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PRIMENESS OF SIMPLE MODULES OVER PATH ALGEBRAS AND LEAVITT PATH ALGEBRAS

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Abstract. Let K be a field and let E be a directed graph, called quiver in the following, and let $A = KE$ be the path algebra that corresponds to E with coefficients in K. An A-module M is a c-prime module in the sense that $rm = 0$ for one $m \in M$ and $r \in A$ implies that either r annihilates all M or $m = 0$. In this article, we prove that for any acyclic graph E, an A-module M is c-prime if and only if it is simple. The primeness of simple modules over Leavitt path algebras is also discussed. We prove that some classes of simple modules over Leavitt path algebras, are not c-prime modules.

1. Introduction and preliminaries

The concept of primeness in algebraic structures was initially developed through the ideal structure of the ring. The concept of a prime ideal was introduced in [12] as the concept of a prime ring. The notion of a prime module was proposed in [9] as a generalization of the prime ideal structure of a ring. Suppose that M is a left module over the ring R (written R-module M). A proper submodule N of M is said to be prime if $rRm = 0$ with r in R and $m \in M$ implies $m \in N$ or $rM \subseteq N$; see [9]. Various different contexts and examples have been studied in [13, 20, 22]. Irawati [10, 11] generalized the concept of the hereditary Noetherian prime ring (HNP) into the concept of the HNP module. In addition, in [19, 21], it was discussed the characterization of prime submodules. A module M is said to be a c-prime module over R if $rm = 0$ with r in R and $m \in M$ implies $rM = 0$. We prove that if M is a c-prime module, then M is a prime module. If R is a commutative ring, then prime modules are c-prime modules. Ranggaswamy [16, 18]

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stated that a Leavitt path algebra behaves the same way as a commutative ring based on its ideal and module structure. In this article, we see, however, that the module structure of a Leavitt path algebra is not the same as module structure of a commutative ring. The aim of this study is to explore the notion of c-prime modules in the setting of path algebras and Leavitt path algebras.

A path algebra KE is an algebra over a field K whose basis is the set of all paths in a quiver, E. A quiver $E = (E^0, E^1, r, s)$ consists of two sets E^0 (whose elements are called vertices) and E^1 (whose elements are called arrows), and two maps $r, s : E^1 \longrightarrow E^0$, which associate to each arrow $e \in E^1$, its source $s(e) \in E^0$ and its range $r(e) \in E^0$, respectively. An algebra can be represented in the form of a quiver, and an algebra (path algebra) can be obtained from a quiver; see [7]. For any algebraically closed field K, each finite-dimensional algebra over K is Morita equivalent to a path algebra modulo an admissible ideal. A Leavitt path algebra is a specific type of path algebra, associated to a directed graph E modulo some relations.

The Leavitt path algebras were introduced in [3,5] as algebraic analogues of C^*-algebras and as natural generalizations of Leavitt algebras of type $(1, n)$, which were investigated by Leavitt [14]. The properties of these algebras, as algebras, have been actively investigated in a series of articles [1–6, 8, 17]. In [1, 2, 4, 8, 17] particularly the module theory of Leavitt path algebras has been studied.

Interestingly, in this article, we find that if KE is a path algebra, where E is an acyclic quiver and M is a module over path algebra KE, then M is simple if and only if it is c-prime. However, some simple modules over a Leavitt path algebra L for more general directed graphs E are not c-prime modules.

In Section 2 of this article, we will give the basic definitions and notation that will be used in this article. In Section 3, it is shown that a simple module over the path algebra of an acyclic quiver is a c-prime module. In Section 4, the primeness of a simple module over Leavitt path algebras is discussed.

2. Path Algebra, representation, and module

This section recalls some basic facts from the theory of path algebras. We refer to [7, 15] for more details.

