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ON MATRIX-VALUED GABOR BESSEL SEQUENCES AND DUAL FRAMES OVER LOCALLY COMPACT ABELIAN GROUPS

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Abstract. We study matrix-valued Gabor Bessel sequences and frames in the matrix-valued space $L^2(G, \mathbb{C}^{n\times n})$, where G is a locally compact abelian (LCA) group and n is a positive integer. First, we show that the Bessel condition (or upper frame condition) can be extended from $L^2(G)$ to its associated matrix-valued signal space $L^2(G, \mathbb{C}^{n\times n})$, and conversely. However, this is not true for the lower frame condition. Secondly, we give sufficient conditions for the extension of a pair of matrix-valued Bessel sequences to matrix-valued dual frames over LCA groups. A special class of matrix-valued dual generators is given. It is shown that the symmetric windows associated with a given matrix-valued Gabor frames constitute a Gabor frame in matrix-valued spaces over LCA groups.

1. Introduction

Duffin and Schaeffer [5] introduced the concept of frame in separable Hilbert spaces in the study of nonharmonic expansion of functions. Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its associated norm is given by $\|f\| = \sqrt{\langle f, f\rangle}$, for $f \in \mathcal{H}$. A sequence of vectors $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is called a *frame* for $\mathcal{H}$, if there exist positive scalars $\alpha_o \leq \beta_o < \infty$ such that

$$\alpha_o \|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq \beta_o \|f\|^2 \tag{1.1}$$

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for all $f \in \mathcal{H}$. The scalars α_o and β_o are called *lower frame bound* and *upper frame bound* of $\{f_k\}_{k=1}^{\infty}$, respectively. If it is possible to choose $\alpha_o = \beta_o$, then we say that $\{f_k\}_{k=1}^{\infty}$ is an α_o-*Parseval frame* (or α_o-*tight frame*). If only upper inequality in (1.1) holds, then we say that $\{f_k\}_{k=1}^{\infty}$ is a *Bessel sequence* with *Bessel bound* β_o.

Let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence in $\mathcal{H}$. Then, the map $T : \ell^2 \to \mathcal{H}$ given by $T(\{\xi_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} \xi_k f_k$ is called the *pre-frame operator* (or *synthesis operator*), and its Hilbert adjoint $T^* : \mathcal{H} \to \ell^2$ given by $T^*(f) = \{\langle f, f_k\rangle\}_{k=1}^{\infty}$ is called the *analysis operator*. Moreover, T and T^* are bounded linear operators on $\mathcal{H}$. The composition $S : \mathcal{H} \to \mathcal{H}$ is known as the *frame operator* that is given by $Sf = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k$. If $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$, then S is a bounded, linear, invertible and positive operator on $\mathcal{H}$. This gives the *reconstruction formula* of each vector $f \in \mathcal{H}$, $f = \sum_{k=1}^{\infty}\langle f, S^{-1}f_k\rangle f_k$. This formula in useful in signal analysis. We refer to books by Christensen [1], Gröchenig [7], Han [8], Heil [9], and Young [20] for basic theory on frames.

In this work, we study matrix-valued Bessel sequences and frames in the matrix-valued signal space $L^2(G, \mathbb{C}^{n\times n})$, where G is a locally compact abelian (LCA) group. The matrix-valued signal spaces are related to video imaging and other applications in signal processing, where signal is multivariate. Xia and Suter [19] classified and constructed vector-valued (matrix-valued) wavelets with sampling property. They also showed that certain linear combinations of known scalar-valued wavelets may yield multiwavelets. Recently, matrix-valued frames are studied in a series of papers [10–15]. We first discuss an interplay between matrix-valued Gabor Bessel sequences over LCA groups and its associated Bessel sequences in atomic spaces. It is shown that the Bessel condition (or upper frame condition) can be carried from $L^2(G, \mathbb{C}^{n\times n})$ to its associated atomic space $L^2(G)$ and vice versa. This is not true for the lower frame condition. We give sufficient conditions for extension of matrix-valued Bessel sequences with Gabor structure to dual frame pair. This generalizes a result of [2] to matrix-valued function spaces over LCA groups. Matrix-valued symmetric Gabor frames are also discussed.

2. Preliminaries

In this section, we review the basic facts and terminology concerning the LCA group and Gabor frames over LCA groups. Symbols $\mathbb{Z}$ and $\mathbb{C}$ denote the set of integers and complex numbers, respectively. Also, $\mathbb{T}$ denotes the unit circle group. Let G denote a second countable LCA group equipped with the Hausdorff topology. A character on G is a map $\gamma \colon G \to \mathbb{T}$ that satisfies $\gamma(x+y) = \gamma(x)\gamma(y)$, for all $x, y \in G$. The collection of all continuous characters on G is denoted by $\hat{G}$, which forms an LCA group under the operation defined by $(\gamma + \gamma')(x) := \gamma(x)\gamma'(x)$, where $\gamma, \gamma' \in \hat{G}$ and $x \in G$ and an appropriate topology. The group $\hat{G}$ is known as a *dual group* of G. It is well known that there exists a Haar measure, unique up to a scalar multiple, on a given LCA group; see [6]. Let μ_G and $\mu_{\hat{G}}$ denote the Haar measure on G and $\hat{G}$, respectively. A *lattice* of G is a discrete subgroup Λ of G for which G/Λ is compact. The annihilator of Λ, denoted by $\Lambda^{\perp}$, is defined by $\Lambda^{\perp} = \{\gamma \in \hat{G} \mid \gamma(x) = 1,\ x \in \Lambda\}$, which is a lattice in $\hat{G}$. The *fundamental domain* associated with the lattice $\Lambda^\perp$ of $\hat{G}$, denoted by V, is a Borel measurable relatively compact set in $\hat{G}$ such that $\hat{G} = \cup_{w\in\Lambda^\perp}(w+V)$, $(w+V)\cap(w'+V)=\emptyset$ for $w\neq w', w, w'\in\Lambda^\perp$. By Aut$G$, we denote the collection of all continuous automorphisms on G. The space of measurable square integrable functions over G, denoted by $L^2(G)$, is defined as $L^2(G) := \big\{f : \int_G |f|^2 d\mu_G < \infty\big\}$. For a function $f \in L^1 \bigcap L^2(G)$, the integral

$$\widehat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}d\mu_G(x), \qquad \gamma\in\widehat{G},$$

is known as the *Fourier transform* of f. It can be extended isometrically to $L^2(G)$. We refer to [6] for basics on LCA groups.

