Khayyam J. Math. 8 (2022), no. 1, 7-16
DOI: 10.22034/KJM.2021.255330.2045

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

NONCOMMUTATIVE CONVEXITY IN MATRICIAL $*$-RINGS

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Communicated by A.M. Peralta

Abstract. For every unital $*$-ring $\mathcal{R}$, we define the notions of $\mathcal{R}$-convexity, as a kind of noncommutative convexity, $\mathcal{R}$-face and $\mathcal{R}$-extreme point, the relative face, and extreme point, for general bimodules over $\mathcal{R}$. The relation between the C^*-convex subsets of $\mathcal{R}$ and $\mathcal{R}$-convex subsets of $M_n(\mathcal{R})$, the set of all $n \times n$ matrices with entries in $\mathcal{R}$, as well as, the relation between the C^*-faces (C^*-extreme points) of these C^*-convex sets and $\mathcal{R}$-faces ($\mathcal{R}$-extreme points) of $\mathcal{R}$-convex sets in $M_n(\mathcal{R})$ is given. Also, we prove the same results for diagonal matrices in $M_n(\mathcal{R})$. Finally, we show that, if the entries are restricted to the positive elements in the unital $*$-ring $\mathcal{R}$, then the set of all diagonal matrices is an $\mathcal{R}$-face of the set of all lower (upper) triangular matrices, and all of these sets are $\mathcal{R}$-faces of $M_n(\mathcal{R}^+)$.

1. Introduction

One of the forms of noncommutative convexity is C^*-convexity. Formal study of C^*-convexity was initiated by Loebl and Paulsen [12]. Farenick and Morenz [8] proved that each irreducible element of the C^*-algebra M_n of complex $n \times n$ matrices, is a C^*-extreme point, the relative extreme point, of the C^*-convex set that it generates. Morenz [14] obtained a right analog of linear extreme points, called structural elements, to prove a generalized Krein Milman theorem for C^*-convex subsets of M_n. Also he extended the notion of face from convexity to C^*-face in C^*-convexity. Kian has worked on this subject in several articles such as [9–11]. In [9], the concept of C^*-convexity has generalized to the sets that have a $B(H)$-module structures, and a generalization of the classical well-known result "f is a convex function if and only if $epi(f)$ is a convex set" has been obtained for operator convex functions. Esslamzadeh and others have investigated the

Date: Received: 1 November 2020; Revised: 22 January 2021; Accepted: 23 January 2021.
2020 *Mathematics Subject Classification.* Primary 52A01; Secondary 16W10.
Key words and phrases. $\mathcal{R}$-convex set, $\mathcal{R}$-face, $\mathcal{R}$-extreme point, $*$-ring.

quantization in $*$-algebras and the structure of quasi-ordered $*$-vector spaces in several articles such as [5–7]. The author and Esslamzadeh [4] generalized the notions of C^*-convexity and C^*-extreme points to $*$-rings. Recently, the author has proved some results on the subject of C^*-convex maps and C^*-affine maps on $*$-rings in [2, 3], and he has investigated the C^*-extreme points of the graph and epigraph of C^*-affine maps. Also, it has shown that for a C^*-convex map f defined on a unital $*$-ring $\mathcal{R}$ with some conditions, the graph of f is a C^*-face of the epigraph of f, and some other results about the C^*-faces of C^*-convex sets in $*$-rings; see [2].

Throughout this article, $\mathcal{R}$ is a unital $*$-ring, that is, a ring with an involution that has an identity element. An element x in $\mathcal{R}$ is called positive, written $x \geq 0$, if $x = y_1^* y_1 + y_2^* y_2 + \cdots + y_n^* y_n$ for some $y_1, y_2, \ldots, y_n$ in $\mathcal{R}$, and the set of all positive elements in $\mathcal{R}$ is denoted by $\mathcal{R}^+$. The self-adjoint elements of $\mathcal{R}$ may be ordered by writing $x \leq y$ in case $y - x \geq 0$. The reference [1] is a basic reference for studying the $*$-rings.