A quiver $E = (E^0, E^1, r, s)$ consists of two sets E^0 (whose elements are called vertices) and E^1 (whose elements are called arrows), and two maps $s : E^1 \to E^0$ and $r : E^1 \to E^0$, which associate to each arrow $\alpha \in E^1$, its source $s(\alpha) \in E^0$, and its range $r(\alpha) \in E^0$, respectively. An arrow $\alpha \in E^1$ of source $a = s(\alpha)$ and range $b = r(\alpha)$ is usually denoted by $\alpha : a \longrightarrow b$. A quiver $E = (E^0, E^1, r, s)$ is usually denoted by $E = (E^0, E^1)$ or even simply by E. A quiver E is said to be finite if E^0 and E^1 are finite sets. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then E is called row finite. A vertex v that emits no arrows is called a sink. A vertex v that emits infinitely many arrows is called an infinite emitter.

A path in the quiver E is a sequence of arrows $p = e_1 e_2 \cdots e_n$ such that $r(e_i) = s(e_{i+1})$ for all i. A finite path in quiver E is a finite sequence of arrows $p = e_1 e_2 \cdots e_n$ where $r(e_i) = s(e_{i+1})$ for all i. In this case, the path p is said to have length n, denoted by $l(p) = n$. If p is a path such that $v = s(p) = r(p)$, then p is called a closed path based at v. If $r(p) = s(p)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then p is called a cycle. A quiver without cycles is called acyclic. For an infinite path $p = e_1e_2\cdots$, for each $n \geq 1$, define $\tau_{\leq n}(p) = e_1e_2\cdots e_n$ and $\tau_{\geq n}(p) = e_{n+1}e_{n+2}\cdots$. Two infinite paths p and q are said to be tail equivalent, denoted by $p \sim q$, if $\tau_{\geq n}(p) = \tau_{\geq m}(q)$, for some integers m, n. Clearly this is an equivalence relation. The tail-equivalence class that contains an infinite path p is denoted by $[p]$.

A path algebra $A = KE$ is an algebra over a field K whose base is the set of all paths in quiver E. Multiplication of two paths is given by concatenation if this is defined, and 0 otherwise. Extending this bilinearly, one gets an algebra structure. Note that A has a unit if and only if E has only finitely many vertices. Let E be a finite connected quiver. The two-sided ideal of A generated by all arrows in E is called an arrow ideal of A and is denoted by R_E. A two-sided ideal $\mathcal{I}$ of KE is said to be admissible if there exists $m \geq 2$ such that

$$R_E^m \subseteq \mathcal{I} \subseteq R_E^2.$$

If $\mathcal{I}$ is an admissible ideal of A, then $(E, \mathcal{I})$ is said to be a bound quiver. The quotient algebra $A/\mathcal{I}$ is called a bound quiver algebra. If A is isomorphic to a bound quiver (E, I), we visualize any (finite dimensional) A-module M as a K-linear representation of (E, I), that is, a family of (finite-dimensional) K-vector spaces M_a, with $a \in E^0$ connected by K-linear maps $\varphi_\alpha : M_a \to M_b$ corresponding to arrows $\alpha : a \to b$ in E and satisfying some relations induced by I.

Let E be a finite quiver. A representation M of E is defined as follows: Each vertex $a \in E^0$ associates to a K-vector space M_a and each arrow $\alpha : a \to b$ in E^1 associates to a K-linear map $\varphi_\alpha : M_a \to M_b$. Such a representation is denoted as $M = (M_a, \varphi_\alpha)_{a\in E^0, \alpha\in E^1}$, or simply as $M = (M_a, \varphi_\alpha)$. It is called a finite-dimensional representation if each vector space M_a is finite-dimensional.

Let $M = (M_a, \varphi_\alpha)$ and $M' = (M'_a, \varphi'_\alpha)$ be two representations of E. A morphism (of representations) $f : M \to M'$ is a family $f = (f_a)_{a\in E^0}$ of K-linear maps $(f_a : M_a \to M'_a)_{a\in E^0}$ that are compatible with the structure maps φ_α, that is, for each arrow $\alpha : a \to b$, we have $\varphi'_\alpha f_a = f_b \varphi_\alpha$ or, equivalently, the following square is commutative:

$$\begin{array}{ccc} M_a & \xrightarrow{\varphi_\alpha} & M_b \\ \downarrow f_a & & \downarrow f_b \\ M'_a & \xrightarrow{\varphi'_\alpha} & M'_b \end{array}$$

Let $f : M \to M'$ and $g : M' \to M"$ be two morphisms of representations of E, where $f = (f_a)_{a\in E^0}$ and $g = (g_a)_{a\in E^0}$. Their composition is defined to be the family $gf = (g_a f_a)_{a\in E^0}$, then gf is easily seen to be a morphism from M to $M"$.