2.1. **The space** $L^2(G,\mathbb{C}^{n\times n})$**.** Throughout the paper, the matrix-valued functions are denoted by bold letters. Let n be a positive integer. The space of matrix-valued functions over G, denoted by $L^2(G,\mathbb{C}^{n\times n})$, is defined as

$$L^2(G,\mathbb{C}^{n\times n}) := \big\{\mathbf{f}(x) : x\in G,\ f_{ij}(x)\in L^2(G)\ (1\le i,\ j\le n)\big\},$$

where

$$\mathbf{f}(x) = \begin{bmatrix} f_{11}(x) & f_{12}(x) & \dots & f_{1n}(x)\\ f_{21}(x) & f_{22}(x) & \dots & f_{2n}(x)\\ \vdots & \vdots & \ddots & \vdots\\ f_{n1}(x) & f_{n2}(x) & \dots & f_{nn}(x)\end{bmatrix}.$$

Functions f_{ij} are called *components* or *atoms* of $\mathbf{f}$. The Frobenius norm on $L^2(G,\mathbb{C}^{n\times n})$ is given by

$$\|\mathbf{f}\| = \Big(\sum_{i,j=1}^{n}\int_G |f_{ij}|^2 d\mu_G\Big)^{\frac{1}{2}}. \tag{2.1}$$

It is easy to see that $L^2(G,\mathbb{C}^{n\times n})$ is a Banach space with respect to the Frobenius norm given in (2.1). The integral of a function $\mathbf{f}\in L^2(G,\mathbb{C}^{n\times n})$ is defined as

$$\int_G \mathbf{f}d\mu_G = \begin{bmatrix} \int_G f_{11}d\mu_G & \int_G f_{12}d\mu_G & \dots & \int_G f_{1n}d\mu_G\\ \int_G f_{21}d\mu_G & \int_G f_{22}d\mu_G & \dots & \int_G f_{2n}d\mu_G\\ \vdots & \vdots & \ddots & \vdots\\ \int_G f_{n1}d\mu_G & \int_G f_{n2}d\mu_G & \dots & \int_G f_{nn}d\mu_G\end{bmatrix}.$$

For $\mathbf{f},\mathbf{g}\in L^2(G,\mathbb{C}^{n\times n})$, the matrix-valued inner product is defined as

$$\langle \mathbf{f},\mathbf{g}\rangle = \int_G \mathbf{f}(x)\mathbf{g}^*(x)d\mu_G.$$

Here $*$ denotes the transpose and the complex conjugate. One may observe that it is not an inner product in usual sense. However, the space $L^2(G,\mathbb{C}^{n\times n})$ forms a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_o$ defined by

$$\langle \mathbf{f},\mathbf{g}\rangle_o = \mathrm{trace}\langle \mathbf{f},\mathbf{g}\rangle, \qquad \mathbf{f},\ \mathbf{g}\in L^2(G,\mathbb{C}^{n\times n}),$$

and $\langle\cdot,\cdot\rangle_o$ generates the Frobenius norm: $||\mathbf{f}||^2 = \mathrm{trace}\langle \mathbf{f},\mathbf{f}\rangle$, $\mathbf{f}\in L^2(G,\mathbb{C}^{n\times n})$.

2.2. **Gabor Frames in** $L^2(G, \mathbb{C}^{n\times n})$**.** Let Λ_0 be a finite subset of $\mathbb{N}$, $B \in \mathrm{Aut}G$, $C \in \mathrm{Aut}\widehat{G}$, Λ a lattice in G, and let Λ' be a lattice in $\widehat{G}$.

Write

$$\Phi_{\Lambda_0} := \{\Phi_l\}_{l\in\Lambda_0} \subset L^2(G, \mathbb{C}^{n\times n}),$$
$$\mathtt{G}(C, B, \Phi_{\Lambda_0}) := \{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'} \subset L^2(G, \mathbb{C}^{n\times n}).$$

For $a \in G$ and $\eta \in \hat{G}$, we consider the following operators on $L^2(G, \mathbb{C}^{n\times n})$:

$$T_a\mathbf{f}(x) = \mathbf{f}(xa^{-1}) \quad \text{(Translation operator)},$$
$$E_\eta\mathbf{f}(x) = \eta(x)\mathbf{f}(x) \quad \text{(Modulation operator)}.$$

For $l \in \Lambda_0$, let $\Phi_l \in L^2(G, \mathbb{C}^{n\times n})$ be given by $\Phi_l(x) = \left[\phi_{ij}^{(l)}(x)\right]_{n\times n}$. Let $B \in \mathrm{Aut}G$ and let $C \in \mathrm{Aut}\widehat{G}$. A family of functions of the form $\mathtt{G}(C, B, \Phi_{\Lambda_0}) := \{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$ in $L^2(G, \mathbb{C}^{n\times n})$ is called a *multiwindow Gabor system* in the space $L^2(G, \mathbb{C}^{n\times n})$ over the LCA group G. The functions Φ_l are called the *matrix-valued Gabor window functions*.

Definition 2.1. A frame of the form $\mathtt{G}(C, B, \Phi_{\Lambda_0})$ for $L^2(G, \mathbb{C}^{n\times n})$ is called a *matrix-valued Gabor frame*. That is, the inequality (*frame inequality*)

$$\alpha_o\|\mathbf{f}\|^2 \leq \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \left\|\left\langle E_{Cm}T_{Bk}\Phi_l, \mathbf{f}\right\rangle\right\|^2 \leq \beta_o\|\mathbf{f}\|^2, \qquad \mathbf{f} \in L^2(G, \mathbb{C}^{n\times n}),$$

holds for some positive scalars α_o and β_o. As in the case of ordinary frames, α_o and β_o are called frame bounds.

