Khayyam J. Math. 6 (2020), no. 2, 155–173
DOI: 10.22034/kjm.2020.109813

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ADMISSIBLE INERTIAL MANIFOLDS FOR SECOND ORDER IN TIME EVOLUTION EQUATIONS

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Communicated by F.H. Ghane

Abstract. We prove the existence of admissible inertial manifolds for the second order in time evolution equations of the form

$$\ddot{x} + 2\varepsilon\dot{x} + Ax = f(t, x)$$

when A is positive definite and self-adjoint with a discrete spectrum and the nonlinear term f satisfies the φ-Lipschitz condition, that is, $\|f(t,x)-f(t,y)\| \leqslant \varphi(t)\left\|A^{\beta}(x-y)\right\|$ for φ belonging to one of the admissible Banach function spaces containing wide classes of function spaces like L_p-spaces, the Lorentz spaces $L_{p,q}$, and many other function spaces occurring in interpolation theory.

1. Introduction and preliminaries

To deal with the asymptotic behavior problem of nonlinear differential equations, one of the interests is to find the conditions for the existence of inertial manifolds. These Lipschitz finite-dimensional manifolds, which were introduced by Foias, Sell, and Temam [8], attract all solutions of the corresponding evolution equations at an exponential rate. Therefore, if such a manifold exists, it allows us to apply the reduction principles to consider the asymptotic behavior of the solutions to the (partial differential) evolution equation by determining the structures of its induced solutions on that inertial manifold, which turn out to be solutions to some induced ordinary differential equations.

The existence of inertial manifolds has been studied by many authors and has been proved for several important classes of evolution equations such as dissipative partial differential equations (see [6]), reaction-diffusion equations (see [10]),

Date: Received: 13 May 2019; Revised: 9 November 2019; Accepted: 28 November 2019.
2010 *Mathematics Subject Classification.* Primary 35B42, 37L25; Secondary 35L90.
Key words and phrases. admissible inertial manifolds, second order in time evolution equations, admissible function spaces, Lyapunov–Perron method.

some specific class of semi-linear parabolic equations (see [4]), and for second order in time equations [4, Section 7]. Lately, the notion of inertial manifolds has been transformed and extended to some other general classes of partial differential equations (PDE) like non-autonomous PDE (see, e.g., [9]), retarded PDE (see, e.g., [2]), and stochastic PDE (see, e.g., [1, 3, 5]). Recently, a more general concept of inertial manifolds has been introduced in [12], namely, the *admissible inertial manifolds*, which are constituted by trajectories of solutions belonging to admissible function spaces (such as L_p spaces, Lorentz spaces $L_{p,q}$, etc). Then, the existence of such an admissible inertial manifold for a class of delay equations with suitable conditions has been proved in our work [13]. In this paper, by a suitable method and assumptions, we extend the obtained results in [12, 13] to second order in time evolution equation of the form

$$\begin{cases} \ddot{x}(t) + 2\varepsilon\dot{x}(t) + Ax(t) = f(t, x(t)), & t > s,\ s \in \mathbb{R},\ \varepsilon > 0, \\ x(s) = x_{s,0}, & s \in \mathbb{R}, \\ \dot{x}(s) = x_{s,1}, \end{cases} \tag{1.1}$$

where A is in general an unbounded linear operator on a separable Hilbert space X, which satisfies Assumption 1.1; $f\colon \mathbb{R} \times X_\beta \to X$ is a continuous nonlinear operator, where $X_\beta := \mathscr{D}(A^\beta)$ is the domain of the fractional power A^β for $0 \leqslant \beta \leqslant 1/2$, in particular $\mathscr{D}(A^0) = X$.

Assumption 1.1. A is a positive definite, self-adjoint operator with a discrete spectrum, say

$$0 < \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_k \leqslant \cdots \quad \text{each with finite multiplicity, and} \quad \lim_{k\to\infty} \mu_k = \infty,$$

and assume that $\{e_k\}_{k=1}^{\infty}$ is the orthonormal basis in X consisting of the corresponding eigenfunctions of the operator A (i.e., $Ae_k = \mu_k e_k$).

Now, let $\mathscr{H} = \mathscr{D}(A^{1/2}) \times X$. It is clear that $\mathscr{H}$ is a separable Hilbert space with the inner product

$$(U, V) = (Ax^0, y^0) + (x^1, y^1),$$

where $U = (x^0, x^1)$ and $V = (y^0, y^1)$ are elements of $\mathscr{H}$. In $\mathscr{H}$, problem (1.1) can be rewritten as a system of first order

$$\begin{cases} \dfrac{dU(t)}{dt} + \mathcal{A}U(t) = \mathcal{F}(t, U(t)), & t > s, \\ U(s) = U_s, \end{cases} \tag{1.2}$$

where

$$U(t) := (x(t), \dot{x}(t)) \quad \text{and} \quad U_s = (x_{s,0}, x_{s,1}).$$

Here the linear operator $\mathcal{A}$ and the mapping $\mathcal{F}$ are defined by

$$\mathcal{A}U = (-x^1, Ax^0 + 2\varepsilon x^1) \quad \text{on the domain} \quad \mathscr{D}(\mathcal{A}) = \mathscr{D}(A) \times \mathscr{D}(A^{1/2}),$$
$$\mathcal{F}(t, U(t)) = (0, f(t, x^0(t))) \quad \text{for} \quad U = (x^0, x^1).$$

It is easy to verify that the eigenvalues and eigenvectors of the operator $\mathcal{A}$ have the form

$$\lambda_N^{\pm} = \varepsilon \pm \sqrt{\varepsilon^2 - \mu_N}, \quad g_N^{\pm} = (e_N, -\lambda_N^{\pm} e_N) \qquad \text{for all } N = 1, 2, \dots,$$

where μ_N and e_N are eigenvalues and eigenvectors of the operator A.

Then, there are three main difficulties when working with abstract problem (1.2): Firstly, since the nonlinearity f is φ-Lipschitz, then $\mathcal{F}$ is not uniform Lipschitz continuous, so the theorem of the existence and uniqueness of solution to (1.2) is not available; Secondly, the semigroup e^{-tA} is defined on X while the surfaces of the inertial manifolds belong to $\mathscr{H}$, that means that the standard perturbation arguments for evolutionary processes under graph transformations cannot be applied directly; Thirdly, the differential operators $\mathcal{A}$ does not satisfy [12, Standing Hypothesis 2.1]. This means that the obtained result in [12] cannot be translated to our case.

To overcome these difficulties, we reformulate the definition of an inertial manifold such that it contains the existence and uniqueness theorem as its property. Next, we build the structure of bounded solutions (in negative half-line) to (1.2) using Lyapunov–Perron's equation such that we can use the dichotomy estimates to prove the existence and uniqueness of solutions to (1.2). So, by using dichotomy estimates on $e^{-t\mathcal{A}}$, admissibility of function spaces, and suitable form of Lyapunov–Perron's equation, we present the structure of an admissible inertial manifold. Consequently, we obtain the existence of an admissible inertial manifold for which it is mentioned above and an example on the damped wave equation is given to illustrate our result.

Now, let the condition $\varepsilon^2 > \mu_{N+1}$ hold for some integer N. We consider the decomposition of the space $\mathscr{H}$ into the orthogonal sum

$$\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2,$$

where

$$\mathscr{H}_1 = \operatorname{span}\{(e_k, 0), (0, e_k) : k = 1, \dots, N\},$$
$$\mathscr{H}_2 = \overline{\operatorname{span}\{(e_k, 0), (0, e_k) : k \geqslant N+1\}} \quad (\text{here } \overline{C} \text{ denotes the closure of the set } C).$$

We will use the following inner products in $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively:

$$\langle U, V\rangle_1 = \varepsilon^2 (x^0, y^0) - (Ax^0, y^0) + (\varepsilon x^0 + x^1, \varepsilon y^0 + y^1),$$
$$\langle U, V\rangle_2 = (Ax^0, y^0) - (\varepsilon^2 - 2\mu_{N+1})(x^0, y^0) + (\varepsilon x^0 + x^1, \varepsilon y^0 + y^1).$$

Here $U = (x^0, x^1)$ and $V = (y^0, y^1)$ are elements from the corresponding subspace $\mathscr{H}_i$. Using (1.3), we define a new inner product and norm in $\mathscr{H}$ by

$$\langle U, V\rangle = \langle U_1, V_1\rangle_1 + \langle U_2, V_2\rangle_2, \quad |U| = \langle U, U\rangle^{1/2}.$$

where $U = U_1 + U_2$ and $V = V_1 + V_2$ are decompositions of the elements U and V into the orthogonal terms V_i, $U_i \in \mathscr{H}_i$, $i = 1$, 2.