A category is a triple $C = (Ob\ C, Hom_C, \bullet)$, where $Ob\ C$ is called the class of objects of C, Hom_C is called the class of morphisms of C, and $\bullet$ is a partial binary operation on morphisms of C satisfying the following conditions:

(1) To each pair of objects X and Y of C, we associate a set $Hom_C(X,Y)$, called the set of morphisms from X to Y, such that if $(X,Y) \neq (Z,U)$, then the intersection of the sets $Hom_C(X,Y)$ and $Hom_C(Z,U)$ is empty;
(2) For each triple of objects X,Y,Z of C, the operation $\bullet : Hom_C(Y,Z) \times Hom_C(X,Y) \to Hom_C(X,Z)$, $(g,f) \longmapsto g \circ f$ (called the composition of f and g), is defined and has the following two properties: (i)$h \circ (g \circ f) = (h \circ g) \circ f$, for every triple $f \in Hom_C(X,Y)$, $g \in Hom_C(Y,Z)$, and $h \in Hom_C(Z,U)$ of morphisms; (ii) For each object X of C, there exists an element $1_X \in Hom_C(X,X)$, called the identity morphism on X, such that if $f \in Hom_C(X,Y)$ and $g \in Hom_C(Z,X)$, then $f \circ 1_X = f$ and $1_X \circ g = g$. We often write $f : X \to Y$ or $X \to Y$ instead of $f \in Hom_C(X,Y)$, and we say that f is a morphism from X to Y. We also write $X \in Ob\ C$ to mean that X is an object of C. We say that a diagram in the category C is commutative whenever the composition of morphisms along any two paths with the same source and target are equal. For instance, we say that the diagram is commutative if $g \circ f = i \circ h$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow h & & \downarrow g \\ V & \xrightarrow{i} & Z \end{array}$$

For a commutative ring K, a category C is called K-linear if for all objects X and Y, we get that $Hom_C(X,Y)$ is a K-module and that composition of morphisms is K-bilinear.

A functor $F : C \to D$ relates two categories C and D in the following way:

(1) To each object $X \in\ Ob\ C$ it associates an object $F(X) \in\ Ob\ D$.
(2) To each map $f \in C(X,Y)$, it associates a map $Ff \in D(F(X),F(Y))$ such that the following properties hold: For each object $X \in\ Ob\ C$, $F1_X = 1_{FX}$, for a map $g \in C(X,Y)$ and $f \in C(Z,Y)$, we have $F(f \circ g) = Ff \circ Fg$.

Let $F : C \to D$ and $G : C \to D$ be two functors. Suppose that for every object $X \in C$, we have a morphism $\eta_X : C(X) \to D(X)$ in D such that for every morphism $\alpha : X \to X'$ in C the diagram is commutative :

$$\begin{array}{ccc} C(X) & \xrightarrow{\eta_X} & D(X) \\ \downarrow C(\alpha) & & \downarrow D(\alpha) \\ C(X') & \xrightarrow{\eta_{X'}} & D(X') \end{array}$$

Then we call η a natural transformation from F to G and we write $\eta : F \to G$.

Let C and D be arbitrary categories; then a functor $F : C \to D$ is a category equivalence in this case, if there are a functor $G : D \to C$ and natural isomorphisms $GF \cong 1_C$ and $FG \cong 1_D$. Two categories are equivalent in this case, if there exists a category equivalence from one to the other. We write $C \approx D$.

Furthermore, the following theorem explains that the category *mod* A whose objects are finitely generated A-modules and whose morphisms are A-linear maps, is equivalent to the category $\text{rep}_K(E, \mathcal{I})$ whose object is a K-linear representation of E that is bounded by $\mathcal{I}$ and is finite-dimensional.

