty)$ defined by $f(t) = t^r$. Then the function f^ν is convex on $[0,\infty)$, and if A is a positive operator on the Hilbert space H, then by inequality (2.1) we have the following refinement of Hölder–McCarthy inequality

$$\langle Ax,x\rangle^r \leq \langle Ax,x\rangle^{(1-\nu)r} \langle A^{\nu r}x,x\rangle \leq \langle A^r x,x\rangle, \tag{4.1}$$

for any $x \in H$ with $\|x\| = 1$.

If $r \geq 2$, then by (4.1), we have

$$\langle Ax,x\rangle^r \leq \langle Ax,x\rangle^{r/2} \left\langle A^{r/2}x,x\right\rangle \leq \langle A^r x,x\rangle, \tag{4.2}$$

for any $x \in H$ with $\|x\| = 1$.

Let $\nu \in (0,1)$ and let $\alpha \in \mathbb{R}$. Consider the function $f(t) = \exp(\alpha t)$, $t \in \mathbb{R}$. Then the function $f^\nu(t) = \exp(\nu\alpha t)$ is convex on $\mathbb{R}$, and for any selfadjoint operator A on H we have

$$\exp(\alpha\langle Ax,x\rangle) \leq \exp((1-\nu)\alpha\langle Ax,x\rangle)\langle \exp(\nu\alpha A)x,x\rangle \leq \langle \exp(\alpha A)x,x\rangle \tag{4.3}$$

for any $x \in H$ with $\|x\| = 1$.

For $\alpha = 1$, we get

$$\exp(\langle Ax,x\rangle) \leq \exp((1-\nu)\langle Ax,x\rangle)\langle \exp(\nu A)x,x\rangle \leq \langle \exp(A)x,x\rangle \tag{4.4}$$

for any $x \in H$ with $\|x\| = 1$ and $\nu \in (0,1)$.

Consider $q \in (0,1)$; then the function $f(t) = t^q$ is concave on $[0,\infty)$, and if A is a positive operator on the Hilbert space H, then by inequality (2.3) we have

$$\begin{aligned}\left\langle A^{q/2}x,x\right\rangle^2 &\leq \left\langle A^{(1-\nu)q}x,x\right\rangle \langle A^{\nu q}x,x\rangle \leq \langle Ax,x\rangle^{(1-\nu)q}\langle A^{\nu q}x,x\rangle \\ &\leq \langle Ax,x\rangle^q\end{aligned} \tag{4.5}$$

for any $x \in H$ with $\|x\| = 1$.

If we take in (4.5) $q \to 1$, $q < 1$, we get

$$\begin{aligned}\left\langle A^{1/2}x,x\right\rangle^2 &\leq \left\langle A^{1-\nu}x,x\right\rangle \langle A^{\nu}x,x\rangle \leq \langle Ax,x\rangle^{1-\nu}\langle A^{\nu}x,x\rangle \\ &\leq \langle Ax,x\rangle\end{aligned} \tag{4.6}$$

for any $x \in H$ with $\|x\| = 1$.

The function $f(t) = \ln(t+1)$ is positive concave on $[0,\infty)$, and if A is a positive operator on the Hilbert space H, then by inequality (2.3), we have

$$\begin{aligned}\left\langle \sqrt{\ln(A+1_H)}x,x\right\rangle^2 &\leq \left\langle \ln^{1-\nu}(A+1_H)x,x\right\rangle \langle \ln^\nu(A+1_H)x,x\rangle \\ &\leq \ln^{1-\nu}(\langle Ax,x\rangle+1)\langle \ln^\nu(A+1_H)x,x\rangle \\ &\leq \ln(\langle Ax,x\rangle+1)\end{aligned} \tag{4.7}$$

for any $x \in H$ with $\|x\| = 1$ and $\nu \in (0,1)$.