Lemma 1.2 ([4, Lemma 7.1])**.** *The estimates*

$$|U|_1 \geqslant \frac{1}{\mu_N^{\beta}} \sqrt{\varepsilon^2 - \mu_N} \|A^{\beta} x^0\|, \quad U = (x^0, x^1) \in \mathscr{H}_1,$$

$$|U|_2 \geqslant \frac{1}{\mu_{N+1}^{\beta}} \delta_{N,\varepsilon} \|A^{\beta} x^0\|, \quad U = (x^0, x^1) \in \mathscr{H}_2,$$

hold for $0 \leqslant \beta \leqslant 1/2$*. Here*

$$\delta_{N,\varepsilon} := \sqrt{\mu_{N+1}} \min \left\{ 1, \sqrt{\frac{\varepsilon^2 - \mu_{N+1}}{\mu_{N+1}}} \right\}. \tag{1.3}$$

In particular, this lemma implies the estimate

$$\|A^{\beta} x^0\| \leqslant \mu_{N+1}^{\beta} \delta_{N,\varepsilon}^{-1} |U|, \tag{1.4}$$

for any $U = (x^0, x^1) \in \mathscr{H}$, where $0 \leqslant \beta \leqslant 1/2$ and $\delta_{N,\varepsilon}$ has the form (1.3).

We now fix an integer N and consider the subspaces

$$\mathscr{H}_1^{\pm} := \operatorname{span}\{g_k^{\pm} : k \leqslant N\},$$

which are orthogonal for the Hermitian inner products $\langle \cdot, \cdot \rangle$, so $\mathscr{H}_1 = \mathscr{H}_1^+ \oplus \mathscr{H}_1^-$. This means that $\mathscr{H} = \mathscr{H}_1^+ \oplus \mathscr{H}_1^- \oplus \mathscr{H}_2$ and $\mathscr{H}_1^+$, $\mathscr{H}_1^-$, $\mathscr{H}_2$ are closed subspaces of $\mathscr{H}$. Then, we denote by $P_{\mathscr{H}_1^-}$, $P_{\mathscr{H}_1^+}$, $P_{\mathscr{H}_2}$ the orthoprojectors onto the subspace $\mathscr{H}_1^-$, $\mathscr{H}_1^+$, $\mathscr{H}_2$ in $\mathscr{H}$ corresponding.

Lemma 1.3 ([4, Lemma 7.2])**.** *We have*

$$\left|e^{-t\mathcal{A}} P_{\mathscr{H}_2}\right| \leqslant e^{-\lambda_{N+1}^- t}, \quad \left|e^{t\mathcal{A}} P_{\mathscr{H}_1^-}\right| \leqslant e^{-\lambda_N^- t}, \quad \left|e^{-t\mathcal{A}} P_{\mathscr{H}_1^+}\right| \leqslant e^{-\lambda_N^+ t} \quad \text{for all } t \geqslant 0,$$

here $|\cdot|$ *is the operator norm induced by the corresponding vector norm.*

Set $P \equiv P_{\mathscr{H}_1^-}$ and $Q := I - P = P_{\mathscr{H}_1^+} + P_{\mathscr{H}_2}$. Lemma 1.3 implies the dichotomy estimates

$$\left|e^{t\mathcal{A}} P\right| \leqslant e^{\lambda_N^- |t|}, \quad t \in \mathbb{R},$$
$$\left|e^{-t\mathcal{A}} (I - P)\right| \leqslant e^{-\lambda_{N+1}^- t}, \quad t > 0.$$

We can define the Green function as follows:

$$\mathcal{G}(t,\tau) = \begin{cases} e^{-(t-\tau)\mathcal{A}}[I - P] & \text{for all } t > \tau, \\ -e^{-(t-\tau)\mathcal{A}} P & \text{for all } t \leqslant \tau. \end{cases} \tag{1.5}$$

Then $\mathcal{G}(t,\tau)$ maps $\mathscr{H}$ into $\mathscr{H}$. Moreover, by dichotomy estimates, we have

$$e^{\gamma(t-\tau)} |\mathcal{G}(t,\tau)| \leqslant e^{-\alpha|t-\tau|} \quad \text{for all } t, \tau \in \mathbb{R}, \tag{1.6}$$

where

$$\alpha := \frac{\lambda_{N+1}^- - \lambda_N^-}{2} \quad \text{and} \quad \gamma := \frac{\lambda_{N+1}^- + \lambda_N^-}{2}.$$

Now, let $\mathbb{I} \subseteq \mathbb{R}$. We recall some notions on *Banach function spaces* as follows (see [12, 13]).

Definition 1.4. A vector space $E_{\mathbb{I}}$ of real-valued Borel-measurable functions on $\mathbb{I}$ (modulo λ-null-functions) is a *Banach function space* (over $(\mathbb{I}, \mathscr{B}, \lambda)$) if

(1) $E_{\mathbb{I}}$ is Banach lattice with respect to a norm $\|\cdot\|_{E_{\mathbb{I}}}$, that is, $(E_{\mathbb{I}}, \|\cdot\|_{E_{\mathbb{I}}})$ is a Banach space, and if $\varphi \in E_{\mathbb{I}}$, ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leqslant |\varphi(\cdot)|$, λ-a.e., then $\psi \in E_{\mathbb{I}}$ and $\|\psi\|_{E_{\mathbb{I}}} \leqslant \|\varphi\|_{E_{\mathbb{I}}}$,

(2) the characteristic functions χ_A belong to $E_{\mathbb{I}}$ for all $A \in \mathscr{B}$ of finite measure, and

$$\sup_{t\in\mathbb{I}} \left\|\chi_{[t,t+1]}\right\|_{E_{\mathbb{I}}} < \infty \quad ; \quad \inf_{t\in\mathbb{I}} \left\|\chi_{[t,t+1]}\right\|_{E_{\mathbb{I}}} > 0,$$

(3) $E_{\mathbb{I}} \hookrightarrow L_{1,loc}(\mathbb{I})$, that is, for each compact interval $\mathbb{J} \subset \mathbb{I}$, there exists a number $\beta_{\mathbb{J}} \geqslant 0$ such that $\int_{\mathbb{J}} |f(t)|\, dt \leqslant \beta_{\mathbb{J}} \|f\|_{E_{\mathbb{I}}}$ for all $f \in E_I$.

Definition 1.5. Let $E_{\mathbb{I}}$ be a Banach function space and let X be a Banach space endowed with the norm $\|\cdot\|$. Then,

$$\mathscr{E}_{\mathbb{I}} := \mathscr{E}(\mathbb{I}, X) := \{h : \mathbb{I} \to X |\ h \text{ is strongly measurable and } \ \|h(\cdot)\| \in E_I\}$$

is a Banach space with respect to the norm

$$\|h\|_{\mathscr{E}_{\mathbb{I}}} := \|\ \|h(\cdot)\|\ \|_{E_{\mathbb{I}}}.$$

We call $\mathscr{E}_{\mathbb{I}}$ *the Banach space corresponding to the Banach function space* $E_{\mathbb{I}}$.

Definition 1.6 (Admissibility)**.** The Banach function space $E_{\mathbb{I}}$ is called *admissible* if the following properties hold:

(i) There is a constant $M \geqslant 1$ such that for every compact interval $[a, b] \subset \mathbb{I}$, and for all $\varphi \in E_{\mathbb{I}}$ we have

$$\int_a^b |\varphi(t)|\, dt \leqslant \frac{M(b-a)}{\left\|\chi_{[a,b]}\right\|_{E_{\mathbb{I}}}} \|\varphi\|_{E_{\mathbb{I}}}. \tag{1.7}$$

(ii) For all $\varphi \in E_{\mathbb{I}}$, the function $\Lambda_1 \in E_{\mathbb{I}}$ where $(\Lambda_1\varphi)(t) = \int_{t-1}^{t} \varphi(\tau)d\tau$.

(iii) $E_{\mathbb{I}}$ is T_τ^+-invariant for all $\tau \in \mathbb{I}$, where

- if $\mathbb{I} = (-\infty, t_0]$ and for some $t_0 \in \mathbb{R}$, then

$$(T_\tau^+\varphi)(t) = \begin{cases} \varphi(t-\tau) & \text{for } t \leqslant \tau + t_0, \\ 0 & \text{for } \ t > t_0; \end{cases}$$

- if $\mathbb{I} = \mathbb{R}$, then

$$(T_\tau^+\varphi)(t) = \varphi(t-\tau), \text{ for } t \in \mathbb{R}.$$

(iv) $E_{\mathbb{I}}$ is T_τ^--invariant for all $\tau \in \mathbb{I}$, where

- if $\mathbb{I} = (-\infty, t_0]$ and for some $t_0 \in \mathbb{R}$, then

$$(T_\tau^-\varphi)(t) = \begin{cases} \varphi(t+\tau) & \text{for } \ t \leqslant t_0 - \tau, \\ 0 & \text{for } \ t > t_0; \end{cases}$$

- if $\mathbb{I} = \mathbb{R}$, then

$$(T_\tau^-\varphi)(t) = \varphi(t+\tau), \text{ for } t \in \mathbb{R}.$$

Furthermore, there are constants N_1, N_2 such that $\|T_\tau^+\| \leqslant N_1$, $\|T_\tau^-\| \leqslant N_2$ for all $\tau \in \mathbb{I}$.