Let $\mathcal{M}_n(\mathbb{C})$ be the complex vector space of all $n \times n$ complex matrices. The space

$$\ell^2(\Lambda_0 \times \Lambda \times \Lambda', \mathcal{M}_n(\mathbb{C})) := \Big\{\{M_{l,j,k}\}_{\substack{l\in\Lambda_0, j\in\Lambda \\ k\in\Lambda'}} \subset \mathcal{M}_n(\mathbb{C}) : \sum_{\substack{l\in\Lambda_0, j\in\Lambda \\ k\in\Lambda'}} \|M_{l,j,k}\|^2 < \infty\Big\}$$

is a Hilbert space and its related norm is given by

$$\|\{M_{l,j,k}\}_{l\in\Lambda_0, j\in\Lambda, k\in\Lambda'}\| = \Big(\sum_{l\in\Lambda_0} \sum_{j\in\Lambda, k\in\Lambda'} \|M_{l,j,k}\|^2\Big)^{\frac{1}{2}}.$$

Let $\mathtt{G}(C, B, \Phi_{\Lambda_0})$ be a frame for $L^2(G, \mathbb{C}^{n\times n})$. The map $V : \ell^2(\Lambda_0 \times \Lambda \times \Lambda', \mathcal{M}_n(\mathbb{C})) \to L^2(G, \mathbb{C}^{n\times n})$ defined by

$$V : \{M_{l,k,m}\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'} \mapsto \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} M_{l,k,m}E_{Cm}T_{Bk}\Phi_l$$

is called the *synthesis operator* or the *pre-frame operator*, associated with the Gabor frame $\mathtt{G}(C, B, \Phi_{\Lambda_0})$. The *analysis operator* is the map $W : L^2(G, \mathbb{C}^{n\times n}) \to \ell^2(\Lambda_0 \times \Lambda \times \Lambda', \mathcal{M}_n(\mathbb{C}))$ given by

$$W : \mathbf{f} \mapsto \Big\{\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\Big\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}.$$

The frame operator of $\mathtt{G}(C, B, \Phi_{\Lambda_0})$ is the composition $S = VW : L^2(G, \mathbb{C}^{n\times n}) \to L^2(G, \mathbb{C}^{n\times n})$ given by

$$S : \mathbf{f} \mapsto \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle E_{Cm}T_{Bk}\Phi_l,$$

where $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$. The frame operator is bounded, linear, and invertible on $L^2(G, \mathbb{C}^{n\times n})$.

3. Main results

We begin with a motivational example.

Example 3.1. Let $G = \mathbb{T}$ be the circle group and let $\phi \in L^2(G)$ be given by $\phi(t) = 1, t \in G$. Let Λ be any lattice in G and let $\Lambda' = \mathbb{Z}$, a lattice in $\widehat{G} = \mathbb{Z}$. Then, the Gabor system $\{E_mT_k\phi\}_{k\in\Lambda, m\in\Lambda'} = \{E_m\phi\}_{m\in\mathbb{Z}} = \{e^{2\pi i m(\cdot)}\}_{m\in\mathbb{Z}}$ is an orthonormal basis and hence a Parseval Gabor frame for $L^2(\mathbb{G})$. Consider the matrix-valued function $\Phi = \begin{bmatrix}\phi & \phi\\ \phi & \phi\end{bmatrix} \in L^2(G, \mathbb{C}^{2\times 2})$. Then, the matrix-valued Gabor system $\{E_mT_k\Phi\}_{k\in\Lambda, m\in\Lambda'}$ does not constitute a frame for $L^2(G, \mathbb{C}^{2\times 2})$. Indeed, for a non-zero function $f_o \in L^2(G)$, let $\mathbf{f} \in L^2(G, \mathbb{C}^{2\times 2})$ be given by $\mathbf{f} = \begin{bmatrix}-f_o & f_o\\ 0 & 0\end{bmatrix}$. Then, $\sum\limits_{k\in\Lambda, m\in\Lambda'} \left\| \left\langle E_mT_k\Phi, \mathbf{f}\right\rangle\right\|^2 = 0$. Hence, $\{E_mT_k\Phi\}_{k\in\Lambda, m\in\Lambda'}$ is not a frame for $L^2(G, \mathbb{C}^{2\times 2})$.

On the other hand, if we take $\Phi = \begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix} = \begin{bmatrix}0 & \phi\\ \phi & 0\end{bmatrix} \in L^2(G, \mathbb{C}^{2\times 2})$, then for any $\mathbf{f} = \begin{bmatrix}f_{11} & f_{12}\\ f_{21} & f_{22}\end{bmatrix} \in L^2(G, \mathbb{C}^{2\times 2})$, we have

$$\sum_{k\in\Lambda, m\in\Lambda'} \left\| \left\langle E_mT_k\Phi, \mathbf{f}\right\rangle\right\|^2 = \|\mathbf{f}\|^2.$$

Thus, $\{E_mT_k\Phi\}_{k\in\Lambda, m\in\Lambda'} = \{E_m\Phi\}_{m\in\mathbb{Z}}$ is a matrix-valued Parseval Gabor frame for $L^2(G, \mathbb{C}^{2\times 2})$. Indeed $\{E_mT_k\phi_{11}\}_{k\in\Lambda, m\in\Lambda'} = \{E_m\phi_{11}\}_{m\in\mathbb{Z}}$ and $\{E_mT_k\phi_{22}\}_{k\in\Lambda, m\in\Lambda'} = \{E_m\phi_{22}\}_{m\in\mathbb{Z}}$ are not Gabor frames for $L^2(G)$.

Example 3.1 shows that the frame conditions of a frame for $L^2(G)$, in general, cannot be carried to its associated matrix-valued signal space $L^2(G, \mathbb{C}^{n\times n})$, and vice versa. In this section, we discuss an interplay between Bessel sequences in $L^2(G)$ and its associated matrix-valued space $L^2(G, \mathbb{C}^{n\times n})$. The following result shows that the Bessel condition (upper frame condition) can be carried from $L^2(G)$ to $L^2(G, \mathbb{C}^{n\times n})$ and conversely.