Theorem 2.1 (see [7, Theorem 1.6])**.** *Let $A = KE/\mathcal{I}$ connected with finite quiver E and $\mathcal{I}$ be an admissible ideal of KE. There exists a K-linear equivalence of categories*

$$F : mod\ A \xrightarrow{\simeq} rep_K(E, \mathcal{I}).$$

By using Theorem 2.1, we can give an interpretation of a simple, projective, and injective module as bound representation. Let $a \in E^0$ define the representation $(S(a)_b, \varphi_\alpha)$ of E, denoted by $S(a)$, as follows:

$$\begin{aligned} S(a)_b &= \begin{cases} 0 & \text{if } b \neq a \\ K & \text{if } b = a, \end{cases} \\ \varphi_\alpha &= 0 \text{ for each } \alpha \in E_1 \end{aligned}$$

Clearly, $S(a)$ is a bound representation of $(E, \mathcal{I})$. Let $(E, \mathcal{I})$ be a bound quiver, let $A = KE/\mathcal{I}$, and let $P(a) = e_a A$, where $a \in E^0$. If $P(a) = (P(a)_b, \varphi_\beta)$, then $P(a)_b$ is the K-vector space with as basis the set of all $\overline{\omega} = \omega + \mathcal{I}$, with ω is a path from a to b, and for an arrow $\beta : b \to c$, the K-linear map $\varphi_\beta : P(a)_b \to P(a)_c$ is given by right multiplication by $\overline{\beta} = \beta + \mathcal{I}$. If $I(a) = (I(a)_b, \varphi_\beta)$, then $I(a)_b$ is the dual of K-vector space with as basis the set of all $\overline{\omega} = \omega + \mathcal{I}$, with ω is a path from b to a and for an arrow $\beta : b \to c$, the K-linear map $\varphi_\beta : I(a)_b \to I(a)_c$ is given by the dual of the left multiplication $\overline{\beta} = \beta + \mathcal{I}$.

3. Indecomposable c-Prime modules over path algebra of an acyclic quiver

This section discusses c-prime modules over the path algebra of an acyclic quiver.

Definition 3.1. Let M be a left A-module. We say that M is a c-prime module, if $rm = 0$ for one $m \in M$ and $r \in A$ implies that either r annihilates all M or $m = 0$.

Dauns [9] gave a general definition of the prime module.

Definition 3.2. Let M be a left R-module. Then M is a prime module if for all $r \in R$ and $m \in M$ with $rRm = 0$, then $rM = 0$.

If R is a commutative ring, then Definition 3.1 is equivalent to Definition 3.2.

Proposition 3.3. *If M is a c-prime module, then M is a prime module.*

Proof. Suppose that M is a c-prime module, and let $r \in R$, $m \in M$, with $rRm = 0$. Then it is clear that $rm = 0$. So $rM = 0$ and M is a prime module. □

Remark 3.4. For the proof of this proposition, we need to assume that R has a unit.

The goal of this section is to show that for the algebra A of an acyclic quiver, an A-module M is a simple module if and only if it is c-prime.

Theorem 3.5. *Let $A = KE$ be a path algebra and let M be an A-module, with E is an acyclic quiver. Then M is c-prime if and only if it is simple.*

Proof. Suppose that M is not a simple module, we will prove that M is not c-prime, that is, there are $0 \neq r \in R$ and $m \in M$ with $rm \neq 0$ but $r \notin Ann\ M$ and $m \neq 0$. Let M be one of the following cases as in Figures 1 and 2:

(1)

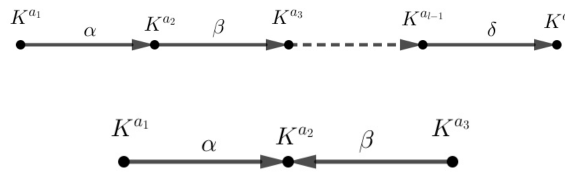

Figure 1. Line graph

(2)