Example 1.7 ([12])**.** The spaces $L_p(\mathbb{R}),\ 1 \leqslant p \leqslant \infty$, the space

$$\mathbf{M}(\mathbb{R}) := \left\{ f \in L_{1,loc}(\mathbb{R}) : \sup_{t\in\mathbb{R}} \int_{t-1}^{t} |f(\tau)|\, d\tau < \infty \right\}$$

endowed with the norm

$$\|f\|_{\mathbf{M}} := \sup_{t\in\mathbb{R}} \int_{t-1}^{t} |f(\tau)|\, d\tau,$$

and many other function spaces occurring in interpolation theory, for example, the Lorentz spaces $L_{p,q},\ 1 < p < \infty, 1 < q < \infty, \ldots$ are admissible Banach function spaces.

Remark 1.8. If $E_{\mathbb{I}}$ is the admissible Banach function space, then $E_{\mathbb{I}} \hookrightarrow \mathbf{M}(\mathbb{I})$.

Proposition 1.9. *Let $E_{\mathbb{I}}$ be an admissible Banach function space. Then the following assertions hold.*

(i) *Let $\varphi \in L_{1,loc}(\mathbb{I})$ such that $\varphi \geqslant 0$ and $\Lambda_1\varphi \in E_{\mathbb{I}}$. For $\sigma > 0$, we define functions $\Lambda'_\sigma\varphi$, $\Lambda''_\sigma\varphi$ by*

$$(\Lambda'_\sigma\varphi)(t) = \int_{-\infty}^{t} e^{-\sigma(t-s)}\varphi(s)ds$$

and

$$(\Lambda''_\sigma\varphi)(t) = \begin{cases} \int_{t}^{\infty} e^{-\sigma(s-t)}\varphi(s)ds, & \text{if } \mathbb{I} = \mathbb{R}, \\ \int_{t}^{t_0} e^{-\sigma(s-t)}\varphi(s)ds & \text{if } \mathbb{I} = (-\infty, t_0]. \end{cases}$$

Then, $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ belong to $E_{\mathbb{I}}$. Moreover, we have

$$\|\Lambda'_\sigma\varphi\|_{E_{\mathbb{I}}} \leqslant \frac{N_1}{1-e^{-\sigma}}\, \|\Lambda_1\varphi\|_{E_{\mathbb{I}}} \ \text{ and } \ \|\Lambda''_\sigma\varphi\|_{E_{\mathbb{I}}} \leqslant \frac{N_2}{1-e^{-\sigma}}\, \|\Lambda_1\varphi\|_{E_{\mathbb{I}}} \tag{1.8}$$

for constants N_1, N_2 defined as in Definition 1.6.

(ii) *$E_{\mathbb{I}}$ contains exponentially decaying functions $e^{-a|t|}$, for all $t \in \mathbb{I}$ and any fixed constant $a > 0$.*

(iii) *$E_{\mathbb{I}}$ does not contain exponentially growing functions $e^{b|t|}$, for all $t \in \mathbb{I}$ and any fixed constant $b > 0$.*

We next recall the definition of associate spaces of admissible Banach spaces on $\mathbb{I}$ as follows.

Definition 1.10. Let $E_{\mathbb{I}}$ be an admissible Banach space, and we denote by $S(E_{\mathbb{I}})$ the unit sphere in $E_{\mathbb{I}}$. Consider, the set $E'_{\mathbb{I}}$ of all measurable real-valued functions ψ on $\mathbb{I}$ such that

$$\varphi\psi \in L_1(\mathbb{I}), \quad \int_{\mathbb{I}} |\varphi(t)\psi(t)|\, dt \leqslant k, \quad \text{for all } \varphi \in S(E_{\mathbb{I}}),$$

where k depends only on ψ and

$$L_1(\mathbb{I}) = \left\{ g : \mathbb{I} \to \mathbb{I} | g \text{ is measurable and } \int\limits_{\mathbb{I}} |g(t)|\, dt < \infty \right\}.$$

Then, $E'_{\mathbb{I}}$ is a normed space with the norm given by

$$\|\psi\|_{E'_{\mathbb{I}}} := \sup \left\{ \int\limits_{\mathbb{I}} |\varphi(t)\psi(t)|\, dt : \ \varphi \in S(E_{\mathbb{I}}) \right\} \quad \text{for } \psi \in E'_{\mathbb{I}},$$

and we call $E'_{\mathbb{I}}$ the *associate space* of $E_{\mathbb{I}}$.

Remark 1.11. Let $E_{\mathbb{I}}$ be an admissible Banach function space and let $E'_{\mathbb{I}}$ be its associate space. Then, we have the following Hölder inequality:

$$\int\limits_{\mathbb{I}} |\varphi(t)\psi(t)|\, dt \leqslant \|\varphi\|_{E_{\mathbb{I}}} \|\psi\|_{E'_{\mathbb{I}}}, \quad \text{for all } \varphi \in E_{\mathbb{I}}, \psi \in E'_{\mathbb{I}}. \tag{1.9}$$

Remark 1.12. In the case when $\mathbb{I} = \mathbb{R}$, we write $E, \mathscr{E}$ instead of $E_{\mathbb{R}}$ and $\mathscr{E}_{\mathbb{R}}$.

Throughout this paper, we assume the following assumption.

Assumption 1.13. (i) The function space $E_{\mathbb{I}}$ and its associate space $E'_{\mathbb{I}}$ are admissible spaces.

(ii) For the function $\varphi \in E'_{\mathbb{I}}$ and fixed $\nu > 0$, the function h_ν defined by

$$h_\nu(t) := \left\| e^{-\nu|t-\cdot|}\varphi(\cdot) \right\|_{E'_{\mathbb{I}}} \text{ for } t \in \mathbb{I}$$

belongs to $E_{\mathbb{I}}$.

Remark 1.14. If we set

$$\mathscr{E}_{\mathbb{I}} := \{ h : \mathbb{I} \to \mathscr{H} \mid h \text{ is strongly measurable and } \|h(\cdot)\| \in E_{\mathbb{I}} \}$$

respect to the norm

$$\|h\|_{\mathscr{E}_{\mathbb{I}}} := \left\| \|h(\cdot)\| \right\|_{E_{\mathbb{I}}},$$

then $\mathscr{E}_{\mathbb{I}}$ is the Banach space corresponding to the Banach function space $E_{\mathbb{I}}$.

Remark 1.15. In the case of infinite-dimensional phase spaces, instead of (1.2), we consider the integral equation

$$U(t) = e^{-(t-s)\mathcal{A}}U(s) + \int_s^t e^{-(t-\xi)\mathcal{A}}\mathcal{F}(\xi, U(\xi))d\xi \quad \text{for a.e. } t \geqslant s. \tag{1.10}$$

By a *solution* of equation (1.10), we mean a *strongly measurable* function $U(t)$ defined on an interval J with the values in $\mathscr{H}$ that satisfies (1.10) for $t,\ s \in J$. We note that the solution U of equation (1.10) is called a *mild solution* of equation (1.2). We refer the reader to [14] for more detailed treatments on the relations between classical and mild solutions of evolution equations (see also [4, 7, 11, 15]).

To obtain the existence of an admissible inertial manifold for equation (1.10), besides Assumptions 1.1 and 1.13, we also need the φ-Lipschitz property of the nonlinear term f in the following definition.

Definition 1.16 (φ-Lipschitz function)**.** Let E be an admissible Banach function space on $\mathbb{R}$ and let φ be a positive function belonging to E. A function $f\colon \mathbb{R}\times X_\beta \to X$ is said to be *φ-Lipschitz* if f satisfies

(i) $\|f(t,x)\| \leqslant \varphi(t)\left(1+\left\|A^\beta x\right\|\right)$ for a.e. $t\in\mathbb{R}$ and for all $x\in X_\beta$,
(ii) $\|f(t,x_1)-f(t,x_2)\| \leqslant \varphi(t)\left\|A^\beta(x_1-x_2)\right\|$ for a.e. $t\in\mathbb{R}$ and for all $x_1,\, x_2\in X_\beta$.