Consider the function $f : \mathbb{R} \to (0,\infty)$, $f(t) = \exp\left(-\frac{1}{2}t^2\right)$. Then

$$f'(t) = -t\exp\left(-\frac{1}{2}t^2\right) \text{ and } f''(t) = \exp\left(-\frac{1}{2}t^2\right)\left(t^2-1\right),$$

which shows that the function is concave on $[-1,1]$.

Now, if A is selfadjoint and $-1_H \le A \le 1_H$, then by (2.3) we have that

$$\begin{aligned}
&\left\langle \exp\left(-\frac{1}{4}A^2\right)x,x\right\rangle^2 \qquad (4.8)\\
&\le \left\langle \exp\left(-\frac{1}{2}(1-\nu)A^2\right)x,x\right\rangle \left\langle \exp\left(-\frac{1}{2}\nu A^2\right)x,x\right\rangle\\
&\le \exp\left(-\frac{1}{2}(1-\nu)\langle Ax,x\rangle^2\right)\left\langle \exp\left(-\frac{1}{2}\nu A^2\right)x,x\right\rangle\\
&\le \exp\left(-\frac{1}{2}\langle Ax,x\rangle^2\right)
\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

Now, let $\nu\in(0,1)$ and let $\frac{1}{\nu} > r \ge 1$. Consider the function $f:[0,\infty)\to[0,\infty)$ defined by $f(t)=t^r$. Then f is convex while the function f^ν is concave on $[0,\infty)$, and if A is a positive operator on the Hilbert space H satisfying condition (3.2), then by taking $\gamma=m^r$ and $\Gamma=M^r$ in (3.9) we get

$$1 \le \frac{\langle Ax,x\rangle^{r\nu}}{\langle A^{r\nu}x,x\rangle} \le S\left(\left(\frac{M}{m}\right)^r\right) \qquad (4.9)$$

for any $x\in H$ with $\|x\|=1$, where $\nu\in(0,1)$ and $\frac{1}{\nu} > r \ge 1$.

If we take $r=1$ in (4.9), then we get

$$1 \le \frac{\langle Ax,x\rangle^{\nu}}{\langle A^{\nu}x,x\rangle} \le S\left(\frac{M}{m}\right) \qquad (4.10)$$

for any $x\in H$ with $\|x\|=1$ and $\nu\in(0,1)$.

Consider $q\in(0,1)$, then the function $f(t)=t^q$ is concave on $[0,\infty)$ and if A is a positive operator on the Hilbert space H satisfying condition (3.2), then by taking $\gamma=m^q$ and $\Gamma=M^q$ in (3.5), we get

$$\langle A^q x,x\rangle \le S\left(\left(\frac{M}{m}\right)^q\right)\langle A^{\nu q}x,x\rangle\langle Ax,x\rangle^{(1-\nu)q} \qquad (4.11)$$

for any $x\in H$ with $\|x\|=1$ and $\nu\in(0,1)$.

This inequality can be written in the equivalent form as

$$\frac{\langle A^q x,x\rangle}{\langle Ax,x\rangle^q} \le S\left(\left(\frac{M}{m}\right)^q\right)\frac{\langle A^{\nu q}x,x\rangle}{\langle Ax,x\rangle^{\nu q}} \qquad (4.12)$$

for any $x\in H$ with $\|x\|=1$ and $\nu\in(0,1)$.

Consider the function $f(t)=\exp\left(-\frac{1}{2}t^2\right)$ on the concavity interval $[-1,1]$. We have that

$$\max_{t\in[-1,1]} f(t) = f(0) = 1 \text{ and } \min_{t\in[-1,1]} f(t) = f(\pm 1) = \exp\left(-\frac{1}{2}\right).$$

If A is selfadjoint and $-1_H \le A \le 1_H$, then by inequality (3.10), we have

$$\frac{\left\langle \exp\left(-\frac{1}{2}A^2\right)x,x\right\rangle}{\exp\left(-\frac{1}{2}\langle Ax,x\rangle^2\right)} \le S\left(\exp\left(\frac{1}{2}\right)\right)\frac{\left\langle \exp\left(-\frac{1}{2}\nu A^2\right)x,x\right\rangle}{\exp\left(-\frac{1}{2}\nu\langle Ax,x\rangle^2\right)} \qquad (4.13)$$

for any $x \in H$ with $\|x\| = 1$.