For C^*-subalgebras $A, B \subseteq B(H)$, Magajna [13] has considered A, B-absolutely convex sets in A, B-subbimodules of $B(H)$, to prove a separation type theorem for C^*-convex subsets of operator bimodules over C^*-algebras and von Neumann algebras. In special case, an extension of C^*-convexity to A-subbimodules of $B(H)$ has defined there. Also, as mentioned above, this generalization has considered in [9] for the sets that have a $B(H)$-module structures. In this article, we consider the same generalization for bimodules over $*$-rings, which we call it $\mathcal{R}$-convexity for bimodules over unital $*$-ring $\mathcal{R}$. More precisely, we define the notions of $\mathcal{R}$-convexity, $\mathcal{R}$-face, and $\mathcal{R}$-extreme point in $\mathcal{R}$-bimodules. Then we focus on the special $\mathcal{R}$-bimodule $M_n(\mathcal{R})$, the set of all $n \times n$ matrices with entries in the unital $*$-ring $\mathcal{R}$, for each $n \in \mathbb{N}$. Note that $M_n(\mathcal{R})$ can be considered also as a $*$-ring with the usual matrix operations and $*$-transposition as an involution, that is, $[a_{ij}]^* = [(a_{ji})^*]$. We show the relation between the C^*-convex subsets of $\mathcal{R}$ and $\mathcal{R}$-convex subsets of $M_n(\mathcal{R})$, as well as, the relation between their relative faces. Also we show that considering an extra condition on $\mathcal{R}$, we have $\mathcal{R} - ext(M_n(K)) = M_n(C^* - ext(K))$ for every C^*-convex subset K of $\mathcal{R}$. Also we prove that the same conclusions hold for diagonal matrices on $\mathcal{R}$. Moreover, we prove that, if the entries restricted to the positive elements in $\mathcal{R}$, then the set of all diagonal matrices is an $\mathcal{R}$-face of the set of all lower (upper) triangular matrices. Furthermore, we show that all of these sets are $\mathcal{R}$-faces of $M_n(\mathcal{R}^+)$.

We use the notation $a^*[a_{ij}]a$ for

$$\operatorname{diag}(a^*) \cdot [a_{ij}] \cdot \operatorname{diag}(a) = \begin{bmatrix} a^* & & 0 \\ & \ddots & \\ 0 & & a^* \end{bmatrix} \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{bmatrix}.$$

2. Definitions and preliminaries

Definition 2.1. A subset K of a unital $*$-ring $\mathcal{R}$ is called C^*-convex, if

$$\sum_{i=1}^{n} a_i^* x_i a_i \in K,$$

whenever $x_i \in K$ and $a_i \in \mathcal{R}$ for all i and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$.

In this case, the summation $\sum_{i=1}^{n} a_i^* x_i a_i$ is called a C^*-convex combination of elements $x_i \in K$.

Definition 2.2. Let K be a C^*-convex subset of $\mathcal{R}$. An element $x \in K$ is called a C^*-extreme point of K if the condition

$$x = \sum_{i=1}^{n} a_i^* x_i a_i, \ \sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}, x_i \in K, a_i \text{ is invertible in } \mathcal{R}, n \in \mathbb{N} \tag{2.1}$$

implies that all x_i are unitarily equivalent to x in $\mathcal{R}$, that is, there exist unitaries $u_i \in \mathcal{R}$ such that $x_i = u_i^* x u_i$ for all i.

The set of all C^*-extreme points of K is denoted by C^*-$ext(K)$. In addition, if Condition (2.1) holds, then we say that x is a proper C^*-convex combination of $x_1, \ldots, x_n$.

In the next two definitions, we extend the notions of C^*-convexity and C^*-extreme points to bimodules over $*$-rings, respectively.