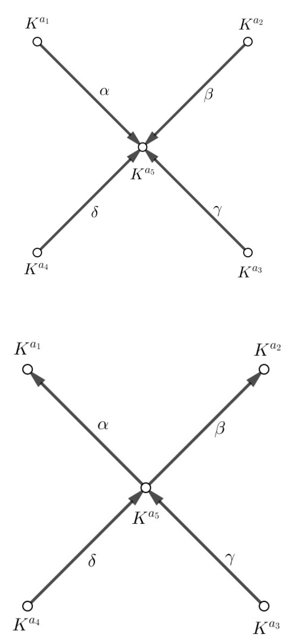

Figure 2. Tree

Suppose that $r \in KE$ and $m \in M$, where $m = (m_1, m_2, m_3, \ldots, m_l)$ with $rm = 0$. Let $m_1 = (m_{(1,1)}, m_{(2,1)}, \ldots, m_{(a_1,1)})$, $m_2 = (m_{(2,1)}, m_{(2,2)}, \ldots, m_{(a_2,2)})$, and $m_l = (m_{(l,1)}, m_{(l,2)}, \ldots, m_{(a_l,l)})$. Without loss of generality, if $r = \alpha$, then $\varphi_\alpha(m_1) = 0$. Thus $m_1 \in\ ker\ (\varphi_\alpha)$. If φ_α is injective, then $m_1 = 0$. Suppose that $m = (0, m_2, m_3, \ldots, m_l) \neq 0$. Let $m_1^{'} \notin ker(\varphi_\alpha)$; then $\alpha(m_1^{'}, 0, 0, 0, \ldots, 0) \neq 0$. Therefore $r \notin\ Ann\ M$. If φ_α is not injective, then let $0 \neq m_1 \in ker\ (\varphi_\alpha)$. Suppose that $m^{'} = (m_1^{'}, m_2^{'}, m_3^{'}, \ldots, m_l^{'})$, with $m_1^{'} \notin\ ker\ (\varphi_\alpha)$; then $rm^{'} \neq 0$, such that $r \notin Ann\ M$. So M is not a c-prime module.

Suppose that M is a simple module; then $M = (0, 0, \ldots, 0, K, 0, \ldots, 0)$. Let $r \in KE$, $m \in M$, where $rm = 0$. Let $m = (0, 0, \ldots, 0, k, 0, 0, \ldots, 0)$ with $k \in K$, where $rm = 0$; then $rm = (0, 0, \ldots, 0, rk, 0, 0, \ldots, 0)$ such that $\varphi_r(k) = 0$. Thus $k \in ker(\varphi_r)$. If φ_r is injective, then $k = 0$, such that $m = 0$. If φ_r is not injective and $n \in M$, where $n = (0, 0, \ldots, 0, s, 0, 0, \ldots, 0)$, then $rn = \varphi_r(n) = (0, 0, \ldots, 0, \varphi_r(s), 0, 0, \ldots, 0) = 0$. Therefore $r \in Ann\ M$. So M is a c-prime module. □

Now, we state Theorem 3.6 that generalizes this section.

Theorem 3.6. *Let K be a field and let A be a finite-dimensional K-algebra. Then any finite-dimensional prime A-module has the form S^n for some integer n and some simple A-module S. If in addition A is basic, then all these modules are c-prime.*

Proof. Suppose M is prime. In the first step, we show that M has to be semisimple. If M is not semisimple, then there is $a \in rad(A)$, the radical of A, with $a.M \neq 0$. Indeed, M is semisimple if and only if $rad(A).M = rad(M) = 0$. Hence, there is $m \in M$ with $a.m \neq 0$. However, the socle of M, the sum of the simple submodules of M, is not 0, that is, $soc(M) \neq 0$, and since $soc(M)$ is semisimple, $rad(A).soc(M) = 0$. Hence, for any nonzero element $x \in soc(M)$, we have $ax = 0$. Therefore M is not c-prime. So, M has to be semisimple.

If M is semisimple and $S_0 \not\cong S_1$ are two nonisomorphic simple A-modules such that $S_0 \oplus S_1$ is a direct factor of M, then $A/rad(A) = \prod_{i=1}^{n} Mat_{n_i}(D_i)$ for some skew field D_i and some integers n_i. Hence there are two idempotents e_0 and e_1 of A (correspondent to the identity in two different matrix components in the Wedderburn decomposition above) such that $e_0 S_0 = S_0$. However $e_0 S_1 = 0$ and $e_1 S_1 = S_1$ whereas $e_1 S_0 = 0$. Hence, taking x nonzero in S_1, then $e_0 x = 0$. However $e_0 M$ contains at least S_0. Hence M is not c-prime.