Proposition 1.17. *Let $\mathcal{F}(t,U)=(0,f(t,u^0))$, where $U=(x^0,x^1)\in\mathscr{H}$ and the nonlinear term $f(t,u^0)$ is φ-Lipschitz. Then we have*

$$|\mathcal{F}(t,U)| \leqslant \varphi(t)+\psi(t)|U|, \tag{1.11}$$

$$|\mathcal{F}(t,U)-\mathcal{F}(t,V)| \leqslant \psi(t)|U-V|, \tag{1.12}$$

where

$$\psi(t)=\varphi(t)\mu_{N+1}^{\beta}\delta_{N,\varepsilon}^{-1}=\varphi(t)\mu_{N+1}^{\beta-\frac{1}{2}}\delta_{N,\varepsilon}^{-1}\max\left\{1,\sqrt{\frac{\mu_{N+1}}{\varepsilon^2-\mu_{N+1}}}\right\}.$$

Proof. Firstly, by inequality (1.11), we have

$$\begin{aligned}|\mathcal{F}(t,U)| = \|f(t,u^0)\| &\leqslant \varphi(t)(1+\|A^\beta u^0\|)\\ &\leqslant \varphi(t)\left(1+\mu_{N+1}^{\beta}\delta_{N,\varepsilon}^{-1}|U|\right)\\ &= \varphi(t)+\psi(t)|U|.\end{aligned}$$

Also, by inequality (1.11), we estimate the following:

$$\begin{aligned}|\mathcal{F}(t,U)-\mathcal{F}(t,V)| &= \|f(t,u^0)-f(t,v^0)\|\\ &\leqslant \varphi(t)\|A^\beta(u^0-v^0)\|\\ &\leqslant \varphi(t)\mu_{N+1}^{\beta}\delta_{N,\varepsilon}^{-1}|U-V|\\ &= \psi(t)|U-V|.\end{aligned}$$

The proof is complete. □

Remark 1.18. For the sake of simplicity, for all $t\in\mathbb{R}$, by putting

$$\kappa(t):=\max\{\varphi(t),\psi(t)\}=\max\left\{\varphi(t),\varphi(t)\mu_{N+1}^{\beta-\frac{1}{2}}\delta_{N,\varepsilon}^{-1}\max\left\{1,\sqrt{\frac{\mu_{N+1}}{\varepsilon^2-\mu_{N+1}}}\right\}\right\},$$

if f is φ-Lipschitz, then $\mathcal{F}$ is κ-Lipschitz, meaning that

$$|\mathcal{F}(t,U)| \leqslant \kappa(t)(1+|U|),$$
$$|\mathcal{F}(t,U)-\mathcal{F}(t,V)| \leqslant \kappa(t)|U-V|.$$

2. Admissible inertial manifolds

Now, we construct the form of the solutions of equation (1.10), which belongs to rescaledly admissible spaces on the half-line $(-\infty,t_0]$ in the following lemma.

Lemma 2.1. *Let the operator A satisfy Assumption 1.1. Let E, E', and $\varphi\in E'$ be as in Assumption 1.13. Let $f:\mathbb{R}\times X_\beta\to X$ be φ-Lipschitz. For fixed $t_0\in\mathbb{R}$,* *let $U(t)$, $t \leqslant t_0$ be a solution to equation (1.10) such that $U(t) \in \mathscr{D}(\mathcal{A})$ for all $t \leqslant t_0$ and the function*

$$Z(t) = \left|e^{-\gamma(t_0-t)}U(t)\right| \quad \text{for all } t \leqslant t_0$$

belongs to $E_{(-\infty,t_0]}$. Then, $U(t)$ satisfies

$$U(t) = e^{-(t-t_0)\mathcal{A}}v_1 + \int_{-\infty}^{t_0} \mathcal{G}(t,\tau)\mathcal{F}(\tau, U(\tau))d\tau \quad \text{for all } t \leqslant t_0, \tag{2.1}$$

where $v_1 \in P\mathscr{H}$, and $\mathcal{G}(t,\tau)$ is Green's function defined as in (1.5).

Proof. Put

$$Y(t) := \int_{-\infty}^{t_0} \mathcal{G}(t,\tau)\mathcal{F}(\tau, U(\tau))d\tau \quad \text{for all } t \leqslant t_0. \tag{2.2}$$

By the definition of $\mathcal{G}(t,\tau)$, we have that $Y(t) \in \mathscr{H}$ for $t \leqslant t_0$. Since $\mathcal{F}$ is κ-Lipschitz, using estimates (1.6), for $t \leqslant t_0$, we obtain

$$\begin{aligned}\left|e^{-\gamma(t_0-t)}Y(t)\right| &\leqslant \int_{-\infty}^{t_0} \left|e^{\gamma(t-\tau)}\mathcal{G}(t,\tau)\right| \kappa(\tau)e^{-\gamma(t_0-\tau)}\left(1+|U(\tau)|\right)d\tau \\ &\leqslant \int_{-\infty}^{t_0} \left|e^{\gamma(t-\tau)}\mathcal{G}(t,\tau)\right| \kappa(\tau)\left(e^{-\gamma(t_0-\tau)}+|Z(\tau)|\right)d\tau. \end{aligned} \tag{2.3}$$

Putting $W(t) := e^{-\gamma(t_0-t)} + |Z(t)|$ for all $t \leqslant t_0$, we have that the function W belongs to $E_{(-\infty,t_0]}$ and

$$\begin{aligned}\int_{-\infty}^{t_0} \left|e^{\gamma(t-\tau)}\mathcal{G}(t,\tau)\right| \kappa(\tau)W(\tau)d\tau &\leqslant \int_{-\infty}^{t_0} e^{-\alpha|t-\tau|}\kappa(\tau)W(\tau)d\tau \\ &\leqslant \left\|e^{-\alpha|t-\cdot|}k(\cdot)\right\|_{E'_{(-\infty,t_0]}} \|W\|_{E_{(-\infty,t_0]}}. \end{aligned} \tag{2.4}$$

Since $K(t) = \left\|e^{-\alpha|t-\cdot|}k(\cdot)\right\|_{E'_{(-\infty,t_0]}}$ belongs to $E_{(-\infty,t_0]}$, using the admissibility of $E_{(-\infty,t_0]}$, we obtain that

$$e^{-\gamma(t_0-\cdot)}Y(\cdot) \in \mathscr{E}_{(-\infty,t_0]}$$

and

$$\left\|e^{-\gamma(t_0-\cdot)}Y(\cdot)\right\|_{\mathscr{E}_{(-\infty,t_0]}} \leqslant \|K(\cdot)\|_{E_{(-\infty,0]}} \|W\|_{E_{(-\infty,0]}}.$$

Next, by computing directly, we verify that $Y(\cdot)$ satisfies the integral equation

$$Y(t_0) = e^{-(t_0-t)\mathcal{A}}Y(t) + \int_t^{t_0} e^{-(t_0-\tau)\mathcal{A}}\mathcal{F}(\tau, U(\tau))d\tau \quad \text{for } t \leqslant t_0. \tag{2.5}$$

Indeed, substituting Y from (2.2) to the right-hand side of (2.5), we get

$$\begin{aligned}&e^{-(t_0-t)\mathcal{A}}Y(t) + \int_t^{t_0} e^{-(t_0-\tau)\mathcal{A}}\mathcal{F}(\tau, U(\tau))d\tau \\ &= e^{-(t_0-t)\mathcal{A}}\int_{-\infty}^{t_0} \mathcal{G}(t,\tau)\mathcal{F}(\tau, U(\tau))d\tau + \int_t^{t_0} e^{-(t_0-\tau)\mathcal{A}}\mathcal{F}(\tau, U(\tau))d\tau \\ &= e^{-(t_0-t)\mathcal{A}}\int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}}(I-P)\mathcal{F}(\tau, U(\tau))d\tau\end{aligned}$$

$$
\begin{aligned}
&-e^{-(t_0-t)\mathcal{A}}\int_t^{t_0} e^{-(t-\tau)\mathcal{A}}P\mathcal{F}(\tau,U(\tau))d\tau+\int_t^{t_0} e^{-(t_0-\tau)\mathcal{A}}\mathcal{F}(\tau,U(\tau))d\tau\\
=&\int_{-\infty}^{t} e^{-(t_0-\tau)\mathcal{A}}(I-P)\mathcal{F}(\tau,U(\tau))d\tau\\
&-\int_t^{t_0} e^{-(t_0-t)\mathcal{A}}e^{-(t-\tau)\mathcal{A}}P\mathcal{F}(\tau,U(\tau))d\tau+\int_t^{t_0} e^{-(t_0-\tau)\mathcal{A}}\mathcal{F}(\tau,U(\tau))d\tau\\
=&\int_{-\infty}^{t_0} e^{-(t_0-\tau)\mathcal{A}}(I-P)\mathcal{F}(\tau,U(\tau))d\tau=\int_{-\infty}^{t_0}\mathcal{G}(t_0,\tau)\mathcal{F}(\tau,U(\tau))d\tau=Y(t_0),
\end{aligned}
$$

here we use the fact that

$$
e^{-(t_0-t)\mathcal{A}}e^{-(t-\tau)\mathcal{A}}P=e^{-(t_0-\tau)\mathcal{A}}P, \quad \text{for all } t\leqslant\tau\leqslant t_0.
$$

Thus, we have

$$
Y(t_0)=e^{-(t_0-t)\mathcal{A}}Y(t)+\int_t^{t_0} e^{-(t_0-\tau)\mathcal{A}}\mathcal{F}(\tau,U(\tau))d\tau.
$$

On the other hand,

$$
U(t_0)=e^{-(t_0-t)\mathcal{A}}U(t)+\int_t^{t_0} e^{-(t_0-\tau)\mathcal{A}}\mathcal{F}(\tau,U(\tau))d\tau.
$$