Theorem 3.2. *The matrix-valued Gabor system $\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$ in the space $L^2(G, \mathbb{C}^{n\times n})$ is a Gabor Bessel sequence if and only if for each i, j $(1 \le i, j \le n)$ and $l \in \Lambda_0$, the family $\{E_{Cm}T_{Bk}\phi_{ij}^{(l)}\}_{k\in\Lambda, m\in\Lambda'}$ is a Gabor Bessel sequence for $L^2(G)$, where $\Phi_l \in L^2(G, \mathbb{C}^{n\times n})$ is given by $\Phi_l(x) = \left[\phi_{ij}^{(l)}(x)\right]_{n\times n}$.*

Proof. For each i, j $(1 \leq i, j \leq n)$ and $l \in \Lambda_0$, let $\{E_{Cm}T_{Bk}\phi_{ij}^{(l)}\}_{k\in\Lambda, m\in\Lambda'}$ be a Gabor Bessel sequence for $L^2(G)$ with Bessel bound $U_{ij}^{(l)}$. Then, for any $f = \big[f_{ij}\big]_{1\leq i,j\leq n} \in L^2(G, \mathbb{C}^{n\times n})$. We compute

$$\begin{aligned}
&\sum_{l\in\Lambda_0}\sum_{k\in\Lambda, m\in\Lambda'} \Big\|\Big\langle E_{Cm}T_{Bk}\Phi_l, \mathbf{f}\Big\rangle\Big\|^2 \\
&= \sum_{l\in\Lambda_0}\sum_{k\in\Lambda, m\in\Lambda'} \Big(\big|\sum_{r=1}^{n}\int_G E_{Cm}T_{Bk}\phi_{1r}^{(l)}\overline{f_{1r}}d\mu_G\big|^2 + \cdots + \big|\sum_{r=1}^{n}\int_G E_{Cm}T_{Bk}\phi_{nr}^{(l)}\overline{f_{1r}}d\mu_G\big|^2 \\
&\quad + \big|\sum_{r=1}^{n}\int_G E_{Cm}T_{Bk}\phi_{1r}^{(l)}\overline{f_{2r}}d\mu_G\big|^2 + \cdots + \big|\sum_{r=1}^{n}\int_G E_{Cm}T_{Bk}\phi_{nr}^{(l)}\overline{f_{2r}}d\mu_G\big|^2 + \cdots \\
&\quad + \big|\sum_{r=1}^{n}\int_G E_{Cm}T_{Bk}\phi_{1r}^{(l)}\overline{f_{nr}}d\mu_G\big|^2 + \cdots + \big|\sum_{r=1}^{n}\int_G E_{Cm}T_{Bk}\phi_{nr}^{(l)}\overline{f_{nr}}d\mu_G\big|^2\Big) \\
&\leq \sum_{l\in\Lambda_0} n\Big(\sum_{r=1}^{n} U_{1r}^{(l)}\|f_{1r}\|^2 + \cdots + \sum_{r=1}^{n} U_{nr}^{(l)}\|f_{1r}\|^2 \\
&\quad + \sum_{r=1}^{n} U_{1r}^{(l)}\|f_{2r}\|^2 + \cdots + \sum_{r=1}^{n} U_{nr}^{(l)}\|f_{2r}\|^2 \\
&\quad + \cdots + \sum_{r=1}^{n} U_{1r}^{(l)}\|f_{nr}\|^2 + \cdots + \sum_{r=1}^{n} U_{nr}^{(l)}\|f_{nr}\|^2\Big) \\
&\leq \sum_{l\in\Lambda_0} n^2\Big((\max_i U_{i1}^{(l)})\|f_{11}\|^2 + \cdots + (\max_i U_{in}^{(l)})\|f_{1n}\|^2 + (\max_i U_{i1}^{(l)})\|f_{21}\|^2 + \cdots \\
&\quad + (\max_i U_{in}^{(l)})\|f_{2n}\|^2 + \cdots + (\max_i U_{i1}^{(l)})\|f_{n1}\|^2 + \cdots + (\max_i U_{in}^{(l)})\|f_{nn}\|^2\Big) \\
&\leq \sum_{l\in\Lambda_0} n^2(\max_{i,j} U_{ij}^{(l)})\|f\|^2 \\
&\leq n^2|\Lambda_0|(\max_{i,j,l} U_{ij}^{(l)})\|f\|^2.
\end{aligned}$$

Hence, $\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$ is a matrix-valued Gabor Bessel sequence in $L^2(G, \mathbb{C}^{n\times n})$ with Bessel bound $\beta_o = n^2|\Lambda_0|(\max_{i,j,l} U_{ij}^{(l)})$.

Conversely, assume that $\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'} \subset L^2(G, \mathbb{C}^{n\times n})$ is a matrix-valued Gabor Bessel sequence with Bessel bound γ_o. Let i, j $(1 \leq i, j \leq n)$, $l \in \Lambda_0$, and $f \in L^2(G)$ be arbitrary but fixed.

Choose

$$\mathbf{h}(\bullet) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \underbrace{f(\bullet)}_{ij^{\text{th}} \text{ place}} & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} \in L^2(G, \mathbb{C}^{n\times n}).$$

Then

$$\begin{aligned}\sum_{k\in\Lambda,m\in\Lambda'}\left|\left\langle E_{Cm}T_{Bk}\phi_{ij}^{(l)},f\right\rangle\right|^2 &\leq \sum_{k\in\Lambda,m\in\Lambda'}\left\|\left\langle E_{Cm}T_{Bk}\Phi_l,\mathbf{h}\right\rangle\right\|^2\\ &\leq \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\left\|\left\langle E_{Cm}T_{Bk}\Phi_l,\mathbf{h}\right\rangle\right\|^2\\ &\leq \gamma_o\|\mathbf{h}\|^2\\ &= \gamma_o\|f\|^2.\end{aligned}$$

Hence, for each i,j $(1\leq i,j\leq n)$ and $l\in\Lambda_0$, the family $\{E_{Cm}T_{Bk}\phi_{ij}^{(l)}\}_{k\in\Lambda,m\in\Lambda'}$ is a Gabor Bessel sequence for $L^2(G)$ with the Bessel bound γ_o. □

Remark 3.3. Example 3.1 shows that the result given in Theorem 3.2 is not true for the lower frame condition in matrix-valued signal spaces.

The next result gives the majorization of energy of window functions. To be precise, we can estimate norms of window functions in terms of Bessel bounds.

Theorem 3.4. *Let $\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}\subset L^2(G,\mathbb{C}^{n\times n})$ be a matrix-valued Gabor Bessel sequence with Bessel bound β_o. Then*

(1) $\sum\limits_{l\in\Lambda_0}\|\Phi_l\|^2\leq n|\Lambda_0|\beta_o$,

(2) $\|\phi_{ij}^{(l)}\|^2\leq\beta_o$ *for every* $1\leq i,j\leq n, l\in\Lambda_o$.

