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# MAPS STRONGLY PRESERVING THE SQUARE ZERO OF $\lambda$-LIE PRODUCT OF OPERATORS

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Abstract. Let $\mathcal{A}$ be a standard operator algebra on a Banach space $\mathcal{X}$ with $\dim \mathcal{X} \geq 2$. In this paper, we characterize the forms of additive maps on $\mathcal{A}$ that strongly preserve the square zero of $\lambda$-Lie product of operators. That is, if $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ is an additive map satisfying

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0,$$

for every $A, B \in \mathcal{A}$ and for a scalar number $\lambda$ with $\lambda \neq -1$, then it is shown that there exists a function $\sigma : \mathcal{A} \to \mathbb{C}$ such that $\phi(A) = \sigma(A)A$ for every $A \in \mathcal{A}$.

## 1. Introduction

In the last decade, many mathematicians have stdied preserving problems. In particular, maps preserving a certain property of products of elements are considered; see [2–11]. We recall some of them which are related to our purpose.

Let $\mathcal{A}$ be a Banach algebra, let $A, B \in \mathcal{A}$, and let $\lambda$ be a scalar. Then $AB+\lambda BA$ is said to be the $\lambda$-Lie product of $A$ and $B$ and is denoted by $[A, B]_\lambda$. The $\lambda$-Lie product is said to be the Jordan product or the Lie product, whenever $\lambda = 1$ or $\lambda = -1$, respectively. The Lie product of $A$ and $B$ is denoted by $[A, B]$. The triple Jordan product of $A$ and $B$ is defined by $ABA$. These products play a rather important role in mathematical physics.

Taghavi et al. [10] considered the maps strongly preserving the $\eta$-Lie product on an algebra $\mathcal{A}$, that is a map $\phi : \mathcal{A} \to \mathcal{A}$ satisfying $\phi(A)\phi(P) + \eta\phi(P)\phi(A) = AP + \eta PA$, for every $A \in \mathcal{A}$, some idempotent $P \in \mathcal{A}$, and some scalar $\eta$.

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Let $\mathcal{B}(\mathcal{X})$ be the Banach algebra of all bounded linear operators on a Banach space $\mathcal{X}$. In [6], the authors characterized unital surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of product of operators, in both directions. Wang et al. [11] characterized linear surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of either products of operators or triple Jordan product of operators. Also Fang [5] characterized linear surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of Jordan product of operators.

We recall that a standard operator algebra $\mathcal{A}$ on a Banach space $\mathcal{X}$ is a norm closed subalgebra of $\mathcal{B}(\mathcal{X})$ that contains the identity and all finite rank operators.

We say that a map $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ strongly preserves the square zero of $\lambda$-Lie product of operators, whenever

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0$$

for every $A, B \in \mathcal{A}$.

In this paper, we characterize the forms of additive maps that strongly preserve the square zero of $\lambda$-Lie products of operators. Our main result is the following theorem.

**Theorem 1.1.** *Assume that $\mathcal{A}$ is a standard operator algebra on a Banach space $\mathcal{X}$ with $\dim \mathcal{X} \geq 2$. Let $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ be an additive map that satisfies*

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0,$$

*for every $A, B \in \mathcal{A}$ and for a scalar $\lambda$ with $\lambda \neq -1$. Then there exists a function $\sigma : \mathcal{A} \to \mathbb{C}$ such that $\phi(A) = \sigma(A)A$ for every $A \in \mathcal{A}$.*

## 2. Proof of main result

First we recall some notations. We assume that $\mathcal{X}$ is a Banach space and $\mathcal{A}$ is a standard operator algebra on $\mathcal{X}$. We denote by $\mathcal{X}^*$, the dual space of $\mathcal{X}$. For every nonzero $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, the symbol $x \otimes f$ stands for the rank one linear operator on $\mathcal{X}$ defined by $(x \otimes f)y = f(y)x$ for any $y \in \mathcal{X}$. Note that every rank one operator in $\mathcal{B}(\mathcal{X})$ can be written in this way. We denote by $\mathcal{F}_1(\mathcal{X})$ the set of all rank one operators in $\mathcal{B}(\mathcal{X})$. The rank one operator $x \otimes f$ is idempotent if and only if $f(x) = 1$ and is nilpotent if and only if $f(x) = 0$.

