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A STUDY OF BESSEL SEQUENCES AND FRAMES VIA PERTURBATIONS OF CONSTANT MULTIPLES OF THE IDENTITY

SIMA MOVAHED[1], HOSSEIN HOSSEINI GIV[1*] AND ALIREZA AHMADI LEDARI[1]

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Abstract. We study those Bessel sequences $\{f_k\}_{k=1}^{\infty}$ in an infinite dimensional, separable Hilbert space H for which operator S defined by $Sf = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k$ is of the form $cI + T$, for some real number c and a bounded linear operator T, where I denotes the identity operator. We use a reverse Schwarz inequality to provide conditions on T and c that allow $\{f_k\}_{k=1}^{\infty}$ to be a frame. Moreover, we study frames whose frame operators are compact (respectively, finite-rank) perturbations of constant multiples of the identity, frames to which we refer as compact-tight (respectively, finite-rank-tight) frames. As our final result, we prove a theorem on the weaving of certain compact-tight frames.

1. Introduction and preliminaries

Frames were defined by Duffin and Schaeffer [10] in 1952 in the context of non-harmonic Fourier series. Later on, various generalizations of frames were proposed; see [1, 3, 4, 12, 16], for example. A frame for a Hilbert space H can be considered as a generalization of an orthonormal basis. Unlike a basis, a frame consists of vectors that are not necessarily linearly independent. Due to the flexibility of frames, they have found many applications in mathematics, physics, and engineering, including signal processing, image processing, and sampling theory. To learn more about the basic theory and applications of frames, we refer the reader to [4, 6, 7, 11, 13, 14].

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*Corresponding author.
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Throughout the paper, H denotes an infinite-dimensional, separable Hilbert space over the field $\mathbb{F}$ of real or complex numbers ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). A sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space H is called a *frame* if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in H.$$

The numbers A and B are called *lower* and *upper frame bounds*, respectively. If we can choose $A = B$, then the frame is called *tight*, and if $A = B = 1$, then the frame is called *Parseval*. The sequence $\{f_k\}_{k=1}^{\infty}$ is said to be a *Bessel sequence* if at least the upper frame condition holds.

Let $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$ be a frame for H. Then, the operators $U_{\mathcal{F}}$ and $U_{\mathcal{F}}^*$ defined by

$$U_{\mathcal{F}} : \ell^2 \to H, \quad U_{\mathcal{F}}(\{c_k\}_{k\in I}) = \sum_{k=1}^{\infty} c_k f_k,$$

and

$$U_{\mathcal{F}}^* : H \to \ell^2, \quad U_{\mathcal{F}}^* f = \{\langle f, f_k\rangle\}_{k\in I}$$

are known as the *synthesis* and *analysis* operators of $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$, respectively. The *frame operator* $S : H \to H$, defined by

$$Sf = U_{\mathcal{F}} U_{\mathcal{F}}^* f = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k,$$

is a positive, self-adjoint, and invertible operator.

The paper is organized as follows. In Section 2, we study those Bessel sequences $\{f_k\}_{k=1}^{\infty}$ in a separable Hilbert space H for which the operator S defined by $Sf := \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k$ is of the form $cI + T$, for some real number c and a bounded linear operator T, where I is the identity operator. We also use a reverse Schwarz inequality to provide conditions on T and c that allow $\{f_k\}_{k=1}^{\infty}$ to be a frame. For more information on reverse Schwarz inequalities, we refer the reader to [8, 9]. In Section 3, we study frames whose frame operators are compact or finite-rank perturbations of constant multiples of the identity. We refer to such frames as compact- and finite-rank-tight frames, respectively. After a careful examination of these notions of frame, we establish a result on the weaving of certain compact-tight frames.

In the rest of this section, we set the stage by discussing some results and notations, which will be used later.

Theorem 1.1 (Spectral theorem for compact self-adjoint operators)**.** *Let T be a compact, self-adjoint operator on an infinite-dimensional, separable Hilbert space H. Then, H has a complete orthonormal system (an orthonormal basis) $\{e_1, e_2, \ldots\}$ consisting of the eigenvectors of T. Moreover, for every $x \in H$,*

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k\rangle e_k,$$

where λ_k is the eigenvalue corresponding to e_k.

In what follows, we present a well-known perturbation theorem (see [6, Theorem 22.1.1]).

Theorem 1.2 ([6]). *Let* $\{f_k\}_{k=1}^{\infty}$ *be a frame for* H *with lower and upper bounds* A, B *respectively. Let* $\{g_k\}_{k=1}^{\infty}$ *be a sequence in* H *and assume that there exist constants* λ, $\mu \geq 0$ *such that* $\lambda + \frac{\mu}{\sqrt{A}} < 1$ *and*

$$\|\sum_{k=1}^{\infty} c_k(f_k - g_k)\| \leq \lambda \|\sum_{k=1}^{\infty} c_k f_k\| + \mu \|c_k\|$$

for all finite scalar sequences $\{c_k\}_{k=1}^{\infty}$. *Then,* $\{g_k\}_{k=1}^{\infty}$ *is a frame for* H *with bounds* $(1 - (\lambda + \frac{\mu}{\sqrt{A}}))^2$ *and* $B(1 + \lambda + \frac{\mu}{\sqrt{B}})^2$.

The following proposition is a special and rephrased version of the previous theorem that will be needed in Definition 1.4 and Theorem 2.3. In fact, it can be obtained from the previous theorem by letting $\lambda = 0$.