Proof. By the hypothesis, we have

$$\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\left\|\left\langle E_{Cm}T_{Bk}\Phi_l,\mathbf{f}\right\rangle\right\|^2\leq\beta_o\|\mathbf{f}\|^2 \text{ for all } \mathbf{f}\in L^2(G,\mathbb{C}^{n\times n}).$$

Therefore, for any (but fixed) $l_o\in\Lambda_0, k_o\in\Lambda, m_o\in\Lambda'$, we have

$$\left\|\left\langle E_{Cm_o}T_{Bk_o}\Phi_{l_o},E_{Cm_o}T_{Bk_o}\Phi_{l_o}\right\rangle\right\|^2\leq\beta_o\|E_{Cm_o}T_{Bk_o}\Phi_{l_o}\|^2. \tag{3.1}$$

We compute

$$\begin{aligned}\|\Phi_{l_o}\|^4 &= \|E_{Cm_o}T_{Bk_o}\Phi_{l_o}\|^4\\ &= |\text{trace}\langle E_{Cm_o}T_{Bk_o}\Phi_{l_o},E_{Cm_o}T_{Bk_o}\Phi_{l_o}\rangle|^2\\ &\leq n\left\|\left\langle E_{Cm_o}T_{Bk_o}\Phi_{l_o},E_{Cm_o}T_{Bk_o}\Phi_{l_o}\right\rangle\right\|^2\\ &\leq n\beta_o\|E_{Cm_o}T_{Bk_o}\Phi_{l_o}\|^2 \qquad \big(\text{by } (3.1)\big)\\ &= n\beta_o\|\Phi_{l_o}\|^2.\end{aligned}$$

This proves (1).

From the first step of the proof of Theorem 3.2, for each i,j $(1\leq i,j\leq n)$ and $l\in\Lambda_0$, the family $\{E_{Cm}T_{Bk}\phi_{ij}^{(l)}\}_{k\in\Lambda,m\in\Lambda'}$ is a Gabor Bessel sequence in $L^2(G)$ with bound β_o. That is,

$$\sum_{k\in\Lambda,m\in\Lambda'}\left|\left\langle E_{Cm}T_{Bk}\phi_{ij}^{(l)},f\right\rangle\right|^2\leq\beta_o\|f\|^2 \text{ for all } f\in L^2(G) \quad \big(l\in\Lambda_0, 1\leq i,j\leq n\big).$$

Therefore, for fixed $k_o \in \Lambda, m_o \in \Lambda'$, we have

$$\left|\left\langle E_{Cm_o}T_{Bk_o}\Phi_{ij}^{(l)}, E_{Cm_o}T_{Bk_o}\Phi_{ij}^{(l)}\right\rangle\right|^2 \leq \beta_o \|E_{Cm_o}T_{Bk_o}\Phi_{ij}^{(l)}\|^2,$$

which implies

$$\|\Phi_{ij}^{(l)}\|^2 \leq \beta_o \quad \big(l \in \Lambda_0, 1 \leq i, j \leq n\big).$$

Hence, (2) is proved. □

3.1. **Extension of Bessel sequences to dual frame pairs.** Extension of Bessel sequences to frames or dual frames is one of fundamental topic for research. This is useful in applications of frames in signal processing. In this direction, Christensen et al. [2] studied the extension of Bessel sequences to frames. Also, see [16], for the extension of a system to tight frames. Extension of Bessel sequences to dual frame pairs was further studied in [3,4,15]. The following result provides sufficient conditions for the extension of a pair of matrix-valued Gabor Bessel sequences to matrix-valued dual Gabor frames for $L^2(G, \mathbb{C}^{n\times n})$. This is inspired by [2, Lemma 4.1].

Theorem 3.5. *Let $\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ and $\{E_{Cm}T_{Bk}\widetilde{\Phi}_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ be matrix-valued Gabor Bessel sequences in $L^2(G, \mathbb{C}^{n\times n})$ with pre-frame operators V_Φ and $V_{\widetilde{\Phi}}$ and analysis operators W_Φ and $W_{\widetilde{\Phi}}$, respectively. Let I be the identity operator on $L^2(G, \mathbb{C}^{n\times n})$. Assume there exist $\Psi_l \in L^2(G, \mathbb{C}^{n\times n}), l \in \Lambda_0$, with the following properties:*

(1) *$\{E_{Cm}T_{Bk}\Psi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ is a matrix-valued Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ with a dual $\{E_{Cm}T_{Bk}\widetilde{\Psi_l}\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$.*
(2) *$V_\Phi W_{\widetilde{\Phi}} E_{Cm}T_{Bk}\Psi_l = E_{Cm}T_{Bk}V_\Phi W_{\widetilde{\Phi}}\Psi_l$, for $l \in \Lambda_0, k \in \Lambda, m \in \Lambda'$.*

Let $\Phi_l', \widetilde{\Phi}_l' \in L^2(G, \mathbb{C}^{n\times n})$ be such that $\Phi_l' = (I - V_\Phi W_{\widetilde{\Phi}})\Psi_l$ and $\widetilde{\Phi}_l' = \widetilde{\Psi_l}, l \in \Lambda_0$.

Then, the families

$$\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'} \bigcup \{E_{Cm}T_{Bk}\Phi_l'\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$$

and

$$\{E_{Cm}T_{Bk}\widetilde{\Phi}_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'} \bigcup \{E_{Cm}T_{Bk}\widetilde{\Phi}_l'\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$$

form a matrix-valued dual frames pair for $L^2(G, \mathbb{C}^{n\times n})$.