**Proposition 2.1.** *Let $A \in \mathcal{A}$, let $x \in \mathcal{X}$, let $f \in \mathcal{X}^*$ such that $f(x) \neq 0$, and let $\lambda \neq 0, -1$. Then $[A, x \otimes f]_\lambda^2 = 0$ if and only if one of the following statements occurs:*

(i) *$Axf(Ax) = -\lambda xf(A^2x)$ and $Axf(x) = -\lambda xf(Ax)$.*
(ii) *$fA = 0$.*

*Proof.* First assume that $Axf(Ax) = -\lambda xf(A^2x)$ and $Axf(x) = -\lambda xf(Ax)$ hold. Hence

$$\begin{aligned}
[A, x \otimes f]_\lambda^2 &= (Ax \otimes f + \lambda x \otimes fA)^2 \\
&= f(Ax)Ax \otimes f + \lambda f(x)Ax \otimes fA + \lambda^2 f(Ax)x \otimes fA + \lambda f(A^2x)x \otimes f \\
&= -\lambda xf(A^2x) \otimes f - \lambda^2 xf(Ax) \otimes fA + \lambda^2 f(Ax)x \otimes fA + \lambda f(A^2x)x \otimes f \\
&= 0.
\end{aligned}$$

Now if $fA = 0$, then

$$\begin{aligned}[A, x\otimes f]_\lambda^2 &= (Ax\otimes f + \lambda x\otimes fA)^2\\ &= (Ax\otimes f)^2 = f(Ax)Ax\otimes f = 0.\end{aligned}$$

Conversely, assume that $[A, x\otimes f]_\lambda^2 = 0$. For an operator $B$, it is clear that

$$B^2 = 0 \Leftrightarrow (B(Bx) = 0, \quad \text{for all } x\in\mathcal{X}) \Leftrightarrow \mathrm{ImB} \subseteq \ker B.$$

This together with the assumptions implies

$$[A, x\otimes f]_\lambda^2 = 0 \Leftrightarrow \mathrm{Im[A, x\otimes f]_\lambda} \subseteq \ker[A, x\otimes f]_\lambda.$$

Let $fA \neq 0$. If $fA$ and $f$ are linearly independent, then $\mathrm{Im[A, x\otimes f]_\lambda} = \mathrm{span}\{Ax, x\}$, and so

$$\mathrm{span}\{Ax, x\} \subseteq \ker(Ax\otimes f + \lambda x\otimes fA),$$

which implies

$$(Ax\otimes f + \lambda x\otimes fA)(Ax) = Axf(Ax) + \lambda xf(A^2x) = 0,$$

$$(Ax\otimes f + \lambda x\otimes fA)(x) = Axf(x) + \lambda xf(Ax) = 0,$$

which are the asserted relations. If $fA$ and $f$ are linearly dependent, then there exists a nonzero scalar $a$ such that $fA = af$, and so

$$[A, x\otimes f]_\lambda = Ax\otimes f + \lambda x\otimes fA = (Ax + ax)\otimes f.$$

Thus $\mathrm{Im}[A, x\otimes f]_\lambda = \mathrm{span}\{Ax + \lambda ax\}$, and so

$$\mathrm{span}\{Ax + \lambda ax\} \subseteq \ker(Ax\otimes f + \lambda x\otimes fA) = \ker((Ax + ax)\otimes f),$$

which implies

$$((Ax + \lambda ax)\otimes f)(Ax + \lambda ax) = 0$$

$$\Rightarrow (Ax + \lambda ax)[f(Ax) + \lambda af(x)] = 0$$

$$\Rightarrow (Ax + \lambda ax)af(x)(1 + \lambda) = 0.$$

Since $f(x) \neq 0$ and $\lambda \neq -1$, we obtain $Ax + \lambda ax = 0$. This together with $fA = af$ implies

$$Axf(Ax) = -\lambda axf(Ax) = -\lambda x(fA)(Ax) = -\lambda xf(A^2x)$$

and

$$Axf(x) = -\lambda axf(x) = -\lambda xf(Ax),$$

and these complete the proof.

□

In the following lemmas, assume that $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ is a map that satisfies

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0,$$

for every $A, B \in \mathcal{A}$ and for a scalar number $\lambda$ with $\lambda \neq 0, -1$.

**Lemma 2.2.** *For every $A \in \mathcal{A}$,* $\ker A \subseteq \ker\phi(A)$*.*

*Proof.* If $x \in \ker A$, then

$$\begin{aligned}[A, x \otimes f]_\lambda^2 &= (Ax \otimes f + \lambda x \otimes fA)^2 \\ &= (\lambda x \otimes fA)^2 = \lambda^2 f(Ax) x \otimes fA = 0,\end{aligned}$$

for every $f \in \mathcal{X}^*$, and so $[\phi(A), x \otimes f]_\lambda^2 = 0$. Applying Proposition 2.1, we have

$$\phi(A)xf(\phi(A)x) = -\lambda x f(\phi(A)^2 x) \tag{2.1}$$

and

$$\phi(A)xf(x) = -\lambda x f(\phi(A)x) \tag{2.2}$$

or $f\phi(A) = 0$, for every $f \in \mathcal{X}^*$ such that $f(x) \neq 0$. We show $\phi(A)x = 0$. First let relations (2.1) and (2.2) hold and let $f(x) = 1$. From (2.2), we obtain $\phi(A)x = -\lambda x f(\phi(A)x)$ and thus

$$f(\phi(A)x) = -\lambda f(x) f(\phi(A)x) = -\lambda f(\phi(A)x).$$

Then $f(\phi(A)x) = 0$ since $\lambda \neq 0, -1$. That is, $\phi(A)x = 0$.