Definition 2.3. Let $\mathcal{R}$ be a unital $*$-ring, let M be an $\mathcal{R}$-bimodule, and let $K \subseteq M$. Then K is called an $\mathcal{R}$-convex subset of M if $\sum\limits_{i=1}^{m} a_i^* X_i a_i \in M$ whenever, $X_i \in M$, $a_i \in \mathcal{R}$ for each i, $\sum\limits_{i=1}^{m} a_i^* a_i = 1_{\mathcal{R}}$ and $m \in \mathbb{N}$.

In this case, the summation $\sum_{i=1}^{m} a_i^* X_i a_i$ is called an $\mathcal{R}$-convex combination of elements $X_i \in M$, and it is called a proper $\mathcal{R}$-convex combination if a_i is invertible in $\mathcal{R}$ for each i.

Definition 2.4. Let K be an $\mathcal{R}$-convex subset of the $\mathcal{R}$-bimodule M. An element $X \in K$ is called an $\mathcal{R}$-extreme point of K if the condition

$$X = \sum_{i=1}^{m} a_i^* X_i a_i, \ \sum_{i=1}^{m} a_i^* a_i = 1_{\mathcal{R}}, X_i \in M, a_i \text{ is invertible in } \mathcal{R}, m \in \mathbb{N},$$

implies that all X_i's are unitarily equivalent to X in M, in the sense that, there exist unitaries $u_i \in \mathcal{R}$ such that $X_i = u_i^* X u_i$ for all i.

The set of all $\mathcal{R}$-extreme points of K is denoted by $\mathcal{R}$-$ext(K)$.

Note that in case $M = \mathcal{R}$, as an $\mathcal{R}$-bimodule, the above two definitions release to the usual definitions of C^*-convexity and C^*-extreme point in $*$-rings, respectively.

Morenz [14] has extended the notion of face from classical convexity to C^*-convexity. Also, this notion has extended to C^*-convexity in $*$-rings in [4], as follows.

Definition 2.5. A nonempty subset F of a C^*-convex set $K \subseteq \mathcal{R}$ is called a C^*-face of K, if the condition $x \in F$ and $x = \sum_{i=1}^{n} a_i^* x_i a_i$ as a proper C^*-convex combination of elements $x_i \in K$, implies that $x_i \in F$ for all i.

In the next definition, we extend the notion of C^*-face to $\mathcal{R}$-convex subsets of $\mathcal{R}$-bimodules.

Definition 2.6. Let $\mathcal{R}$ be a unital $*$-ring, let M be an $\mathcal{R}$-bimodule, and let K be an $\mathcal{R}$-convex subset of M. A nonempty subset F of K is called an $\mathcal{R}$-face of K, if the conditions $X \in F$ and $X = \sum_{i=1}^{n} a_i^* X_i a_i$ as a proper $\mathcal{R}$-convex combination of elements $X_i \in K$, imply that $X_i \in F$ for all i.

3. Main results

In this section, we concentrate on the set $M_n(\mathcal{R})$, the set of all $n \times n$ matrices with entries in $\mathcal{R}$, as an $\mathcal{R}$-bimodule, where $\mathcal{R}$ is an arbitrary unital $*$-ring, and we attempt to obtain the relation between the C^*-convex subsets of $\mathcal{R}$ and $\mathcal{R}$-convex subsets of $M_n(\mathcal{R})$. Also we investigate the relations between the C^*-faces (C^*-extreme points) of the C^*-convex subsets of $\mathcal{R}$ and $\mathcal{R}$-faces ($\mathcal{R}$-extreme points) of the $\mathcal{R}$-convex subsets of $M_n(\mathcal{R})$.

