Khayyam J. Math. 8 (2022), no. 1, 85-93
DOI: 10.22034/kjm.2021.229343.1817

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

INITIAL VALUE PROBLEMS FOR NONLINEAR CAPUTO FRACTIONAL RELAXATION DIFFERENTIAL EQUATIONS

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Communicated by P.I. Naumkin

Abstract. The main purpose of this article is to establish the existence and uniqueness of solutions for a class of fractional relaxation differential equations. Existence and uniqueness results are based on the Krasnoselskii fixed point theorem and Banach contraction mapping principle. Finally, an example is given to illustrate this work.

1. Introduction

The concept of fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary noninteger order. Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, and so on. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors; see [1–24], [25, 27–31] and the references therein. Chidouh, Guezane-Lakoud, and Bebbouchi [14] discussed the existence and uniqueness of positive solutions of a nonlinear Caputo fractional relaxation differential equation. Ardjouni and Djoudi [6] studied the positivity of solutions of a nonlinear Caputo fractional differential equation. Seemab and Ur Rehman [25]

Date: Received: 30 April 2020; Revised: 6 October 2021; Accepted: 10 October 2021.
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2020 *Mathematics Subject Classification.* Primary 34A08; Secondary 34A12.

Key words and phrases. Fixed points, fractional relaxation differential equations, Mittag-Leffler, existence, uniqueness.

investigated the existence and stability analysis for a nonlinear Caputo fractional differential equation.

Inspired and motivated by the works mentioned above, we study the existence and uniqueness of solutions for the following initial value problem of the fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}\left(u\left(t\right)-g\left(t,u\left(t\right)\right)\right)+\omega u\left(t\right)=f\left(t,u\left(t\right)\right), \ t\in\left(0,T\right], \\ u\left(0\right)=u_{0}, \end{cases} \tag{1.1}$$

where ${}^{C}D^{\alpha}$ is the standard Caputo's fractional derivative of order $0<\alpha\leq 1$, $\omega>0$, and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ and $g:[0,T]\times\mathbb{R}\to\mathbb{R}$ are given continuous functions. To show the existence and uniqueness of solutions, we transform (1.1) into an equivalent integral equation and then use the Krasnoselskii and Banach fixed point theorems.

The rest of this article is organized as follows. In Section 2, we introduce some definitions and lemmas and state some preliminaries results needed in later sections. Also, we present the Banach and Krasnoselskii fixed point theorems. In Section 3, we prove the existence and uniqueness of solutions for (1.1). Finally, an example is given to illustrate our main results.

2. Preliminaries

In this section, we present some basic definitions, notations, and results of fractional calculus, which are used throughout this article.

Let $T>0$, and let $J=[0,T]$. By $C\left(J,\mathbb{R}\right)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|u\|=\sup\left\{u\left(t\right):t\in J\right\}.$$

Definition 2.1 ([19]). The fractional integral of order $\alpha>0$ of a function $u:J\to\mathbb{R}$ is given by

$$I^{\alpha}u\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-s\right)^{\alpha-1}u\left(s\right)ds,$$

provided the right side is pointwise defined on J.

Definition 2.2 ([19]). *The Caputo fractional derivative of order* $\alpha>0$ of function $u:J\to\mathbb{R}$ is given by

$${}^{C}D^{\alpha}u(t)=I^{n-\alpha}D^{(n)}u\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_{0}^{t}\left(t-s\right)^{n-\alpha-1}u^{(n)}\left(s\right)ds,$$

where $n=[\alpha]+1$, provided the right side is pointwise defined on J.