Now let $f\phi(A) = 0$ for every $f$ such that $f(x) \neq 0$. Since $\phi(A)x \neq 0$, there exists a linear functional $f$ such that $f(x) \neq 0$ and $f(\phi(A)x) = 1$, a contradiction, because $f(\phi(A)x) = (f\phi(A))x = 0$. Therefore, $\phi(A)x = 0$. □

Next assume that $\phi$ is additive.

**Lemma 2.3.** *For every rank one operator $A$, $\phi(A) = 0$ or $\phi(A) = \kappa(A)A$, where $\kappa : \mathcal{A} \to \mathbb{C}$ is a function.*

*Proof.* Let $A = x \otimes f$, for some $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$. From Lemma 2.2, we have

$$\ker x \otimes f \subseteq \ker \phi(x \otimes f),$$

which implies that $\ker f \subseteq \ker \phi(x \otimes f)$ and since $\ker f$ is a hyperspace of $\mathcal{X}$, $\ker \phi(x \otimes f) = \mathcal{X}$ or $\ker \phi(x \otimes f) = \ker f$. Therefore $\phi(x \otimes f)$ is a zero operator or there exists a vector $y$ such that $\phi(x \otimes f) = y \otimes f$. We divide the rest of the proof into two cases:

Case 1. Let $f(x) \neq 0$ and let $g$ be a functional such that $g(x) = 0$. We have

$$\begin{aligned}[x \otimes f, x \otimes g]_\lambda^2 &= [f(x)x \otimes g + \lambda g(x) x \otimes f]^2 \\ &= [f(x) x \otimes g]^2 = 0\end{aligned}$$

and then

$$[\phi(x \otimes f), x \otimes g]_\lambda^2 = [y \otimes f, x \otimes g]_\lambda^2 = 0,$$

which implies

$$\begin{aligned}[f(x)y \otimes g + \lambda g(y) x \otimes f]^2 =& f(x)g(y) y \otimes g + \lambda^2 g(y) f(x) x \otimes f \\ &+ \lambda g(y) f(x) f(y) x \otimes g = 0.\end{aligned}$$

Since $f(x) \neq 0$, we obtain

$$y \otimes g(y) g = x \otimes (-\lambda^2 g(y) f - \lambda g(y) f(y) g).$$

This implies that $x$ and $y$ are linearly dependent or $g(y) = 0$. If $g(y) = 0$ and $x$ and $y$ are linearly independent, we get a contradiction, since in this case by $\dim \mathcal{X} \geq 2$, there exists a functional $g$ such that $g(x) = 0$ but $g(y) = 1$.

Therefore $x$ and $y$ are linearly dependent and then there is a scalar $\kappa(A)$ such that $\phi(A) = \kappa(A)A$.

Case 2. Let $f(x) = 0$. There exists a linear functional $h$ such that $h(x) = 1$ and then by Case 1, we have

$$\phi(x \otimes (f + h)) = kx \otimes (f + h),$$

where $k = \kappa(x \otimes (f + h))$. On the other hand, the additivity of $\phi$ together with Case 1 implies

$$\phi(x \otimes (f + h)) = \phi(x \otimes f) + \phi(x \otimes h) = \phi(x \otimes f) + tx \otimes h,$$

where $t = \kappa(x \otimes h)$. Thus

$$\phi(x \otimes f) = kx \otimes (f + h) - tx \otimes h = x \otimes (kf + kh - th).$$

This together with $\phi(x \otimes f) = y \otimes f$ implies that $x$ and $y$ are linearly dependent and this completes the proof. □

*Proof of Theorem 1.1.* We divide the proof into two cases.

Case 1. Let $\lambda = 0$. First we show $\ker A \subseteq \ker \phi(A)$ for every $A \in \mathcal{A}$. Assume $Ax = 0$. This together with the assumption yields $(\phi(A)x \otimes f)^2 = 0$ for every $f \in \mathcal{X}^*$. Thus $\phi(A)x = 0$ or $f(\phi(A)x) = 0$, for every $f \in \mathcal{X}^*$. Since $f$ is arbitrary, in the second case, we obtain $\phi(A)x = 0$, too. Thus by the first paragraph of the proof of Lemma 2.3, for every $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, we have $\phi(x \otimes f) = 0$ or there exists a vector $y$ such that $\phi(x \otimes f) = y \otimes f$. If $f(x) \neq 0$, then $[(x \otimes f)(x \otimes g)]^2 = 0$, for every functional $g$ such that $g(x) = 0$. This implies that

$$[(\phi(x \otimes f))(x \otimes g)]^2 = 0$$
$$\Rightarrow [(y \otimes f)(x \otimes g)]^2 = 0$$
$$\Rightarrow (f(x)y \otimes g)^2 = 0 \Rightarrow g(y) = 0.$$