Proof. For all $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$, we have

$$\begin{aligned} V_{\widetilde{\Phi}} W_\Phi \mathbf{f} &= V_{\widetilde{\Phi}}(\{\langle \mathbf{f}, E_{Cm}T_{Bk}\Phi_l\rangle\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}) \\ &= \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'} \langle f, E_{Cm}T_{Bk}\Phi_l\rangle E_{Cm}T_{Bk}\widetilde{\Phi}_l. \end{aligned}$$

By condition (1), $\{E_{Cm}T_{Bk}\Psi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ and $\{E_{Cm}T_{Bk}\widetilde{\Psi_l}\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ form a pair of dual frames for $L^2(G, \mathbb{C}^{n\times n})$. Therefore, for all $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$, we have

$$(I - V_{\widetilde{\Phi}} W_\Phi)\mathbf{f} = \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'} \langle (I - V_{\widetilde{\Phi}} W_\Phi)\mathbf{f}, E_{Cm}T_{Bk}\Psi_l\rangle E_{Cm}T_{Bk}\widetilde{\Psi_l}$$

$$= \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\langle \mathbf{f},(I-V_\Phi W_{\widetilde{\Phi}})E_{Cm}T_{Bk}\Psi_l\rangle E_{Cm}T_{Bk}\widetilde{\Psi_l}. \quad (3.2)$$

By using condition (2), we have

$$(I-V_\Phi W_{\widetilde{\Phi}})E_{Cm}T_{Bk}\Psi_l = E_{Cm}T_{Bk}(I-V_\Phi W_{\widetilde{\Phi}})\Psi_l. \quad (3.3)$$

For any $\mathbf{f}\in L^2(G,\mathbb{C}^{n\times n})$, we have

$$(I-V_{\widetilde{\Phi}}W_\Phi)\mathbf{f} = \mathbf{f} - \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\langle f,E_{Cm}T_{Bk}\Phi_l\rangle E_{Cm}T_{Bk}\widetilde{\Phi}_l. \quad (3.4)$$

By using (3.2), (3.3), (3.4), and $\Phi'_l = (I-V_\Phi W_{\widetilde{\Phi}})\Psi_l$, $\widetilde{\Phi}'_l = \widetilde{\Psi_l}, l\in\Lambda_0$, for all $\mathbf{f}\in L^2(G,\mathbb{C}^{n\times n})$, we have

$$\mathbf{f} = \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\langle \mathbf{f},E_{Cm}T_{Bk}\Phi_l\rangle E_{Cm}T_{Bk}\widetilde{\Phi_l} + \sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\langle f,E_{Cm}T_{Bk}\Phi'_l\rangle E_{Cm}T_{Bk}\widetilde{\Phi'_l}.$$

Therefore, the families $\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'} \bigcup \{E_{Cm}T_{Bk}\Phi'_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ and $\{E_{Cm}T_{Bk}\widetilde{\Phi}_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'} \bigcup \{E_{Cm}T_{Bk}\widetilde{\Phi}'_l\}_{l\in\Lambda_0,k\in\Lambda,m\in\Lambda'}$ form matrix-valued dual Gabor frames for $L^2(G,\mathbb{C}^{n\times n})$. □

Remark 3.6. Conditions given in Theorem 3.5 are sufficient but not necessary. First we recall that a sequence $\{\mathbf{f}_k\}_{k\in\mathbb{N}}$ in $L^2(G,\mathbb{C}^{n\times n})$ is called an orthonormal sequence if

$$\langle \mathbf{f}_k,\mathbf{f}_l\rangle = \begin{cases} I_{n\times n}, & k=l;\\ O_{n\times n}, & k\neq l. \end{cases} \quad (3.5)$$

Furthermore, a sequence $\{\mathbf{f}_k\}_{k\in\mathbb{N}} \subset L^2(G,\mathbb{C}^{n\times n})$ is an orthonormal basis for $L^2(G,\mathbb{C}^{n\times n})$ if it satisfies (3.5) and every $\mathbf{f}\in L^2(G,\mathbb{C}^{n\times n})$ can be written as $\mathbf{f} = \sum_{k\in\mathbb{N}}\langle \mathbf{f},\mathbf{f}_k\rangle\mathbf{f}_k$.

Now we show that the conditions given in Theorem 3.5 are only sufficient but not necessary. Let $\phi\in L^2(\mathbb{T})$ be given by $\phi(t)=1$, $t\in\mathbb{T}$. Let Λ be any lattice in $\mathbb{T}$ and let $\Lambda'=\mathbb{Z}$, a lattice in $\widehat{\mathbb{T}}=\mathbb{Z}$. Then, the Gabor system $\{E_mT_k\phi\}_{k\in\Lambda,m\in\Lambda'} = \{E_m\phi\}_{m\in\mathbb{Z}} = \{e^{2\pi i m(\cdot)}\}_{m\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$.

Consider

$$\Phi = \begin{bmatrix}\phi & 0\\ 0 & \phi\end{bmatrix} \in L^2(\mathbb{T},\mathbb{C}^{2\times 2}).$$

Then, $\{E_mT_k\Phi\}_{k\in\Lambda,m\in\Lambda'} = \{E_m\Phi\}_{m\in\mathbb{Z}}$ is an orthonormal system and for any $\mathbf{f}\in L^2(\mathbb{T},\mathbb{C}^{2\times 2})$. We have $\mathbf{f} = \sum_{m\in\mathbb{Z}}\langle \mathbf{f},E_m\Phi\rangle E_m\Phi$. That is, $\{E_m\Phi\}_{m\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T},\mathbb{C}^{2\times 2})$. In particular, we have a pair of matrix-valued Gabor Bessel sequences $\{E_m\Phi\}_{m\in\mathbb{Z}}$ and $\{E_m\widetilde{\Phi}\}_{m\in\mathbb{Z}} = \{E_m\Phi\}_{m\in\mathbb{Z}}$ in $L^2(\mathbb{T},\mathbb{C}^{2\times 2})$ with pre-frame operators V_Φ and $V_{\widetilde{\Phi}}$ and analysis operators W_Φ and $W_{\widetilde{\Phi}}$, respectively. Furthermore, for any $\mathbf{f}\in L^2(\mathbb{T},\mathbb{C}^{2\times 2})$, we have

$$V_\Phi W_{\widetilde{\Phi}}(\mathbf{f}) = V_\Phi(\{\langle \mathbf{f},E_m\widetilde{\Phi}\rangle\}_{m\in\mathbb{Z}}) = \sum_{m\in\mathbb{Z}}\langle \mathbf{f},E_m\Phi\rangle E_m\Phi = \mathbf{f}.$$

Thus, $V_\Phi W_{\widetilde{\Phi}} = I$. That is, $I - V_\Phi W_{\widetilde{\Phi}} = 0$. Note that $\{E_m T_k \Psi\}_{k\in\Lambda, m\in\Lambda'} = \{E_m T_k \Phi\}_{k\in\Lambda, m\in\Lambda'} = \{E_m \Phi\}_{m\in\mathbb{Z}}$ is a matrix-valued Gabor frame for $L^2(\mathbb{T}, \mathbb{C}^{2\times 2})$ with a dual $\{E_m T_k \widetilde{\Psi}\}_{k\in\Lambda, m\in\Lambda'} = \{E_m T_k \widetilde{\Phi}\}_{k\in\Lambda, m\in\Lambda'} = \{E_m \widetilde{\Phi}\}_{m\in\mathbb{Z}}$.