Then $U(t_0)-U(t_0)=e^{-(t_0-t)\mathcal{A}}[U(t)-Y(t)]$.
Now, we need to prove that $U(t_0)-Y(t_0)\in P\mathscr{H}$. Indeed, by applying the operator $I-P$ to the expression

$$
U(t_0)-Y(t_0)=e^{-(t_0-t)\mathcal{A}}U(t)-Y(t)],
$$

we have

$$
\begin{aligned}
\|(I-P)[U(t_0)-Y(t_0)]\|&=\left\|e^{-(t_0-t)\mathcal{A}}(I-P)[U(t)-Y(t)]\right\|\\
&\leqslant e^{-\lambda_{N+1}^{-}(t_0-t)}\cdot\|I-P\|\cdot\left|e^{-\gamma(t_0-t)}[U(t)-Y(t)]\right|.
\end{aligned}
\tag{2.6}
$$

Since

$$
\operatorname*{esssup}_{t\leqslant t_0}\left|e^{-\gamma(t_0-t)}[U(t)-Y(t)]\right|<+\infty,
$$

letting $t\to-\infty$, we obtain

$$
\|(I-P)[U(t_0)-Y(t_0)]\|=0 \qquad \text{hence} \qquad (I-P)[U(t_0)-Y(t_0)]=0.
$$

Thus, $v_1:=U(t_0)-Y(t_0)\in P\mathscr{H}$. Using the fact that the restriction of $e^{-(t_0-t)\mathcal{A}}$ on $P\mathscr{H}$ is invertible with the inverse $e^{-(t-t_0)\mathcal{A}}$, we obtain

$$
\begin{aligned}
U(t)&=e^{-(t-t_0)\mathcal{A}}v_1+Y(t)\\
&=e^{-(t-t_0)\mathcal{A}}v_1+\int_{-\infty}^{t_0}\mathcal{G}(t,\tau)\mathcal{F}(\tau,U(\tau))d\tau \quad \text{for } t\leqslant t_0.
\end{aligned}
$$

The proof is completed. □

Remark 2.2. Equation (2.1) is called *Lyapunov–Perron equation*, which will be used to determine the inertial manifold for equation (1.10). By computing directly, we can see that the converse of Lemma 2.1 is also true. This means that all solutions of equation (2.1) satisfy equation (1.10) for $t \leqslant t_0$.

We now show the existence of rescaling bounded solutions to (1.10) on negative half-line in the following lemma.

Lemma 2.3. *Let the operator A satisfy Assumption 1.1. Let E, E', and $\varphi \in E'$ be as in Assumption 1.13. For all $t \in \mathbb{R}$, we define the function K by*

$$K(t) = \left\| e^{-\alpha|t-\cdot|}\kappa(\cdot)\right\|_{E'}, \tag{2.7}$$

where

$$\kappa(\cdot) := \max\left\{\varphi(\cdot), \varphi(\cdot)\mu_{N+1}^{\beta-\frac{1}{2}}\delta_{N,\varepsilon}^{-1}\max\left\{1, \sqrt{\frac{\mu_{N+1}}{\varepsilon^2-\mu_{N+1}}}\right\}\right\}.$$

If $f : \mathbb{R}\times X_\beta \to X$ is φ-Lipschitz such that $k = \|K(\cdot)\|_E < 1$, then there corresponds to each $v_1 \in P\mathscr{H}$ one and only one solution $U(\cdot)$ of equation (1.10) on $(-\infty, t_0]$ satisfying the condition $PU(t_0) = v_1$, and

$$Z(t) = |e^{-\gamma(t_0-t)}U(t)|, \quad t \leqslant t_0$$

belongs to $E_{(-\infty,t_0]}$ for each $t_0 \in \mathbb{R}$.

Proof. Denote

$$\mathscr{E}^{\gamma,t_0} := \left\{V\colon (-\infty,t_0] \to \mathscr{H} \mid V \text{ is strongly measurable, } \left|e^{-\gamma(t_0-\cdot)}V(\cdot)\right| \in E_{(-\infty,t_0]}\right\}$$

endowed with the norm

$$\|V\|_\gamma := \left\|\left|e^{-\gamma(t_0-\cdot)}V(\cdot)\right|\right\|_{E_{(-\infty,t_0]}}. \tag{2.8}$$

For each $t_0 \in \mathbb{R}$ and $v_1 \in P\mathscr{H}$, we prove that the linear transformation T defined by

$$(TU)(t) = e^{-(t-t_0)\mathcal{A}}v_1 + \int_{-\infty}^{t_0}\mathcal{G}(t,\tau)\mathcal{F}(\tau,U(\tau))d\tau \quad \text{for } t\leqslant t_0 \tag{2.9}$$

acts from $\mathscr{E}^{\gamma,t_0}$ into itself and is a contraction. In fact, for $U(\cdot) \in \mathscr{E}^{\gamma,t_0}$, we have that

$$|\mathcal{F}(t,U(t))| \leqslant \kappa(t)\left(1+|U(t)|\right).$$

Therefore, putting

$$Y(t) := e^{-(t-t_0)\mathcal{A}}v_1 + \int_{-\infty}^{t_0}\mathcal{G}(t,\tau)\mathcal{F}(\tau,U(\tau))d\tau \quad \text{for } t\leqslant t_0,$$

we derive that

$$\left|e^{-\gamma(t_0-t)}Y(t)\right| \leqslant e^{-\alpha(t_0-t)}\|v_1\| + \int_{-\infty}^{t_0}\left|e^{\gamma(t-\tau)}\mathcal{G}(t,\tau)\right|\kappa(\tau)e^{-\gamma(t_0-\tau)}\left(1+|U(\tau)|\right)d\tau \tag{2.10}$$

for all $t \leqslant t_0$.
Using the estimate (2.4), we obtain

$$\left|e^{-\gamma(t_0-t)}Y(t)\right| \leqslant \|v\| + K(t)\left\|W\right\|_{E_{(-\infty,t_0]}} \text{ for all } t \leqslant t_0,$$

where $W(t) := e^{-\gamma(t_0-t)}(1+|U(t)|)$.

Since $e^{-\alpha(t_0-\cdot)}$ and $K(\cdot)$ belong to $E_{(-\infty,t_0]}$, then $Y(\cdot) \in \mathscr{E}^{\gamma,t_0}$ and

$$\|Y(\cdot)\|_\gamma \leqslant \|v\| + k\,\|W\|_{E_{(-\infty,t_0]}}.$$

Therefore, the linear transformation T acts from $\mathscr{E}^{\gamma,t_0}$ to $\mathscr{E}^{\gamma,t_0}$.

For $X,\ Z \in \mathscr{E}^{\gamma,t_0}$, we now estimate

$$\begin{aligned}\left|e^{-\gamma(t_0-t)}(TX(t)-TZ(t))\right| &\leqslant \int_{-\infty}^{t_0} \left|e^{-\gamma(t_0-t)}\mathcal{G}(t,\tau)\right| \|\mathcal{F}(\tau,X(\tau))-\mathcal{F}(\tau,Z(\tau))\| d\tau \\ &\leqslant \int_{-\infty}^{t_0} \left|e^{-\gamma(t_0-t)}\mathcal{G}(t,\tau)\right| \kappa(\tau)e^{-\gamma(t_0-\tau)}\left|X(\tau)-Z(\tau)\right| d\tau.\end{aligned}$$

Again, using (2.4), we derive

$$\|TX(\cdot)-TZ(\cdot)\|_\gamma \leqslant k\|X(\cdot)-Z(\cdot)\|_\gamma.$$

Hence, since $k<1$, we obtain that $T\colon \mathscr{E}^{\gamma,t_0} \to \mathscr{E}^{\gamma,t_0}$ is a contraction. Thus, there exists a unique $U(\cdot) \in \mathscr{E}^{\gamma,t_0}$ such that $TU = U$. By the definition of T, we have that $U(\cdot)$ is the unique solution in $\mathscr{E}^{\gamma,t_0}$ of equation (1.10) for $t \leqslant t_0$. By Lemma 2.1 and Remark 2.2, we have that $U(\cdot)$ is the unique solution in $\mathscr{E}^{\gamma,t_0}$ of equation (1.10) for $t \leqslant t_0$. □

By the above results, we can define the admissible inertial manifold for (1.10) as follows.