Theorem 3.1. *K is a C^*-convex subset of $\mathcal{R}$ if and only if $M_n(K)$ is an $\mathcal{R}$-convex subset of $M_n(\mathcal{R})$.*

Proof. Let K be a C^*-convex subset of $\mathcal{R}$, and let $X = \sum\limits_{t=1}^{m} a_t^* Y_t a_t$ be an $\mathcal{R}$-convex combination of elements $Y_t \in M_n(K)$. We must show that $X = [X_{ij}] \in M_n(K)$. We have $X_{ij} = \sum\limits_{t=1}^{m} a_t^* (Y_t)_{ij} a_t$ for each i, j $(1 \leq i, j \leq n)$. So X_{ij} is written as a C^*-convex combination of elements $(Y_t)_{ij} \in K$, and hence $X_{ij} \in K$.

Conversely, suppose that $M_n(K)$ is an $\mathcal{R}$-convex subset of $M_n(\mathcal{R})$ and that $x = \sum\limits_{t=1}^{m} a_t^* y_t a_t$ is a C^*-convex combination of elements $y_t \in K$. Put

$$X = \begin{bmatrix} x & \dots & x \\ \vdots & \ddots & \vdots \\ x & \dots & x \end{bmatrix}_{n\times n} \quad \text{and} \quad Y_t = \begin{bmatrix} y_t & \dots & y_t \\ \vdots & \ddots & \vdots \\ y_t & \dots & y_t \end{bmatrix}_{n\times n},$$

for all $t\,(1 \leq t \leq m)$. Then $Y_t \in M_n(K)$ and $X = \sum\limits_{t=1}^{m} a_t^* Y_t a_t$. So by the $\mathcal{R}$-convexity of $M_n(K)$, we conclude that $X \in M_n(K)$, and hence $x \in K$. Thus, K is a C^*-convex subset of $\mathcal{R}$. $\square$

Theorem 3.2. *F is a C^*-face of the C^*-convex set K in $\mathcal{R}$ if and only if $M_n(F)$ is an $\mathcal{R}$-face of $M_n(K)$ in $M_n(\mathcal{R})$.*

Proof. Let F be a C^*-face of K, let $X = [x_{ij}] \in M_n(F)$, and let $X = \sum\limits_{t=1}^{m} a_t^* Y_t a_t$ be a proper $\mathcal{R}$-convex combination of elements $Y_t = [(y_t)_{ij}] \in M_n(K)$. We must show that $Y_t \in M_n(F)$ for all t. For each $i, j\,(1 \leq i, j \leq n)$, we have $x_{ij} = \sum\limits_{t=1}^{m} a_t^* (y_t)_{ij} a_t$, and hence $(y_t)_{ij} \in F$ for each i, j and each $t\,(1 \leq t \leq m)$. So $Y_t \in M_n(F)$ for all t, and hence $M_n(F)$ is an $\mathcal{R}$-face of $M_n(K)$.

Conversely, suppose that $M_n(F)$ is an $\mathcal{R}$-face of $M_n(K)$, that $x \in F$, and that $x = \sum_{i=1}^{m} a_i^* y_i a_i$ is a proper C^*-convex combination of elements $y_i \in K$. Put

$$X = \begin{bmatrix} x & \dots & x \\ \vdots & \ddots & \vdots \\ x & \dots & x \end{bmatrix}_{n\times n} \quad \text{and} \quad Y_i = \begin{bmatrix} y_i & \dots & y_i \\ \vdots & \ddots & \vdots \\ y_i & \dots & y_i \end{bmatrix}_{n\times n},$$

for each $i\,(1 \le i \le m)$. Then $X \in M_n(F)$ and $X = \sum_{i=1}^{m} a_i^* Y_i a_i$ is a proper $\mathcal{R}$-convex combinations of elements $Y_i \in M_n(K)$. So $Y_i \in M_n(F)$, and hence $y_i \in F$ for each i. Therefore, F is a C^*-face of K. □

Note that the above conclusions hold for all rectangular matrices $M_{m,n}(F)$ and $M_{m,n}(K)$ by the same proofs.