Suppose now that A is a basic. Then $n_i = 1$ for all i. Let $M = S^n$ for some simple A-module S and some integer n. Let e be the unit in the Wedderburn decomposition that acts as identity on S and let $\pi : A \to D_i e = D_i$ be the projection of A onto this component. Then for any $x \in M - 0$, we get $ax \neq 0$ if and only if $\phi(x) \neq 0$. Hence $ax = 0$ implies $\phi(a) = 0$ and this implies $a \in Ann_A(M)$. □

4. Leavitt path algebras and primeness of simple modules over a Leavitt path algebra

The graph (directed graph) E can be extended by adding arrows in the opposite direction. The arrows in E^1 are called the real edges while the edges in the opposite direction from the real edges are called the ghost edges. The set of all ghost edges in E is expressed as $(E^1)^*$. A Leavitt path algebra can be identified by using the extended graph E of the path algebra KE. Given $E = (E^0, E^1, r, s)$, the extended graph E as a new graph $\hat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$ with $(E^1)^* = \{e_i^* | e_i \in E^1\}$ and the functions r' and s', is defined as follows: $r'|_{E^1} = r, s_{\prime}|_{E^1} = s, r'(e_i^*) = s(e_i)$, and $s'(e_i^*) = r(e_i)$.

Following [17], we define the Leavitt path algebra of E with coefficients in K as the K-algebra generated by sets $\{v : v \in E^0\}$ and $\{e, e^* : e \in E^1\}$ that satisfy the following relations:

(1) $s(e)e = e = er(e)$, for all $e \in E^1$,
(2) $r(e)e^* = e^* = e^*s(e)$,for all $e^* \in E^1$,
(3) (CK1) $e^*f = \delta_{ef}r(e)$ for all $e, f \in E^1$,
(4) (CK2) $v = \sum_{\{e \in E_1 | s(e)=v\}} ee^*$ whenever $v \in E^0$ is not a sink.

This algebra is denoted by $L_K(E)$. The relations (3) and (4) are called the Cuntz Krieger relations. For the sake of convenience in writing, the Leavitt path algebra $L_K(E)$ will be abbreviated by L if no confusion may occur. Every element of L can be written as $a = \sum_{i=1}^{n} k_i \alpha_i {\beta_i}^*$, where $k_i \neq 0$, $k_i \in K$, and α_i, β_i are paths in E.

As noted in [1], if M is a left L-module, then we may define for each $m \in M$, the L-homomorphism $\rho : L \to M$, with $\rho_m(r) = rm$. By using ρ, Chen [8] introduced a class of simple modules over Leavitt path algebras. We will describe the general method used in [8,17,18] to construct simple modules over L by using special vertices or cycles in the graph E.

Definition 4.1. Let u be a vertex in a graph E that is either a sink or an infinite emitter. Let A_u be the K-vector space having as basis the set $B = \{p : p \textit{ paths in } E \textit{ with } r(p) = u\}$. We make A a left L-module as follows: Define, for each vertex v and each edge e in E, linear transformations P_v, S_e, and S_{e^*} on A by defining their actions on basis B as follows:

(1) For all $p \in B$,
$$P_v(p) = \begin{cases} p & if \ v = s(p), \\ 0 & others, \end{cases}$$
(2)
$$S_e(p) = \begin{cases} ep & if \ r(e) = s(p), \\ 0 & others, \end{cases}$$
(3)
$$S_{e^*}(u) = 0,$$
(4)
$$S_{e^*}(p) = \begin{cases} p' & if \ p = ep', \\ 0 & others. \end{cases}$$

Then it is straightforward to check that the endomorphisms $\{P_u, S_e, S_e^* : u \in E^0, e \in E^1\}$ satisfy the defining relations (1)–(4) of the Leavitt path algebra L. This induces an algebra homomorphism ϕ from L to $End_K(S_{v\infty})$, mapping u to P_u, e to S_e, and e^* to S_{e^*}. Then A_u can be made a left module over L via the homomorphism ϕ.