Choose $\Phi' = \begin{bmatrix} \phi & 0 \\ 0 & 0 \end{bmatrix}$, $\widetilde{\Phi}' = 0$. Then

$$\{E_m \Phi\}_{m\in\mathbb{Z}} \bigcup \{E_m \Phi'\}_{m\in\mathbb{Z}} \quad \text{and} \quad \{E_m \widetilde{\Phi}\}_{m\in\mathbb{Z}} \bigcup \{E_m \widetilde{\Phi}'\}_{m\in\mathbb{Z}}$$

constitute matrix-valued dual Gabor frames for $L^2(\mathbb{T}, \mathbb{C}^{2\times 2})$. Indeed, for any $\Phi \in L^2(\mathbb{T}, \mathbb{C}^{2\times 2})$, we have $\Phi' \neq (I - V_\Phi W_{\widetilde{\Phi}})\Psi = (I - V_\Phi W_{\widetilde{\Phi}})\Phi = 0$.

The next result gives sufficient conditions for the extension of a pair of matrix-valued Gabor Bessel sequences with several generator to matrix-valued Gabor dual frames for $L^2(G, \mathbb{C}^{n\times n})$. This type of extension for matrix-valued wave packet system over $\mathbb{R}^d$ can be found in [15].

Theorem 3.7. *Let $\{E_{Cm} T_{Bk} \Phi_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$ and $\{E_{Cm} T_{Bk} \widetilde{\Phi}_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$ be matrix-valued Gabor Bessel sequences in $L^2(G, \mathbb{C}^{n\times n})$ with pre-frame operators V_Φ and $V_{\widetilde{\Phi}}$ and analysis operators W_Φ and $W_{\widetilde{\Phi}}$, respectively. Assume there exist $\Psi_l \in L^2(G, \mathbb{C}^{n\times n}), l \in \Lambda_0$, with the following properties:*

(1) *$\{E_{Cm} T_{Bk} \Psi_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$ is a matrix-valued Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ with a dual $\{E_{Cm} T_{Bk} \widetilde{\Psi_l}\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$.*

(2) *$V_\Phi W_{\widetilde{\Phi}} T_{Bk} \Psi_l = T_{Bk} V_\Phi W_{\widetilde{\Phi}} \Psi_l$, for $l \in \Lambda_0, k \in \Lambda$.*

Then, there exist $\Phi'_l, \widetilde{\Phi}'_l \in L^2(G, \mathbb{C}^{n\times n}), l \in \Lambda_0$, such that the families

$$\{E_{Cm} T_{Bk} \Phi_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'} \bigcup \{E_{Cm} T_{Bk} \Phi'_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$$

and

$$\{E_{Cm} T_{Bk} \widetilde{\Phi}_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'} \bigcup \{E_{Cm} T_{Bk} \widetilde{\Phi}'_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$$

form dual matrix-valued Gabor frames for $L^2(G, \mathbb{C}^{n\times n})$.

Proof. For any $m' \in \Lambda'$, we compute

$$\begin{aligned} V_\Phi W_{\widetilde{\Phi}} E_{Cm'} \Psi_l &= \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \langle E_{Cm'} \Psi_l, E_{Cm} T_{Bk} \widetilde{\Phi}_l \rangle E_{Cm} T_{Bk} \Phi_l \\ &= \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \langle \Psi_l, E_{-Cm'} E_{Cm} T_{Bk} \widetilde{\Phi}_l \rangle E_{Cm} T_{Bk} \Phi_l \\ &= E_{Cm'} E_{-Cm'} \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \langle \Psi_l, E_{C(m-m')} T_{Bk} \widetilde{\Phi}_l \rangle E_{Cm} T_{Bk} \Phi_l \\ &= E_{Cm'} \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \langle \Psi_l, E_{C(m-m')} T_{Bk} \widetilde{\Phi}_l \rangle E_{C(m-m')} T_{Bk} \Phi_l \\ &= E_{Cm'} \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \langle \Psi_l, E_{Cm} T_{Bk} \widetilde{\Phi}_l \rangle E_{Cm} T_{Bk} \Phi_l \\ &= E_{Cm'} V_\Phi W_{\widetilde{\Phi}} \Psi_l. \end{aligned}$$

The result now follows from Theorem 3.5. ∎

3.2. **Matrix-valued symmetric Gabor frames over LCA groups.** Wu and Cao [18] and Wang and Wu [17] studied symmetric frames with wave packet structure in the Lebesgue space $L^2(\mathbb{R}^2)$. Let $\{\varphi^m\}_{m=1}^M$ be a finite set in $L^2(\mathbb{R}^2)$. The Wang and Wu [17] considered the following functions:

$$\varphi_1^m(x) = \frac{\varphi^m(x) + \varphi^m(-x)}{2} \quad \text{and} \quad \varphi_2^m(x) = \frac{\varphi^m(x) - \varphi^m(-x)}{2}, \qquad 1 \le m \le M.$$

Wang and Wu [17] showed that if the wave packet system

$$\Big\{D_2^j E_\ell T_k g^m : j \in \mathbb{Z}, k, \ell \in \mathbb{Z}^2, m \in \{1, 2, \dots, M\}\Big\}$$

is a frame for $L^2(\mathbb{R}^2)$ with frame bounds a_o, b_o, then the collection

$$\Big\{D_2^j E_\ell T_k g_1^m \cup D_2^j E_\ell T_k g_2^m : j \in \mathbb{Z}, k, \ell \in \mathbb{Z}^2, m \in \{1, 2, \dots, M\}\Big\}$$

is a symmetric or antisymmetric frame about origin with the same frame bounds. On the other hand, the frame properties, in general, cannot be carried from $L^2(G)$ to $L^2(G, \mathbb{C}^{n\times n})$ and vice-versa. In [15], the authors studied matrix-valued symmetric wave packet frames over $\mathbb{R}^d$. The following result shows that the frame properties of matrix-valued symmetric Gabor frames over LCA associated with a given matrix-valued Gabor frame for the underlying function space are preserved. This is inspired by [17, Theorem 1].