Theorem 3.3. *Let K be a C^*-convex subset of the unital $*$-ring $\mathcal{R}$, such that for each $x \in K$, $C^* - co(\{x\}) = \{x\}$. Then $M_n(C^*{-}ext(K)) = \mathcal{R}{-}ext(M_n(K))$.*

Proof. Let $X = [x_{ij}] \in \mathcal{R} - ext(M_n(K))$ and let $x_{lk} = \sum_{t=1}^{m} a_t^* y_t a_t$ be a proper C^*-convex combination of elements $y_t \in K$ for fixed $l, k \in \{1, \dots, n\}$. Also, suppose that $Y_t = [(Y_t)_{ij}]$, where

$$(Y_t)_{ij} = \begin{cases} x_{ij}, & (i,j) \neq (l,k), \\ y_t, & (i,j) = (l,k). \end{cases}$$

Then, $X = \sum_{t=1}^{m} a_t^* Y_t a_t$ is a proper $\mathcal{R}$-convex combination of elements $Y_t \in M_n(K)$, and hence X is unitarily equivalent to Y_t for each $t\,(1 \le t \le m)$. So there exist unitary elements $u_t \in \mathcal{R}$ such that $X = u_t^* Y_t u_t$ for each t. Thus, $x_{lk} = u_t^* y_t u_t$, and hence x_{lk} is a C^*-extreme point of K. Since l and k are arbitrary in $\{1, \dots, n\}$, so $X \in M_n(C^* - ext(K))$.

Conversely, suppose that for each $i, j\,(1 \le i, j \le n)$, $x_{ij} \in C^*{-}ext(K)$ and that $X = [x_{ij}] = \sum_{t=1}^{m} a_t^* Y_t a_t$ is a proper $\mathcal{R}$-convex combination of elements $Y_t \in M_n(K)$. Then $x_{ij} = \sum_{t=1}^{m} a_t^* (Y_t)_{ij} a_t$ is a proper C^*-convex combination of elements $(Y_t)_{ij} \in K$, and hence there are unitaries $(u_t)_{ij} \in \mathcal{R}$ such that $x_{ij} = ((u_t)_{ij})^* (Y_t)_{ij} (u_t)_{ij}$. Since $C^* - co(\{x\}) = \{x\}$ for each $x \in K$, so we have $x_{ij} = (Y_t)_{ij}$, and hence $X = Y_t$ for all $t\,(1 \le t \le m)$. Therefore, $X \in \mathcal{R} - ext(M_n(K))$. □

Example 3.4. Let $\mathcal{R}$ be a unital $*$-ring. Then the following sets are $\mathcal{R}$-convex in $M_n(\mathcal{R})$:

(1) The set $M_n(\mathcal{R})$ of all $n \times n$ matrices with entries in $\mathcal{R}$.
(2) The set $UT_n(\mathcal{R})$ of all $n \times n$ upper triangular matrices with entries in $\mathcal{R}$.
(3) The set $LT_n(\mathcal{R})$ of all $n \times n$ lower triangular matrices with entries in $\mathcal{R}$.
(4) The set $D_n(\mathcal{R})$ of all $n \times n$ diagonal matrices with entries in $\mathcal{R}$.

(5) The set of all $n \times n$ symmetric (antisymmetric) matrices with entries in $\mathcal{R}$.

In the next proposition, we give some other examples of $\mathcal{R}$-convex subsets of $M_n(\mathcal{R})$.

Proposition 3.5. *The following sets are $\mathcal{R}$-convex subsets of $M_n(\mathcal{R})$ for every unital $*$-ring $\mathcal{R}$.*

1. *The set $M_n^{sa}(\mathcal{R})$ of all $n \times n$ self-adjoint matrices with entries in $\mathcal{R}$.*
2. *The set $M_n^{+}(\mathcal{R})$ of all $n \times n$ positive matrices with entries in $\mathcal{R}$.*
3. *The set of all $n \times n$ (column) row stochastic matrices with entries in $\mathcal{R}$.*
4. *The set of all $n \times n$ doubly stochastic matrices with entries in $\mathcal{R}$.*