Lemma 4.2. *If the vertex u is either a sink or infinite emitter, then A_u is a simple left L-module.*

By using Definition 4.1 and Lemma 4.2, we obtain the following theorem.

Theorem 4.3. *If u is a sink, then A_u is not a c-prime module.*

Proof. To prove this theorem, without loss of generality, we reduce the case to a graph with three vertices and two edges as shown below. Let E be the graph in Figure 3. The simple module A_{v_2} is the module over L with basis as the

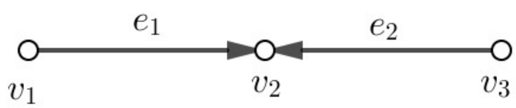

FIGURE 3. Graph E with the sink

set $B = \{p : p \textit{ paths in } E \textit{ with } r(p) = v_2\}$ and a sink v_2, and B actually only contains $\{v_2, e_1, e_2\}$. Let $r = -3v_1 + 2v_2 + 3e_1 - 2e_1^* - 2e_2^*, r \in L$, and $m = v_2 + e_1 + e_2, m \in A_{v_2}$. Then $rm = 0$. However, if we take $m_1 = e_2$ and $m_1 \in A_{v_2}$, then $rm_1 \neq 0$. So $r \notin \ Ann\ A_{v_2}$. Therefore, there is no $m \neq 0$ and $r \notin \ Ann\ A_{v_2}$ such that $rm = 0$. Thus, A_{v_2} is not a c-prime module.

The case for E with subgraph in Figure 4 has the same argument. □

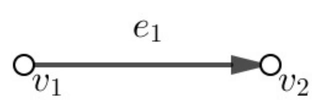

FIGURE 4. Subgraph E with sink

Theorem 4.4. *If v is an infinite emitter in E, then A_v is not a c-prime module.*

Proof. We reduce to the case with graph in Figure 5. The simple module A_{v_2} is the module over L with the set $B = \{p : p \textit{ paths in } E \textit{ with } r(p) = v_2\}$ as basis, and v_2 is an infinite emitter. It is clear $B = \{v_2, e_1, e_2\}$. Let $r = v_1 + 2e_1, r \in L$, and $m = v_2 - 2e_1 + e_2, m \in A_{v_2}$. Thus $rm = 0$. However, if we take $m_1 = e_1, m_1 \in A_{v_2}$, then $rm_1 \neq 0$. So $r \notin \ Ann\ A_{v_2}$. Therefore, there is no $m \neq 0$ and $r \notin \ Ann\ A_{v_2}$ such that $rm = 0$. Thus, A_{v_2} is not a c-prime module.

The case where E has a subgraph F as in Figure 6, has the same argument. □

Remark 4.5. Even if v_2 is not infinite emitter, then the module A_{v_2} is still not a c-prime module.

Definition 4.6. (Chen simple module $A_{[p]}$) Let $[p]$ denote the tail equivalence class of all infinite paths equivalent to p. Let $A_{[p]}$ denote the K-vector space with $[p]$ as basis. As in the definition of A_u, for each vertex v and each edge e in E,

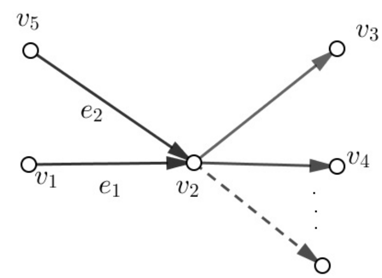

FIGURE 5. Graph E with infinite emitter

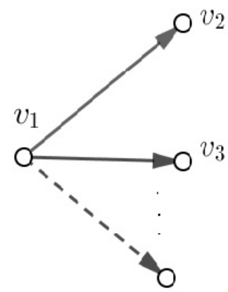

FIGURE 6. Subgraph F with infinite emitter

define the linear transformations P_v, S_e, and S_{e^*} on A by defining their actions on basis $[p]$ satisfying the conditions (1),(2),(3), but not (4), in Definition 4.1. As before, they satisfy the defining relations of a Leavitt path algebra and thus induce the homomorphism $\varphi : L \longrightarrow A_{[p]}$. The vector space $A_{[p]}$ then becomes a left L-module via the map φ.