Theorem 3.8. *Let $\{E_{Cm}T_{Bk}\Phi_l\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$ be a matrix-valued Gabor frame for $L^2(G, \mathbb{C}^{n\times n})$ with frame bounds L_Φ, U_Φ. Then*

$$\{E_{Cm}T_{Bk}\Phi_l^1\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'} \bigcup \{E_{Cm}T_{Bk}\Phi_l^2\}_{l\in\Lambda_0, k\in\Lambda, m\in\Lambda'}$$

forms a symmetric matrix-valued frame for $L^2(G, \mathbb{C}^{n\times n})$ with the frame bounds L_Φ, U_Φ, where $\Phi_l^1, \Phi_l^2 \in L^2(G, \mathbb{C}^{n\times n})$ are defined as

$$\Phi_l^1(\xi) = \frac{\Phi_l(\xi) + \Phi_l(-\xi)}{2}, \quad \Phi_l^2(\xi) = \frac{\Phi_l(\xi) - \Phi_l(-\xi)}{2} \quad (l \in \Lambda_0, \xi \in G).$$

.

Proof. For $M_1, M_2 \in \mathcal{M}_n(\mathbb{C})$, it is easy to see that the Frobenius norm satisfies the following property:

$$\|M_1 + M_2\|^2 = \|M_1\|^2 + \|M_2\|^2 + \operatorname{trace}(M_1{}^* M_2) + \operatorname{trace}(M_1 M_2{}^*). \tag{3.6}$$

Using (3.6), for any $\mathbf{f} \in L^2(G, \mathbb{C}^{n\times n})$, we compute

$$\begin{aligned}
&\sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \Big\|\Big\langle E_{Cm}T_{Bk}\Phi_l^1, \mathbf{f}\Big\rangle\Big\|^2 \\
&= \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \Big\|\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(\cdot)}{2}, \mathbf{f}\Big\rangle + \Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(-\cdot)}{2}, \mathbf{f}\Big\rangle\Big\|^2 \\
&= \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \Big\|\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(\cdot)}{2}, \mathbf{f}\Big\rangle\Big\|^2 + \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \Big\|\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(-\cdot)}{2}, \mathbf{f}\Big\rangle\Big\|^2 \\
&\quad + \sum_{l\in\Lambda_0} \sum_{k\in\Lambda, m\in\Lambda'} \operatorname{trace}\Big(\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(\cdot)}{2}, \mathbf{f}\Big\rangle^* \Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(-\cdot)}{2}, \mathbf{f}\Big\rangle\Big)
\end{aligned}$$

$$+\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\operatorname{trace}\Big(\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(\cdot)}{2},\mathbf{f}\Big\rangle\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(-\cdot)}{2},\mathbf{f}\Big\rangle^*\Big). \tag{3.7}$$

Similarly, for any $\mathbf{f}\in L^2(G,\mathbb{C}^{n\times n})$, we have

$$\begin{aligned}
&\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\Big\|\Big\langle E_{Cm}T_{Bk}\Phi_l^2,\mathbf{f}\Big\rangle\Big\|^2\\
&=\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\Big\|\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(\cdot)}{2},\mathbf{f}\Big\rangle\Big\|^2+\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\Big\|\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(-\cdot)}{2},\mathbf{f}\Big\rangle\Big\|^2\\
&\quad-\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\operatorname{trace}\Big(\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(\cdot)}{2},\mathbf{f}\Big\rangle^*\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(-\cdot)}{2},\mathbf{f}\Big\rangle\Big)\\
&\quad-\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\operatorname{trace}\Big(\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(\cdot)}{2},\mathbf{f}\Big\rangle\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(-\cdot)}{2},\mathbf{f}\Big\rangle^*\Big).
\end{aligned}\tag{3.8}$$

From (3.7) and (3.8), we have

$$\begin{aligned}
&\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\Big\|\Big\langle E_{Cm}T_{Bk}\Phi_l^1,\mathbf{f}\Big\rangle\Big\|^2+\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\Big\|\Big\langle E_{Cm}T_{Bk}\Phi_l^2,\mathbf{f}\Big\rangle\Big\|^2\\
&=2\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\Big\|\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(\cdot)}{2},\mathbf{f}\Big\rangle\Big\|^2+2\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\Big\|\Big\langle E_{Cm}T_{Bk}\frac{\Phi_l(-\cdot)}{2},\mathbf{f}\Big\rangle\Big\|^2\\
&=\frac{1}{2}\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\Big\|\Big\langle E_{Cm}T_{Bk}\Phi_l(\cdot),\mathbf{f}\Big\rangle\Big\|^2+\frac{1}{2}\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\Big\|\Big\langle E_{Cm}T_{Bk}\Phi_l,\mathbf{f}(-\cdot)\Big\rangle\Big\|^2\\
&\geq\frac{1}{2}L_\Phi\|\mathbf{f}\|^2+\frac{1}{2}L_\Phi\|\mathbf{f}(-\cdot)\|^2\\
&=L_\Phi\|\mathbf{f}\|^2,\ \ \mathbf{f}\in L^2(G,\mathbb{C}^{n\times n}).
\end{aligned}\tag{3.9}$$

Similarly,

$$\begin{aligned}
&\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\Big\|\Big\langle E_{Cm}T_{Bk}\Phi_l^1,\mathbf{f}\Big\rangle\Big\|^2+\sum_{l\in\Lambda_0}\sum_{k\in\Lambda,m\in\Lambda'}\Big\|\Big\langle E_{Cm}T_{Bk}\Phi_l^2,\mathbf{f}\Big\rangle\Big\|^2\\
&\leq U_\Phi\|\mathbf{f}\|^2,\ \ \mathbf{f}\in L^2(G,\mathbb{C}^{n\times n}).
\end{aligned}\tag{3.10}$$

The proof now follows from (3.9) and (3.10). □

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