Proof. (1) Let $X_i \in M_n^{sa}(\mathcal{R})$ and let $a_i \in \mathcal{R}$ such that $\sum_{i=1}^{m} a_i^* a_i = 1_{\mathcal{R}}$. We must show that $X = \sum_{i=1}^{m} a_i^* X_i a_i \in M_n^{sa}(\mathcal{R})$. Since $(a_i^* X_i a_i)_{kl} = a_i^*(X_i)_{kl} a_i$, so

$$\begin{aligned}\left(\sum_{i=1}^{m} a_i^* X_i a_i\right)_{kl} &= \sum_{i=1}^{m} (a_i^* X_i^* a_i)_{kl} = \sum_{i=1}^{m} a_i^* (X_i^*)_{kl} a_i \\ &= \sum_{i=1}^{m} a_i^* ((X_i)_{lk})^* a_i = \left(\left(\sum_{i=1}^{m} a_i^* X_i a_i\right)_{lk}\right)^* \\ &= \left(\left(\sum_{i=1}^{m} a_i^* X_i a_i\right)^*\right)_{kl} = (X^*)_{kl}.\end{aligned}$$

Thus $X_{kl} = (X^*)_{kl}$ for each $1 \leq k, l \leq m$ and hence $X = X^*$. Therefore, $X \in M_n^{sa}(\mathcal{R})$.
(2) If $X_i \geq 0$, then $a_i^* X_i a_i = (a_i I_n)^* X_i (a_i I_n) \geq 0$, and hence

$$\sum_{i=1}^{m} a_i^* X_i a_i = \sum_{i=1}^{m} (a_i I_n)^* X_i (a_i I_n) \geq 0.$$

Therefore, $M_n^{+}(\mathcal{R})$ is an $\mathcal{R}$-convex set.
(3) Suppose that X_i is an $n \times n$ row stochastic matrix for each i $(1 \leq i \leq m)$, that is, $\sum_{l=1}^{n} (X_i)_{kl} = 1_{\mathcal{R}}$, for each k $(1 \leq k \leq n)$. Then,

$$\begin{aligned}\sum_{l=1}^{n} \left[\sum_{i=1}^{m} a_i^* X_i a_i\right]_{kl} &= \sum_{l=1}^{n} \sum_{i=1}^{m} a_i^* [X_i]_{kl} a_i = \sum_{i=1}^{m} \sum_{l=1}^{n} a_i^* [X_i]_{kl} a_i \\ &= \sum_{i=1}^{m} a_i^* \left(\sum_{l=1}^{n} [X_i]_{kl}\right) a_i = \sum_{i=1}^{m} a_i^* 1_{\mathcal{R}} a_i = 1_{\mathcal{R}}.\end{aligned}$$

Similarly the set of all column stochastic matrices is also an $\mathcal{R}$-convex set.
(4) It is a straightforward conclusion of part (3). □

Theorem 3.6. *Let $\mathcal{R}$ be a unital $*$-ring. Then the following properties hold:*

(i) *K is a C^*-convex subset of $\mathcal{R}$ if and only if $D_n(K)$ is an $\mathcal{R}$-convex subset of $D_n(\mathcal{R})$.*
(ii) *F is a C^*-face of K if and only if $D_n(F)$ is an $\mathcal{R}$-face of $D_n(K)$.*
(iii) *$\{x_1, \ldots, x_n\} \subseteq C^*{-}ext(K)$ if and only if $\operatorname{diag}(x_1, \ldots, x_n) \in \mathcal{R}{-}ext(D_n(K))$, provided that, $C^* - co(\{x\}) = \{x\}$ for all $x \in K$.*