Definition 2.3 ([9]). The two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}\left(z\right)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+\beta)}, \quad \alpha>0,\ \beta\in\mathbb{C},\ z\in\mathbb{C}.$$

For $\beta = 1$, we obtain the Mittag-Leffler function in one parameter

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0,\ z \in \mathbb{C}.$$

Lemma 2.4 ([9]). *For $0 < \alpha \leq 1$, the Mittag-Leffler type function $E_{\alpha,\alpha}(-\omega t^{\alpha})$ satisfies*

$$0 \leq E_{\alpha,\alpha}(-\omega t^{\alpha}) \leq \frac{1}{\Gamma(\alpha)}, \qquad t \in [0,\infty),\ \omega \geq 0,$$

and

$$\lim_{t \to 0^+} E_{\alpha,\alpha}(-\omega t^{\alpha}) = E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}.$$

Lemma 2.5 ([15]). *For $t \in [0,\infty)$ and $0 < \alpha \leq 1$, the one-parameter Mittag-Leffler function $E_{\alpha,1}(-t^{\alpha})$ is a decreasing function of t and it is bounded from above by 1, that is,*

$$E_{\alpha,1}(-\omega t^{\alpha}) \leq 1.$$

Furthermore, it is to be noted that

$$\lim_{t \to \infty} E_{\alpha,1}(-\omega t^{\alpha}) = 0.$$

Lastly in this section, we state the fixed point theorems, which enable us to prove the existence and uniqueness of a solution of (1.1).

Theorem 2.6 (Banach's fixed point theorem [26]). *Let Ω be a nonempty closed subset of a Banach space $(S, \|\cdot\|)$. Then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Theorem 2.7 (Krasnoselskii's fixed point theorem [26]). *Let Ω be a nonempty bounded closed convex subset of a Banach space $(S, \|\cdot\|)$. Suppose that F_1 and F_2 map Ω into S such that*

(i) *$F_1 u + F_2 v \in \Omega$ for all $u, v \in \Omega$,*
(ii) *F_1 is continuous and compact,*
(iii) *F_2 is a contraction.*

Then there is $u \in \Omega$ with $F_1 u + F_2 u = u$.

3. Existence and uniqueness

Let us start by defining what we mean by a solution of Problem (1.1).

Definition 3.1. A function $u \in C^1(J, \mathbb{R})$ is said to be a solution of Problem (1.1) if u satisfies ${}^{C}D^{\alpha}(u(t) - g(t, u(t))) + \omega u(t) = f(t, u(t))$ for any $t \in J$ and $u(0) = u_0$.

For the existence of solutions for Problem (1.1), we need the following auxiliary lemma.

Lemma 3.2. *Let $u \in C(J, \mathbb{R})$ and let u' exist. Then u is a solution of the initial value Problem (1.1) if and only if it is a solution of the integral equation*

$$u(t) = (u_0 - g(t, u_0)) E_{\alpha,1}(-\omega t^{\alpha}) + g(t, u(t))$$

$$+\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^{\alpha}) \left[f(s,u(s)) - \omega g(s,u(s))\right] ds. \quad (3.1)$$

Proof. It is easy to prove by the Laplace transform. □

In the following subsections, we prove existence, as well as the existence and uniqueness results for Problem (1.1) by using a variety of fixed point theorems.

The following assumptions will be used in our main results

(H1) There exists a constant $L_f \in \mathbb{R}^+$ such that
$$|f(t,u) - f(t,v)| \leq L_f |u-v|,$$
for $t \in J$, $u, v \in \mathbb{R}$.

(H2) There exists a constant $L_g \in (0,1)$ such that
$$|g(t,u) - g(t,v)| \leq L_g |u-v|,$$
for $t \in J$, $u, v \in \mathbb{R}$.

3.1. Existence and uniqueness results via Banach's fixed point theorem.

Theorem 3.3. *Assume that the assumptions (H1)–(H2) are satisfied. If*
$$L_g + \frac{T^{\alpha}}{\Gamma(\alpha+1)}(L_f + L_g\omega) < 1, \quad (3.2)$$
then there exists a unique solution for Problem (1.1) *on* J.