Hence $x$ and $y$ are linearly dependent. If $f(x) = 0$, then by Case 2 in the proof of Lemma 2.3, we obtain that $x$ and $y$ are linearly dependent, too. Therefore, $\phi(x \otimes f) = 0$ or $\phi(x \otimes f) = kx \otimes f$ for some scalar $k$.

Let $A \in \mathcal{A} \setminus \mathcal{F}_1(\mathcal{X})$ and let $x \in \mathcal{X}$. We know $(Ax \otimes f)^2 = 0$, for every $f \in \mathcal{X}^*$ with $f(Ax) = 0$. Thus $(\phi(A)x \otimes f)^2 = 0$ and then $\phi(A)x = 0$ or $f(\phi(A)x) = 0$, which implies that $Ax$ and $\phi(A)x$ are linearly dependent for every $x \in \mathcal{X}$. Hence by [1, Theorem 2.3], there exists a scalar number $k$ such that $\phi(A) = kA$. This together with the previous discussion implies that there exists a function $\sigma : \mathcal{A} \to \mathbb{C}$ such that $\phi(A) = \sigma(A)A$ for every $A \in \mathcal{A}$.

Case 2. Let $\lambda \neq 0$. Let $A \in \mathcal{A} \setminus \mathcal{F}_1(\mathcal{X})$ and let $x \in \mathcal{X}$. There exists a linear functional $f$ such that $f(x) = 1$. Set $P = Ax \otimes f$. It is clear that $(A - P)x = 0$, and so Lemma 2.2 implies

$$(\phi(A) - \phi(P))x = 0 \Rightarrow \phi(A)x = \phi(P)x.$$

By Lemma 2.3, we have $\phi(P) = 0$ or $\phi(P) = \kappa(P)P$. If $\phi(P) = 0$, then $\phi(A)x = 0$. In the second case, if $\phi(P) = \kappa(P)P$, then $\phi(A)x = \kappa(P)Px = \kappa(P)Ax$. However, in both cases, $\phi(A)x$ and $Ax$ are linearly dependent, for every $x \in \mathcal{X}$, and so by [1, Theorem 2.3], there exists a scalar number $k$ such that $\phi(A) = kA$. This together with Lemma 2.3 follows that there exists a function $\sigma : \mathcal{A} \to \mathbb{C}$ such that $\phi(A) = \sigma(A)A$ for every $A \in \mathcal{A}$. □

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## References


1. M. Brešar and P. Šemrl, *On locally linearly dependent operators and derivations*, Trans. Amer. Math. Soc. **351** (1999) 1257–1275.
2. G.M.A. Chebotar, W.-F. Ke, P.-H. Lee and N.-C. Wong, *Mappings preserving zero products*, Studia Math. **155** (2003) 77–94.
3. M. Dobovišek, B. Kuzma, G. Lešnjak, C.K. Li and T. Petek, *Mappings that preserve pairs of operators with zero triple Jordan Product*, Linear Algebra Appl. **426** (2007) 255–279.
4. G. Dolinar, S. Du, J. Hou and P. Legiša, *General preservers of invariant subspace lattices*, Linear Algebra Appl. **429** (2008) 100–109.
5. L. Fang, *Linear maps preserving the idempotency of Jordan products of operators*, Electron. J. Linear Algebra **22** (2011) 767–779.
6. L. Fang, G. Ji and Y. Pang, *Maps preserving the idempotency of products of operators*, Linear Algebra Appl. **426** (2007) 40–52.
7. L. Molnăr, *Non-linear Jordan triple automorphisms of sets of self-adjoint matrices and operators*, Studia Math. **173** (2006) 39–48.
8. A. Taghavi and R. Hosseinzadeh, *Maps preserving the dimension of fixed points of products of operators*, Linear Multilinear Algebra **62** (2013) 1285–1292.
9. A. Taghavi, R. Hosseinzadeh and H. Rohi, *Maps preserving the fixed points of sum of operators*, Oper. Matrices **9** (2015), no. 3, 563–569.
10. A. Taghavi, F. Kolivand and H. Rohi, *A note on strong $\eta$-Lie products preserving Maps on some algebras*, Mediterr. J. Math. **14** (2017) 1285–1292.
11. M. Wang, L. Fang, G. Ji and Y. Pang, *Linear maps preserving idempotency of products or triple Jordan products of operators*, Linear Algebra Appl. **429** (2008) 181–189.


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