Proof. The proof of (i) and (ii) is straightforward by noting the following equalities:

$$\sum_{i=1}^{m} a_i^* \begin{bmatrix} x_{i1} & & & 0 \\ & x_{i2} & & \\ & & \ddots & \\ 0 & & & x_{in} \end{bmatrix} a_i = \sum_{i=1}^{m} \begin{bmatrix} a_i^* x_{i1} a_i & & & 0 \\ & a_i^* x_{i2} a_i & & \\ & & \ddots & \\ 0 & & & a_i^* x_{in} a_i \end{bmatrix}$$
$$= \begin{bmatrix} \sum\limits_{i=1}^{m} a_i^* x_{i1} a_i & & & 0 \\ & \sum\limits_{i=1}^{m} a_i^* x_{i2} a_i & & \\ & & \ddots & \\ 0 & & & \sum\limits_{i=1}^{m} a_i^* x_{in} a_i \end{bmatrix}$$

(iii) Let $x_i \in C^* - ext(K)$, let $y_{it} \in K$, and let a_t be an invertible element in $\mathcal{R}$ for each t $(1 \le t \le n)$ such that $\sum\limits_{t=1}^{m} a_t^* a_t = 1_{\mathcal{R}}$ and

$$\begin{bmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n \end{bmatrix} = \sum_{t=1}^{m} a_t^* \begin{bmatrix} y_{1t} & & & 0 \\ & y_{2t} & & \\ & & \ddots & \\ 0 & & & y_{nt} \end{bmatrix} a_t.$$

Then for each $i\,(1 \le i \le n)$, $x_i = \sum\limits_{t=1}^{m} a_t^* y_{it} a_t$, and hence x_i is unitarily equivalent to y_{it} for all t. So $y_{it} = x_i$, by the assumption that the C^*-convex hull of each element x in K is the singleton $\{x\}$. Therefore,

$$\begin{bmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n \end{bmatrix} = \begin{bmatrix} y_{1t} & & & 0 \\ & y_{2t} & & \\ & & \ddots & \\ 0 & & & y_{nt} \end{bmatrix},$$

for each $t\,(1 \le t \le m)$. So $\operatorname{diag}(x_1, \ldots, x_n) \in \mathcal{R} - ext(D_n(K))$.

Conversely, suppose that $\operatorname{diag}(x_1, \ldots, x_n) \in \mathcal{R} - ext(D_n(K))$ and that $x_i = \sum\limits_{t=1}^{m} a_t^* y_t a_t$ is a proper C^*-convex combination of elements $y_t \in K$. Put

$$Y_t = \operatorname{diag}(x_1, \ldots, x_{i-1}, y_t, x_{i+1}, \ldots, x_n),$$

for each $t\,(1 \leq t \leq m)$. Then,

$$\begin{bmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n \end{bmatrix} = \sum_{t=1}^{m} a_t^* Y_t a_t, \tag{3.1}$$

since $\sum_{t=1}^{m} a_t^* x_j a_t = x_j$ by the assumption $C^* - co(\{x\}) = \{x\}$. On the other hand, $\operatorname{diag}(x_1, \ldots, x_n)$ is an $\mathcal{R}$-extreme point of $D_n(K)$, so the relation (3.1) implies that

$$\operatorname{diag}(x_1, \ldots, x_n) = u_t^* Y_t u_t \quad \text{for all } t,$$

for unitaries $u_t \in \mathcal{R}$. Hence, $x_i = u_t^* y_t u_t$, and therefore $x_i \in C^* - ext(K)$. □

Proposition 3.7. *Let $UT_n(\mathcal{R}^+)$, $LT_n(\mathcal{R}^+)$, and $D_n(\mathcal{R}^+)$ denote the sets of all upper triangular, lower triangular, and diagonal $n \times n$ matrices with positive entries in $\mathcal{R}$, respectively. Then $D_n(\mathcal{R}^+)$ is an $\mathcal{R}$-convex set in $M_n(\mathcal{R})$, and also an $\mathcal{R}$-face of the $\mathcal{R}$-convex sets $UT_n(\mathcal{R}^+)$ and $LT_n(\mathcal{R}^+)$ provided that, for each $x_i \in \mathcal{R}$, and $n \in \mathbb{N}$, $x_1^* x_1 + \cdots + x_n^* x_n = 0$ implies that $x_1 = \cdots = x_n = 0$.*