Proof. We define the operator $\Phi : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by
$$\begin{aligned}(\Phi u)(t) =& (u_0 - g(t,u_0))E_{\alpha,1}(-\omega t^{\alpha}) + g(t,u(t)) \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^{\alpha}) \left[f(s,u(s)) - \omega g(s,u(s))\right] ds.\end{aligned}$$
Clearly, the fixed points of operator Φ are solutions of Problem (1.1). For any $u, v \in C([0,T],\mathbb{R})$ and $t \in J$, we have
$$\begin{aligned}&|(\Phi u)(t) - (\Phi v)(t)| \\ &\leq |g(t,u(t)) - g(t,v(t))| \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^{\alpha}) |f(s,u(s)) - f(s,v(s))| ds \\ &\quad + \omega \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^{\alpha}) |g(s,u(s)) - g(s,v(s))| ds.\end{aligned}$$
By (H1) and (H2), we have
$$|(\Phi u)(t) - (\Phi v)(t)| \leq L_g \|u-v\| + \frac{T^{\alpha}L_f}{\Gamma(\alpha+1)} \|u-v\| + \frac{T^{\alpha}L_g\omega}{\Gamma(\alpha+1)} \|u-v\|,$$
thus
$$\|\Phi u - \Phi v\| \leq \left(L_g + \frac{T^{\alpha}}{\Gamma(\alpha+1)}(L_f + L_g\omega)\right) \|u-v\|.$$
From (3.2), Φ is a contraction. As a consequence of Banach's fixed point theorem, we get that Φ has a unique fixed point which is a unique solution of Problem (1.1) on J. □

3.2. **Existence results via Krasnoselskii's fixed point theorem.**

Theorem 3.4. *Assume* (H2) *and the following hypotheses:*

(H3) *There exists* $p_1 \in C(J, \mathbb{R}^+)$ *such that*

$$|f(t,u)| \leq p_1(t),$$

for $t \in J$ *and each* $u \in \mathbb{R}$.

(H4) *There exists* $p_2 \in C(J, \mathbb{R}^+)$ *such that*

$$|g(t,u)| \leq p_2(t),$$

for $t \in J$ *and each* $u \in \mathbb{R}$.

Then Problem (1.1) *has at least one solution in* Ω.

Proof. Let us fix

$$\rho \geq |u_0| + q + p_2^* + \frac{T^\alpha}{\Gamma(\alpha+1)}(p_1^* + \omega p_2^*),$$

where $p_1^* = \sup_{t\in J} p_1(t)$, $p_2^* = \sup_{t\in J} p_2(t)$, and $q = \sup_{t\in J} |g(t,u_0)|$. Consider the nonempty closed convex subset

$$\Omega = \{u \in C(J, \mathbb{R}),\ \|u\| \leq \rho\},$$

and define two operators F_1 and F_2 on Ω, as follows:

$$(F_1 u)(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha)\left[f(s,u(s)) - \omega g(s,u(s))\right] ds,$$

and

$$(F_2 u)(t) = (u_0 - g(t,u_0))E_{\alpha,1}(-\omega t^\alpha) + g(t,u(t)).$$

We shall use the Krasnoselskii fixed point theorem to prove that there exists at least one fixed point of the operator $F_1 + F_2$ in Ω. The proof will be given in several steps.

Step 1. We prove $F_1 u + F_2 v \in \Omega$ for all $u, v \in \Omega$.

For any $u, v \in \Omega$, we have

$$\begin{aligned}
&|(F_1 u)(t) + (F_2 v)(t)| \\
&= \Big|\big(u_0 - g(t,u_0)\big)E_{\alpha,1}(-\omega t^\alpha) + g(t,v(t)) \\
&\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha)\left[f(s,u(s)) - \omega g(s,u(s))\right] ds\Big| \\
&\leq |u_0| + q + p_2^* + \frac{T^\alpha}{\Gamma(\alpha+1)}(p_1^* + \omega p_2^*).
\end{aligned}$$

Thus

$$\|F_1 u + F_2 v\| \leq |u_0| + q + p_2^* + \frac{T^\alpha}{\Gamma(\alpha+1)}(p_1^* + \omega p_2^*) \leq \rho.$$

Hence, $F_1 u + F_2 v \in \Omega$, for all $u, v \in \Omega$.

Step 2. We prove that F_1 is compact and continuous.

For all $u \in \Omega$, we have

$$|(F_1 u)(t)|$$

$$= \left| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) \left[f(s,u(s)) - \omega g(s,u(s)) \right] ds \right|$$
$$\leq \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) \left[|f(s,u(s))| + \omega |g(s,u(s))| \right] ds$$
$$\leq \frac{T^\alpha}{\Gamma(\alpha+1)} (p_1^* + \omega p_2^*).$$

Thus

$$\|F_1 u\| \leq \frac{T^\alpha}{\Gamma(\alpha+1)} (p_1^* + \omega p_2^*).$$

Hence, $F_1$is uniformly bounded on Ω.