If A is a positive operator on the Hilbert space H satisfying the condition (3.2), then by (3.12) we get

$$1 \leq \frac{\langle Ax, x\rangle^{r\nu}}{\langle A^{r\nu}x, x\rangle} \leq K^R\left(\left(\frac{M}{m}\right)^r\right) \tag{4.14}$$

for any $x \in H$ with $\|x\| = 1$, where $\nu \in (0, 1)$ with $\frac{1}{\nu} > r \geq 1$ and $R = \max\{\nu, 1 - \nu\}$.

From (3.13), we have

$$\frac{\langle A^q x, x\rangle}{\langle Ax, x\rangle^q} \leq K^R\left(\left(\frac{M}{m}\right)^q\right)\frac{\langle A^{\nu q}x, x\rangle}{\langle Ax, x\rangle^{\nu q}}, \tag{4.15}$$

where $q \in (0, 1)$, $\nu \in (0, 1)$ and $R = \max\{\nu, 1 - \nu\}$.

If A is selfadjoint and $-1_H \leq A \leq 1_H$, then by inequality (3.13), we have

$$\frac{\left\langle \exp\left(-\frac{1}{2}A^2\right)x, x\right\rangle}{\exp\left(-\frac{1}{2}\langle Ax, x\rangle^2\right)} \leq K^R\left(\exp\left(\frac{1}{2}\right)\right)\frac{\left\langle \exp\left(-\frac{1}{2}\nu A^2\right)x, x\right\rangle}{\exp\left(-\frac{1}{2}\nu\langle Ax, x\rangle^2\right)} \tag{4.16}$$

for any $x \in H$ with $\|x\| = 1$, where $\nu \in (0, 1)$ and $R = \max\{\nu, 1 - \nu\}$.

References

1. H. Alzer, C.M. da Fonseca and A. Kovačec, *Young-type inequalities and their matrix analogues*, Linear Multilinear Algebra, **63** (2015), no. 3, 622–635.
2. S.S. Dragomir, *Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces*, J. Inequal. Appl. **2010** (2010), Art. ID 496821, 15 pp.
3. S.S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type*, Springer, New York, 2012.
4. S.S. Dragomir, *Two new reverses of Young's inequality*, Preprint RGMIA Res. Rep. Coll. **18** (2015), Art. 158. [`http://rgmia.org/papers/v18/v18a158.pdf`].
5. S.S. Dragomir, *A note on new refinements and reverses of Young's inequality*, Transylv. J. Math. Mech. **8** (2016), no. 1, 45–49.
6. S.S. Dragomir, *A note on Young's inequality*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **111** (2017), no. 2, 349–354.
7. S.S. Dragomir and N.M. Ionescu, *Some converse of Jensen's inequality and applications,* Rev. Anal. Numér. Théor. Approx. **23** (1994), no. 1, 71–78.
8. W. Liao, J. Wu and J. Zhao, *New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant,* Taiwanese J. Math. **19** (2015), no. 2, 467–479.
9. C.A. McCarthy, c_p, Israel J. Math. **5** (1967) 249–271.
10. B. Mond and J. Pečarić, *Convex inequalities in Hilbert space*, Houston J. Math. **19** (1993) 405–420.
11. J. Pečarić, T. Furuta, J. Mićić Hot and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Monographs in Inequalities 1, Element, Zagreb, 2005.
12. W. Specht, *Zer theorie der elementaren Mittel*, Math. Z. **74** (1960) 91–98.
13. M. Tominaga, *Specht's ratio in the Young inequality*, Sci. Math. Jpn. **55** (2002) 583–588.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`