Proof. It is clear that $D_n(\mathcal{R}^+)$, $LT_n(\mathcal{R}^+)$, and $UT_n(\mathcal{R}^+)$ are $\mathcal{R}$-convex sets since $\left[\sum_{i=1}^{m} a_i^* A_i a_i\right]_{kl} = \sum_{i=1}^{m} a_i^* [A_i]_{kl} a_i$. Let $D \in D_n(\mathcal{R}^+)$ and let $D = \sum_{i=1}^{m} a_i^* B_i a_i$, where $a_i \in \mathcal{R}$ is invertible and $B_i \in UT_n(\mathcal{R}^+)$ for each i. We must show that $B_i \in D_n(\mathcal{R}^+)$ for each $i\,(1 \leq i \leq m)$. Since $(B_i)_{kl} \in \mathcal{R}^+$, so there are $x_{i,kl,j} \in \mathcal{R}$ for $1 \leq j \leq N_{i,kl}$ such that $(B_i)_{kl} = \sum_{j=1}^{N_{i,kl}} x_{i,kl,j}^* x_{i,kl,j}$ for each $i \in \{1, \ldots, m\}$ and $k, l \in \{1, \ldots, n\}$. If $k \neq l$, then

$$\begin{aligned} 0 = D_{kl} &= \sum_{i=1}^{m} (a_i^* (B_i)_{kl} a_i) \\ &= \sum_{i=1}^{m} a_i^* \left(\sum_{j=1}^{N_{i,kl}} x_{i,kl,j}^* x_{i,kl,j} \right) a_i \\ &= \sum_{i=1}^{m} \sum_{j=1}^{N_{i,kl}} \left(a_i^* x_{i,kl,j}^* x_{i,kl,j} a_i \right) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{N_{i,kl}} (x_{i,kl,j} a_i)^* (x_{i,kl,j} a_i). \end{aligned}$$

Therefore, $x_{i,kl,j} a_i = 0$. Invertibility of a_i implies that $x_{i,kl,j} = 0$ for each $j\,(1 \leq j \leq N_{i,kl})$. Thus, $(B_i)_{kl} = 0$, and therefore $B_i \in D_n(\mathcal{R}^+)$ for each $i\,(1 \leq i \leq m)$. Similarly, we can prove that $D_n(\mathcal{R}^+)$ is an $\mathcal{R}$-face of $LT_n(\mathcal{R}^+)$. □

Proposition 3.8. *The sets $D_n(\mathcal{R}^+)$, $UT_n(\mathcal{R}^+)$, and $LT_n(\mathcal{R}^+)$ are $\mathcal{R}$-faces of the $\mathcal{R}$-convex set $M_n(\mathcal{R}^+)$ if for each $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$, the following implication holds:*

$$x_1^* x_1 + \cdots + x_n^* x_n = 0 \Longrightarrow x_1 = \cdots = x_n = 0.$$

Proof. The proposition can be proved similar to the previous proposition, and hence we omit the proof. □

Remark 3.9. We can replace the following condition instead of the condition that $\sum_{i=1}^{n} x_i^* x_i = 0$ implies that $x_i = 0$ for all i, in the above proposition:
For each $x_i \in \mathcal{R}^+$ and every invertible elements $a_i \in \mathcal{R}$ satisfying $\sum_{i=1}^{m} a_i^* a_i = 1_{\mathcal{R}}$, the condition $\sum_{i=1}^{m} a_i^* x_i a_i = 0$ implies that $x_i = 0$ for all i.

Open problem. Are there exist the same conclusions for the general bimodules over the unital $*$-rings, that is, is there any relation between the C^*-convex subsets of $\mathcal{R}$ and $\mathcal{R}$-convex subsets of the $\mathcal{R}$-bimodule M, and also between their appropriate faces and extreme points in general case?

Acknowledgement. The author would like to express his sincere gratitude for referees valuable comments and suggestions to improve the article.

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