Lemma 4.7. *If p is an infinite path, then $A_{[p]}$ is a simple left L-module.*

Theorem 4.8. *Let E be the graph in Figure 7. If p is an infinite path, then $A_{[p]}$ is not a c-prime module.*

Proof. Let E be a graph A_∞. Let $r = 5v_1 + 3v_2 - 5e_1 - 3(e_1)^*, r \in L$,

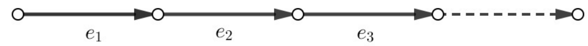

FIGURE 7. Graph A_∞

$p_1 = e_1e_2e_3\cdots$, $p_2 = e_2e_3e_4\cdots$, $m = p_1 + p_2$, and $m \in A_{[p]}$. Then $rm = 0$. However, if we take $m_1 = p_1$, then $rm_1 \neq 0$. So $r \notin Ann\ A_{[p]}$. Therefore, there is no $m \neq 0$ and $r \notin Ann\ A_{[p]}$ such that $rm = 0$. Thus, $A_{[p]}$ is not a c-prime module. □

Theorem 4.9. *Let E be the graph in Figure 8. If p is a rational infinite path,*

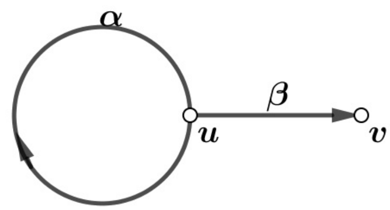

FIGURE 8. Loops with one exit

then $A_{[\alpha^\infty]}$ is a c-prime module.

Proof. Suppose that B is the basis of $A_{[\alpha^\infty]}$ and that $B = \{v, \beta, \alpha\beta, \alpha^2\beta, \ldots\}$, with $rm = x(k_1\alpha^\infty) = 0$. Then $xA_{[\alpha^\infty]} = 0$, $r \in L$, and $m \in A_{[\alpha^\infty]}$. Thus $A_{[\alpha^\infty]}$ is a c-prime module □

Theorem 4.10. *Let E be a graph excluding a single vertex and loop. There exists a simple module over Leavitt path algebras that is c-prime if and only if E is the graph in Figure 9.*

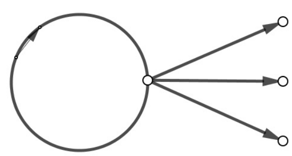

FIGURE 9. Loops with some exit

Proof. Suppose that M is a simple and c-prime module. Let $r \in L$ and let $m \in M$, and define $\varphi : L \to M$ with $\varphi(r) = rm$. It is clear that φ is surjective because M is a simple module, so $L/Ann_L M \cong M$ is a division ring. Then $Ann_L M$ is a maximal ideal; thus $Ann_L M = I$ is an ideal. Next we will show that if L/I is a division ring, then E is a graph with a single vertex or a single loop. Since $I = Ann_L M$ is an ideal of L, so $I \neq L$. Let $u \in L$ and let $u \notin I$. Let $v \in L$ and let $u \neq v$; then $(I + u)(I + v) = I + uv = 0$, so $u \in I$ or $v \in I$. However $u \notin I$, so $v \in I$ (because L/I is a division ring). So L/I has only one vertex. Let α and β be loops at u. Consider $u = \alpha\alpha^*$ and $u = \beta\beta^*$. Clearly $\alpha, \alpha^*, \beta, \beta^* \notin I$, but $(I + \alpha^*)(I + \beta) = I + \alpha^*\beta = 0$. Because L/I is a division ring, $\alpha^* \in I$ or $\beta \in I$ (contradiction). So L/I has only one loop. Because E is not only one loop and there is no other loop at u, E is the graph in Figure 9.

Suppose that E is the graph in Figure 9. Let p be a rational infinite path in graph E. By using Definition 4.6 and Lemma 4.7, it is clear that M is a simple module over a Leavitt path algebra. Similarly, using Theorem 4.9, we can see that M is a c-prime module. □

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