Definition 2.4. Let E be an admissible function space and let $\mathscr{E}$ be a Banach space corresponding to E. An *admissible inertial manifold* of $\mathscr{E}$-class for (1.10) is a collection of Lipschitz surfaces $\mathbb{M} = \{\mathscr{M}_t\}_{t\in\mathbb{R}}$ in X such that $\mathscr{M}_t$ is the graph of a Lipschitz function

$$\mathbf{\Phi}_t\colon P\mathscr{H} \to (I-P)\mathscr{H},$$

that is,

$$\mathscr{M}_t = \{U + \mathbf{\Phi}_t U : U \in P\mathscr{H}\} \quad \text{for } t\in\mathbb{R} \tag{2.11}$$

and the following conditions are satisfied:

(i) The Lipschitz constants of $\mathbf{\Phi}_t$ are independent of t, that is, there exists a constant C independent of t such that

$$|\mathbf{\Phi}_t U_1 - \mathbf{\Phi}_t U_2| \leqslant C|U_1-U_2| \quad \text{for all } t\in\mathbb{R} \text{ and } U_1,\ U_2 \in P\mathscr{H}. \tag{2.12}$$

(ii) There exists $\gamma>0$ such that to each $U_0 \in \mathscr{M}_{t_0}$ there corresponds one and only one solution $U(\cdot)$ to (1.10) on $(-\infty,t_0]$ satisfying that $U(t_0)=U_0$ and the function

$$V(t) = e^{-\gamma(t_0-t)}U(t) \tag{2.13}$$

belongs to $\mathscr{E}_{(-\infty,t_0]}$ for each $t_0\in\mathbb{R}$.

(iii) $\{\mathscr{M}_t\}_{t\in\mathbb{R}}$ is positively invariant under (1.10), that is, if a solution $U(t)$, $t\geqslant s$ of (1.10) satisfies $U_s \in \mathscr{M}_s$, then we have that $U(t)\in\mathscr{M}_t$ for $t \geqslant s$.

(iv) $\{\mathscr{M}_t\}_{t\in\mathbb{R}}$ exponentially attracts all the solutions to (1.10), that is, for any solution $U(\cdot)$ of (1.10) and any fixed $s\in\mathbb{R}$, there is a positive constant H such that

$$\operatorname{dist}_{\mathscr{H}}(U(t),\mathscr{M}_t) \leqslant He^{-\gamma(t-s)} \quad \text{for } t \geqslant s, \tag{2.14}$$

where γ is the same constant as the one in (2.13), and $\mathrm{dist}_{\mathscr{H}}$ denotes the Hausdorff semi-distance generated by the norm in $\mathscr{H}$.

Now, we show the existence of admissible inertial manifolds for (1.10) in the following theorem.

Theorem 2.5. *Let the operator A satisfy Assumption 1.1. Let E, E', and $\varphi \in E'$ be as in Assumption 1.13. Let f be φ-Lipschitz. If*

$$k < 1 \quad \text{and} \quad \frac{kN_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k < 1, \tag{2.15}$$

then equation (1.10) has an admissible inertial manifold of $\mathscr{E}$-class.

Proof. We start by defining a collection of surfaces $\{\mathscr{M}_{t_0}\}_{t_0\in\mathbb{R}}$ by

$$\mathscr{M}_{t_0} := \{V + \mathbf{\Phi}_{t_0}V \mid V \in P\mathscr{H}\},$$

here $\mathbf{\Phi}_{t_0} \colon P\mathscr{H} \to (I-P)\mathscr{H}$ is defined by

$$\mathbf{\Phi}_{t_0}(V) = \int_{-\infty}^{t_0} e^{-(t_0-\tau)\mathcal{A}}(I-P)\mathcal{F}(\tau, U(\tau))d\tau = (I-P)U(t_0), \tag{2.16}$$

where $U(\cdot)$ is the unique solution in $\mathscr{E}^{\gamma,t_0}$ of equation (1.10) satisfying that $PU(t_0) = V$ (note that the existence and uniqueness of such $U(\cdot)$ is proved in Lemma 2.3).

Next, we prove $\mathbf{\Phi}_{t_0}$ is Lipschitz continuous with Lipschitz constant independent of t_0. Indeed, for V_1 and V_2 belonging to $P\mathscr{H}$, we have

$$\begin{aligned}
|\mathbf{\Phi}_{t_0}(V_1) - \mathbf{\Phi}_{t_0}(V_2)| &\leqslant \int_{-\infty}^{t_0} \left|e^{-(t_0-s)\mathcal{A}}(I-P)\right| |\mathcal{F}(s,U_1(s)) - \mathcal{F}(s,U_2(s))|\, ds \\
&= \int_{-\infty}^{t_0} |\mathcal{G}(t_0,s)|\, |\mathcal{F}(s,U_1(s)) - \mathcal{F}(s,U_2(s))|\, ds \\
&\leqslant \int_{-\infty}^{t_0} \left|e^{\gamma(t_0-s)}\mathcal{G}(t_0,s)\right| \kappa(s) \left|e^{-\gamma(t_0-s)}\left(U_1(s) - U_2(s)\right)\right| ds \\
&\leqslant k|U_1(\cdot) - U_2(\cdot)|_\gamma \quad \text{(here we use the estimate (2.4)).}
\end{aligned} \tag{2.17}$$

We now estimate $|U_1(\cdot) - U_2(\cdot)|_\gamma$. Since $U_i(\cdot)$ is the unique solution in $\mathscr{E}^{\gamma,t_0}$ of equation (1.10) on $(-\infty, t_0]$ satisfying $PU_i(t_0) = V_i$ with $i = 1, 2$, respectively, we have that

$$\begin{aligned}
&\left|e^{-\gamma(t_0-t)}\left(U_1(t) - U_2(t)\right)\right| \\
&= \left|e^{-\gamma(t_0-t)}\left(e^{-(t-t_0)\mathcal{A}}(V_1 - V_2) + \int_{-\infty}^{t_0} \mathcal{G}(t,\tau)[\mathcal{F}(\tau,U_1(\tau)) - \mathcal{F}(\tau,U_2(\tau))]d\tau\right)\right| \\
&\leqslant |V_1 - V_2| + k|U_1(\cdot) - U_2(\cdot)|_\gamma \quad \text{for all } t \leqslant t_0.
\end{aligned}$$

Hence, we obtain

$$|U_1(\cdot) - U_2(\cdot)|_\gamma \leqslant |V_1 - V_2| + k|U_1(\cdot) - U_2(\cdot)|_\gamma.$$

Therefore, since $k < 1$, we get

$$|U_1(\cdot) - U_2(\cdot)|_\gamma \leqslant \frac{1}{1-k}\,|V_1 - V_2|\,.$$

Substituting this inequality to (2.17), we obtain

$$|\mathbf{\Phi}_{t_0}(V_1) - \mathbf{\Phi}_{t_0}(V_2)| \leqslant \frac{k}{1-k}\,|V_1 - V_2|\,.$$

This yields that $\mathbf{\Phi}_{t_0}$ is Lipschitz continuous with the Lipschitz constant $\dfrac{k}{1-k}$ independent of t_0. We thus obtain the property (i) in Definition 2.4 of the Admissible Inertial Manifold.

The property (ii) of the Admissible Inertial Manifold follows from Lemmas 2.3 and 2.1 and Remark 2.2.

To prove the property (iii), let $U(\cdot)$ be a solution to equation (1.10) satisfying $U(s) = U_s \in \mathscr{M}_s$, that is, $U(s) = PU(s) + \mathbf{\Phi}_s(PU(s))$. Then, we fix an arbitrary number $t_0 \in [s, \infty)$ and define a function $W(\cdot)$ on $(-\infty, t_0]$ by

$$W(t) = \begin{cases} U(t) & \text{if } t \in [s, t_0], \\ V(t) & \text{if } t \in (-\infty, s], \end{cases}$$

where $V(\cdot)$ is the unique solution in $\mathscr{E}^{\gamma, t_0}$ of equation (1.10) satisfying $V(s) = U(s) \in \mathscr{M}_s$. Then, using equation (1.10) and (2.16), we obtain

$$\begin{aligned} W(t) &= e^{-(t-s)\mathcal{A}}\left(PU(s) + \mathbf{\Phi}_s(PU(s))\right) + \int_s^t e^{-(t-\tau)\mathcal{A}}\mathcal{F}(\tau, W(\tau))d\tau \\ &= e^{-(t-s)\mathcal{A}}\left(PU(s)\right) + \int_{-\infty}^t e^{-(t-\tau)\mathcal{A}}(I-P)\mathcal{F}(\tau, W(\tau))d\tau \\ &\quad + \int_s^t e^{-(t-\tau)\mathcal{A}}P\mathcal{F}(\tau, W(\tau))d\tau \quad \text{ for } s \leqslant t \leqslant t_0. \end{aligned} \tag{2.18}$$

Obviously, equation (2.18) also remains true for $t \in (-\infty, s]$. Now, in equation (2.18) setting $t = t_0$ and applying the projection P, we obtain

$$PW(t_0) = e^{-(t_0-s)\mathcal{A}}\left(PU(s)\right) + \int_s^{t_0} e^{-(t_0-\tau)\mathcal{A}}P\mathcal{F}(\tau, W(\tau))d\tau \quad \text{ for } s \leqslant t_0.$$

It follows from the above equation that

$$\begin{aligned} PU(s) &= e^{(t_0-s)\mathcal{A}}\left(PU(t_0)\right) - \int_s^{t_0} e^{(t_0-s)\mathcal{A}}e^{-(t_0-\tau)\mathcal{A}}P\mathcal{F}(\tau, W(\tau))d\tau \\ &= e^{-(s-t_0)\mathcal{A}}\left(PU(t_0)\right) - \int_s^{t_0} e^{-(s-\tau)\mathcal{A}}P\mathcal{F}(\tau, W(\tau))d\tau \quad \text{ for } s \leqslant t_0. \end{aligned} \tag{2.19}$$

Substituting this form of $PU(s)$ to equation (2.18), we obtain

$$\begin{aligned} W(t) &= e^{-(t-t_0)\mathcal{A}}PU(t_0) + \int_{t_0}^t e^{-(t-\tau)\mathcal{A}}P\mathcal{F}(\tau, W(\tau))d\tau \\ &\quad + \int_{-\infty}^t e^{-(t-\tau)\mathcal{A}}(I-P)\mathcal{F}(\tau, W(\tau))d\tau \end{aligned}$$

$$= e^{-(t-t_0)\mathcal{A}}PU(t_0) + \int_{-\infty}^{t_0} \mathcal{G}(t,\tau)\mathcal{F}(\tau, W(\tau))d\tau \quad \text{for } t \leqslant t_0. \tag{2.20}$$

Therefore, $U(t_0) = W(t_0) = PU(t_0) + \mathbf{\Phi}_{t_0}(PU(t_0))$ for all $t_0 \geqslant s$.