It remains to show that $F_1(\Omega)$ is equicontinuous. Let $x \in \Omega$. Then for any $0 < t_1 < t_2 \leq T$, we have

$$|(F_1 u)(t_2) - (F_1 u)(t_1)|$$
$$= \left| \int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t_2-s)^\alpha) \left[f(s,u(s)) - \omega g(s,u(s)) \right] ds \right.$$
$$\left. - \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t_1-s)^\alpha) \left[f(s,u(s)) - \omega g(s,u(s)) \right] ds \right|$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} \right) \left(|f(s,u(s))| + \omega |g(s,u(s))| \right) ds$$
$$+ \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \left(|f(s,u(s))| + \omega |g(s,u(s))| \right) ds$$
$$\leq \frac{(p_1^* + \omega p_2^*)}{\Gamma(\alpha+1)} \left(2(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha \right)$$
$$\leq \frac{2(p_1^* + \omega p_2^*)}{\Gamma(\alpha+1)} (t_2-t_1)^\alpha. \tag{3.3}$$

As $t_1 \to t_2$, the right-hand side of inequality (3.3) tends to zero and the convergence is independent of u in Ω, which means that $F_1(\Omega)$ is equicontinuous. The Arzela–Ascoli theorem implies that F_1 is compact. Moreover, the continuity of f and g implies that F_1 is continuous.

Step 3. We prove that $F_2 : \Omega \to C(J, \mathbb{R})$ is a contraction mapping.

For all $u, v \in \Omega$ and $t \in J$, we have

$$\begin{aligned} |(F_2 u)(t) - (F_2 v)(t)| &= |g(t,u(t)) - g(t,v(t))| \\ &\leq L_g \|u - v\|. \end{aligned}$$

Thus

$$\|F_2 u - F_2 v\| \leq L_g \|u - v\|.$$

Hence, the operator F_2 is a contraction.

Clearly, all the hypotheses of the Krasnoselskii fixed point theorem are satisfied. Hence, there exists a fixed point $u \in \Omega$ such that $u = F_1 u + F_2 u$, which is a solution of Problem (1.1). □

Example 3.5. We consider the fractional initial value problem

$$\begin{cases} {}^{C}D^{\frac{1}{2}}\left(u\left(t\right)-\frac{1}{4}u\left(t\right)\cos(t)\right)+\frac{1}{2}u\left(t\right)=\frac{1}{(\exp(t)+4)(|u(t)|+1)}, \ t\in J=\left[0,1\right], \\ u\left(0\right)=1, \end{cases} \tag{3.4}$$

where $T=1$, $u_0=1$, $\alpha=\omega=\frac{1}{2}$, $g(t,u)=\frac{1}{4}t\sin(u)$, and $f(t,u)=\frac{1}{(\exp(t)+4)(|u|+1)}$. For each $u,v\in\mathbb{R}$ and $t\in J$, we have

$$\begin{aligned} \left|f\left(t,u\right)-f\left(t,v\right)\right| &= \left|\frac{1}{\left(\exp(t)+4\right)\left(|u|+1\right)}-\frac{1}{\left(\exp(t)+4\right)\left(|v|+1\right)}\right| \\ &\leq \frac{|u-v|}{\left(\exp(t)+4\right)\left(1+|u|\right)\left(1+|v|\right)} \\ &\leq \frac{1}{5}\left|u-v\right|, \end{aligned}$$

and

$$|g(t,u)-g(t,v)|\leq\frac{1}{4}\left|u-v\right|.$$

Hence, assumptions (H1) and (H2) are satisfied with $L_f=\frac{1}{5}$ and $L_g=\frac{1}{4}$. The condition

$$L_g+\frac{T^{\alpha}}{\Gamma\left(\alpha+1\right)}\left(L_g\omega+L_f\right)\simeq 0.62<1$$

is satisfied. It follows from Theorem 3.3 that Problem (3.4) has a unique solution on J.

Acknowledgement. The authors would like to thank the anonymous referee for his/her valuable comments and good advice.

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