Lately, we prove the property (iv) of the admissible inertial manifold. To do this, we prove that for any solution $U(\cdot)$ to equation (1.10) and any $s \in \mathbb{R}$ there is a solution $U^\star(\cdot)$ of (1.10) such that $U^\star(t) \in \mathscr{M}_t$ for $t \geqslant s$ and

$$|U(t) - U^\star(t)| \leqslant \frac{\eta}{1-L}e^{-\gamma(t-s)} \quad \text{for all } t \geqslant s, \text{ and some constant } \eta, \tag{2.21}$$

where $L := \dfrac{kN_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k < 1$ is given as in (2.15). The solution $U^\star(\cdot)$ is called an *induced trajectory.*

We find the induced trajectory in the form $U^\star(t) = U(t) + W(t)$ with

$$\|W\|_{s,+} = \operatorname*{esssup}_{t\geqslant s} e^{\gamma(t-s)}|W(t)| < \infty. \tag{2.22}$$

Substituting $U^\star(\cdot)$ into (1.10), we obtain that $U^\star(\cdot)$ is a solution to (1.10) for $t \geqslant s$ if and only if $W(\cdot)$ is a solution to the equation

$$W(t) = e^{-(t-s)\mathcal{A}}W(s) + \int_s^t e^{-(t-\xi)\mathcal{A}}[\mathcal{F}(\xi, U(\xi)+W(\xi)) - \mathcal{F}(\xi, U(\xi))]d\xi. \tag{2.23}$$

For the sake of simplicity in the presentation, we put

$$F(t,W) = \mathcal{F}(t, U+W) - \mathcal{F}(t,U)$$

and set

$$L_\infty^{s,+} = \left\{V\colon [s,\infty) \to \mathscr{H} \mid V \text{ is strongly measurable and} \operatorname*{esssup}_{t\geqslant s} e^{\gamma(t-s)}|V(t)| < \infty\right\}$$

endowed with the norm $\|\cdot\|_{s,+}$ defined as in (2.22).

Then, by the same way as in Lemma 2.1 and Remark 2.2, we can prove that a function $W(\cdot) \in L_\infty^{s,+}$ is a solution to (2.23) if and only if it satisfies

$$W(t) = e^{-(t-s)\mathcal{A}}X_0 + \int_s^\infty \mathcal{G}(t,\tau)F(\tau, W(\tau))d\tau \quad \text{for } t \geqslant s \text{ and } X_0 \in (I-P)\mathscr{H}. \tag{2.24}$$

Here the value $X_0 \in (I-P)\mathscr{H}$ is chosen such that $U^\star(s) = U(s) + W(s) \in \mathscr{M}_s$, that is,

$$(I-P)(U(s) - W(s)) = \mathbf{\Phi}_s(P(U(s)+W(s))).$$

From (2.24), it follows that

$$W(s) = X_0 - \int_s^\infty e^{-(s-\tau)\mathcal{A}}PF(\tau, W(\tau))d\tau. \tag{2.25}$$

Hence

$$P(U(s)+W(s)) = PU(s) - \int_s^\infty e^{-(s-\tau)\mathcal{A}}PF(\tau, W(\tau))d\tau,$$

and therefore

$$X_0 = (I-P)W(s) = -(I-P)U(s) + \mathbf{\Phi}_s\left(PU(s) - \int_s^\infty e^{-(s-\tau)\mathcal{A}}PF(\tau, W(\tau))d\tau\right). \tag{2.26}$$

Substituting this form of X_0 into (2.24), we obtain

$$W(t) = e^{-(t-s)\mathcal{A}}\left[-(I-P)U(s) + \mathbf{\Phi}_s\left(PU(s) - \int_s^\infty e^{-(s-\tau)\mathcal{A}}PF(\tau, W(\tau))d\tau\right)\right] + \int_s^\infty \mathcal{G}(t,\tau)F(\tau, W(\tau))d\tau \quad \text{for } t \geqslant s. \tag{2.27}$$

What we have to do now to prove the existence of $U^\star$ satisfying (2.21) is to prove that equation (2.27) has a solution $W(\cdot) \in L_\infty^{s,+}$. To do this we prove that the linear transformation T defined by

$$(TX)(t) = e^{-(t-s)\mathcal{A}}\left[-(I-P)U(s) + \mathbf{\Phi}_s\left(PU(s) - \int_s^\infty e^{-(s-\tau)\mathcal{A}}PF(\tau, X(\tau))d\tau\right)\right] + \int_s^\infty \mathcal{G}(t,\tau)F(\tau, X(\tau))d\tau \quad \text{for } t \geqslant s.$$

acts from $L_\infty^{s,+}$ into itself and is a contraction.

Indeed, for $X(\cdot) \in L_\infty^{s,+}$, we have that $|F(t,X(t))| \leqslant \kappa(t)|X(t)|$. Therefore, putting

$$q(X) := -(I-P)U(s) + \mathbf{\Phi}_s\left(PU(s) - \int_s^\infty e^{-(s-\tau)\mathcal{A}}PF(\tau, X(\tau))d\tau\right),$$

we can estimate

$$\begin{aligned} e^{\gamma(t-s)}\left|(TX)(t)\right| &\leqslant e^{\gamma(t-s)}\left|e^{-(t-s)\mathcal{A}}q(X)\right| + \int_s^\infty \left|e^{\gamma(t-\tau)}\mathcal{G}(t,\tau)\right|\kappa(\tau)e^{\gamma(\tau-s)}\left|X(\tau)\right|d\tau \\ &\leqslant \left\|e^{\gamma(t-s)}e^{-(t-s)\mathcal{A}}q(X)\right\| + \int_s^\infty \left|e^{\gamma(t-\tau)}\mathcal{G}(t,\tau)\right|\kappa(\tau)d\tau\|X(\cdot)\|_{s,+}. \end{aligned} \tag{2.28}$$

Using the Lipschitz property of $\mathbf{\Phi}_s$ and for $t \geqslant s$, we now estimate the first term in the right-hand side of the above formula. In fact,

$$\begin{aligned} \left|e^{\gamma(t-s)}e^{-(t-s)\mathcal{A}}q(X)\right| &\leqslant \left|e^{\gamma(t-s)}e^{-(t-s)\mathcal{A}}(-(I-P)U(s) + \mathbf{\Phi}_s(PU(s)))\right| \\ &\quad + \left|e^{\gamma(t-s)}e^{-(t-s)\mathcal{A}}(q(X) + (I-P)U(s) - \mathbf{\Phi}_s(PU(s)))\right| \\ &\leqslant e^{(\gamma-\lambda_{N+1}^-)(t-s)}\Big(\left|(-(I-P)U(s) + \mathbf{\Phi}_s(PU(s)))\right| \\ &\quad + \left|(q(X) + (I-P)U(s) - \mathbf{\Phi}_s(PU(s)))\right|\Big) \\ &\leqslant \eta + \left|(q(X) + (I-P)U(s) - \mathbf{\Phi}_s(PU(s)))\right| \\ &\leqslant \eta + \left|\mathbf{\Phi}_s\left(Pu(s) - \int_s^\infty e^{-(s-\tau)\mathcal{A}}PF(\tau, X(\tau))d\tau\right) - \mathbf{\Phi}_s(PU(s))\right| \\ &\leqslant \eta + \frac{k}{1-k}\left|\int_s^\infty e^{-(s-\tau)\mathcal{A}}PF(\tau, X(\tau))d\tau\right| \\ &\leqslant \eta + \frac{k}{1-k}\int_s^\infty e^{-\alpha(\tau-s)}\kappa(\tau)\left|e^{\gamma(\tau-s)}X(\tau)\right|d\tau \\ &\leqslant \eta + \left[\frac{kN_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty\right]\|X(\cdot)\|_{s,+}. \end{aligned}$$

Substituting these estimates to (2.28), we obtain $TX \in L_\infty^{s,+}$ and

$$\|TX\|_{s,+} \leqslant \eta + \left[\frac{kN_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k\right]\|X(\cdot)\|_{s,+}. \tag{2.29}$$

Therefore, the linear transformation T acts from $L_\infty^{s,+}$ to $L_\infty^{s,+}$.

Now, using the fact that

$$|F(t,W_1) - F(t,W_2)| \leqslant \kappa(t)\,|W_1 - W_2|$$

and for $X,\, Z \in L_\infty^{s,+}$, for all $t \geqslant s$, we now estimate

$$\begin{aligned}
\left|e^{\gamma(t-s)}\left(TX(t) - TZ(t)\right)\right| &\leqslant \frac{k}{1-k}\left|\int_s^\infty e^{-(s-\tau)\mathcal{A}}P\big(F(\tau,X(\tau)) - F(\tau,Z(\tau))\big)d\tau\right| \\
&\quad + \int_s^\infty \left|e^{\gamma(t-s)}\mathcal{G}(t,\tau)\right|\left|F(\tau,X(\tau)) - F(\tau,Z(\tau))\right|d\tau \\
&\leqslant \frac{k}{1-k}\int_s^\infty e^{-\alpha(\tau-s)}\kappa(\tau)\left|e^{\gamma(\tau-s)}\left[X(\tau) - Z(\tau)\right]\right|d\tau \\
&\quad + \int_s^\infty \left|e^{\gamma(t-\tau)}\mathcal{G}(t,\tau)\right|\kappa(\tau)e^{\gamma(\tau-s)}\left|X(\tau) - Z(\tau)\right|d\tau \\
&\leqslant \left[\frac{kN_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k\right]\|X(\cdot) - Z(\cdot)\|_{s,+}.
\end{aligned}$$

Therefore,

$$\|TX(\cdot) - TZ(\cdot)\|_{s,+} \leqslant \left[\frac{kN_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k\right]\|X(\cdot) - Z(\cdot)\|_{s,+}.$$

Hence, if

$$\frac{kN_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k < 1,$$

then we obtain that $T\colon L_\infty^{s,+} \to L_\infty^{s,+}$ is a contraction. Thus, there exists a unique $W(\cdot) \in L_\infty^{s,+}$ such that $TW = W$. By the definition of T, we have that $W(\cdot)$ is the unique solution in $L_\infty^{s,+}$ of equation (2.27) for $t \geqslant s$. Also, using (2.29), we have the estimate for $\|W(\cdot)\|_{s,+}$ as

$$\|W(\cdot)\|_{s,+} \leqslant \frac{\eta}{1-L}.$$

Furthermore, by the determination of W, we obtain the existence of the solution $U^\star = U + W$ to equation (1.10) such that $U^\star(t) \in \mathscr{M}_t$ for $t \geqslant s$, and $U^\star$ satisfies inequality (2.21) yielding that

$$|U^\star(t) - U(t)| = |W(t)| \leqslant \frac{\eta}{1-L}e^{-\gamma(t-s)} \qquad \text{for all } t \geqslant s.$$

By putting $H := \dfrac{\eta}{1-L}$, it follows from this inequality that

$$\operatorname{dist}_{\mathscr{H}}(U(t), \mathscr{M}_t) \leqslant He^{-\gamma(t-s)}.$$

Therefore, $\{\mathscr{M}_t\}_{t\in\mathbb{R}}$ exponentially attracts every solution $U(\cdot)$ of integral equation (1.10). □

Remark 2.6. By the definition of the constant k, the condition (2.15) is fulfilled if the difference $\lambda_{N+1} - \lambda_N$ is sufficiently large, and/or the norm $\|\Lambda_1\varphi\|_\infty = \sup_{t\in\mathbb{R}} \int_{t-1}^{t} \varphi(\tau)d\tau$ is sufficiently small.

3. Example

Consider the semilinear damped wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) + 2\varepsilon \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + a(t)\ln\left(1+|u(t,x)|\right), \quad 0<x<\pi, \quad t \geqslant t_0 \\ u(t,0) = u(t,\pi) = 0 \quad t > t_0 \\ u(t_0,x) = \phi_1(x), \quad \dfrac{\partial u}{\partial t}(t_0,x) = \phi_2(x), \quad 0<x<\pi, \end{cases} \tag{3.1}$$

where ϕ_1, ϕ_2 are given initial functions and $a(t)$ is defined by

$$a(t) = \begin{cases} n & \text{if} \quad t \in \left[n - \dfrac{1}{2^{n+c}}, n + \dfrac{1}{2^{n+c}}\right] \quad \text{for } n = 1, 2, \ldots. \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

We now take $X = L^2(0,\pi)$, and let $A : X \to X$ be defined by

$$Au = -\frac{\partial^2 u}{\partial x^2}$$

on the domain

$$\mathscr{D}(A) = H_0^1(0,\pi) \cap H^2(0,\pi).$$

Then, A satisfies Assumption 1.1, that is, A is a positive operator with discrete point spectrum

$$1^2, 2^2, \ldots, n^2, \ldots$$

and $\lambda_{N+1} - \lambda_N = 2N+1$.

Note that, $\mathscr{D}(A^{1/2}) = H_0^1(0,\pi)$.

Let $f : \mathbb{R} \times \mathscr{D}(A^{1/2})$ be defined by $f(t,u) = a(t)\ln(1+|u|)$. It is obvious that f is φ-Lipschitz with $\varphi(t) = a(t)$ for all $t \in \mathbb{R}$. Furthermore, since φ can take any arbitrarily large value, then $\varphi \notin L_\infty$. Now, if we take $E = L^p(\mathbb{R})$ with $1 < p < \infty$, then $E' = L^q(\mathbb{R})$ for $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and we have

$$\int_{\mathbb{R}} |\varphi(t)|^q \, dt = \sum_{n\in\mathbb{N}} \int_{n-\frac{1}{2^{n+c}}}^{n+\frac{1}{2^{n+c}}} n^q dt = \sum_{n\in\mathbb{N}} n^q \frac{1}{2^{n+c-1}} < +\infty,$$

that is, $\varphi \in E'$. But

$$\|\Lambda_1\varphi\|_\infty = \sup_{t\in\mathbb{R}} \int_t^{t+1} \varphi(\tau)d\tau = \sup_{t\geqslant 0} \int_t^{t+1} a(\tau)d\tau \leqslant 2\sup_{n\in\mathbb{N}} \int_{n-\frac{1}{2^{n+c}}}^{n+\frac{1}{2^{n+c}}} nd\tau \leqslant \frac{1}{2^{c-2}}.$$

So, by Remark 2.6, equation (3.1) has an admissible inertial manifold of $\mathscr{E}$-class if N **and/or** c are large enough (here, $\mathscr{E}$ is the Banach space corresponding to $L^p(\mathbb{R})$).

References

1. A. Bensoussan and F. Landoli, *Stochastic inertial manifolds*, Stochastics and Stochastics Rep. **53** (1995) 13–39.
2. L. Boutet de Monvel, I.D. Chueshov and A.V. Rezounenko, *Inertial manifolds for retarded semilinear parabolic equations*, Nonlinear Anal. **34** (1998) 907–925.
3. I.D. Chueshov, *Approximate inertial manifolds of exponential order for semilinear parabolic equations subjected to additive white noise*, J. Dynam. Differential Equations **7** (1995) 549–566.
4. I.D. Chueshov, *Introduction to the Theory of Infinite-Dimensional Dissipative Systems*, "ACTA" Scientific Publishing House, 2002.
5. I.D. Chueshov and M. Scheutzow, *Inertial manifolds and forms for stochastically perturbed retarded semilinear parabolic equations*, J. Dyn. Differ. Equ. **13** (2001) 355–380.
6. P. Constantin, C. Foias, B. Nicolaenko and R. Temam, *Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations*, Springer-Verlag, New York, 1989.
7. K.J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Grad. Texts in Math. 194, Springer-Verlag, Berlin-Heidelberg, 2000.
8. C. Foias, G.R. Sell and R. Temam *Variétés inertielles des équations différentielles dissipatives*, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 5, 139–141.
9. N. Koksch and S. Siegmund, *Pullback attracting inertial manifols for nonautonomous dynamical systems*, J. Dynam. Differential Equations **14** (2002) 889–941.
10. M. Kwak, *Finite dimensional description of convective reaction diffusion equations*, J. Dynam. Differential Equations **4** (1992) 515–543.
11. A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Basel, 1995.
12. T.H. Nguyen, *Admissibly inertial manifolds for a class of semi-linear evolution equations*, J. Differential Equations **254** (2013), 2638–2660.
13. T.H. Nguyen and A. Minh Le, *Admissible inertial manifolds for delay equations and applications to Fisher-Kolmogorov model*, Acta Appl. Math. **156** (2018) 15–31.
14. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, 1983.
15. G.R. Sell and Y. You, *Dynamics of Evolutionary Equations*, Applied Mathematical Sciences, vol. 143, Springer-Verlag, 2002.

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