Proposition 1.3 ([2]). *Let* $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$ *be a frame for* H *with bounds* A, B, *respectively. Let* $\mathcal{G} = \{g_k\}_{k=1}^{\infty}$ *be a sequence in* H. *If there exists* $\mu \geq 0$ *such that* $\|U_{\mathcal{F}} - U_{\mathcal{G}}\| \leq \mu < \sqrt{A}$, *then* $\{g_k\}_{k=1}^{\infty}$ *is a frame with lower and upper bounds* $A(1 - \frac{\mu}{\sqrt{A}})^2$ *and* $B(1 + \frac{\mu}{\sqrt{A}})^2$, *and* $\|U_{\mathcal{F}} - U_{\mathcal{G}}\| \leq \mu$.

Next, we present the important concept of μ-perturbation. We will use this concept in Theorem 2.3 (3).

Definition 1.4. Let $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$ and $\mathcal{G} = \{g_k\}_{k=1}^{\infty}$ be Bessel sequences in a Hilbert space H. For $\mu > 0$, we say that $\mathcal{G}$ is a μ-perturbation of $\mathcal{F}$ if

$$\|U_{\mathcal{F}} - U_{\mathcal{G}}\| \leq \mu.$$

The following theorem is that part of [15, Corollary 1], which is appropriate for our destinations. It presents a reverse Schwarz inequality, which will be used later in Theorem 2.3.

Theorem 1.5 ([15]). *Let* $(V, \langle \cdot, \cdot \rangle)$ *be an inner product space over the real or complex number field* $\mathbb{F}$ *(*$\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*). Fix* $\tau \geq 0$ *and* $\kappa \geq 1$ *with* $\kappa^2 = \tau^2 + 1$. *For nonzero vectors* $x, y \in V$, *the following conditions are equivalent:*

(1) $||x||\ ||y|| \leq \kappa |\langle x, y \rangle|$.
(2) *There exists* $\lambda \in \mathbb{F} \backslash \{0\}$ *such that* $||\lambda x - y|| \leq \frac{\tau}{\kappa} ||y||$.

Note that $\kappa = \sqrt{1 + \tau^2}$. Therefore, the equivalence stated in Theorem 1.5 can be written as

$$||x||\ ||y|| \leq \sqrt{1 + \tau^2} |\langle x, y \rangle| \Longleftrightarrow ||\lambda x - y|| \leq \frac{\tau}{\sqrt{1 + \tau^2}} ||y||.$$

Our next definition will be utilized in Theorem 3.11.

Definition 1.6 ([5]). A finite family of frames $\{f_{ij}\}_{j=1, i \in I}^{M}$ in a Hilbert space H is said to be woven if there exist universal constants A and B such that for every partition $\{\sigma_j\}_{j=1}^{M}$ of I, the family $\{f_{ij}\}_{j=1, i \in \sigma_j}^{M}$ is a frame for H with bounds A and B, respectively. Each family $\{f_{ij}\}_{j=1, i \in \sigma_j}^{M}$ is called a weaving.

If $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence in H, then we let S denote the operator, which is defined by

$$Sf := \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \tag{1.1}$$

for every $f \in H$.

2. From Bessel sequences to frames

Let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence in an infinite-dimensional, separable Hilbert space H, and let S be the operator defined as in (1.1). Assume that $S = cI + T$ for some real number c and an operator T. Let us begin with some questions.

1) Assuming that (1.1) holds, what can be proved for T? For instance, is T necessarily bounded? If yes, what is an estimate for $\|T\|$?
2) Under what conditions on c and T, $\{f_k\}_{k=1}^{\infty}$ can be a frame?
3) If $\{f_k\}_{k=1}^{\infty}$ is a frame, what further properties can be proved for T?

In the remainder of this section, we try to answer these questions. However, before doing so, let us describe the reason such questions are really worthy of consideration. It is known that the frame operator of a tight frame is a constant multiple of the identity. So, it would be a rewarding idea to study those frames for which the frame operator is a perturbation of some constant multiple of the identity, for example, a compact perturbation. The latter case is of great importance, because by studying such perturbations, we obtain frames which are, in a sense, the closest relatives of tight frames.

Proposition 2.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence in H. Also, assume that S is the operator defined as in (1.1). If $S = cI + T$ for some real number c and a linear operator T, then the following statements are true:*

(1) *The operator T is bounded and self-adjoint.*
(2) *If T is a positive operator and $c > 0$, then $\{f_k\}_{k=1}^{\infty}$ is a frame with c as a lower bound.*
(3) *If $\{f_k\}_{k=1}^{\infty}$ is a frame with lower bound A, then T is a positive operator whenever $c \le A$.*

Proof. (1) Since $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence, S is a bounded operator. Hence, $T = S - cI$ is also bounded. That T is self-adjoint is obvious, because c is a real number.

(2) Given $f \in H$,

$$\begin{aligned} \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 &= \langle Sf, f \rangle \\ &= \langle cf, f \rangle + \langle Tf, f \rangle \\ &\ge c||f||^2. \end{aligned}$$

(3) Note that for every $f \in H$,

$$\begin{aligned} \langle Tf, f \rangle &= \langle Sf, f \rangle - \langle cf, f \rangle \\ &\ge (A - c)||f||^2. \end{aligned}$$

Therefore, $c \leq A$ implies that $\langle Tf, f\rangle \geq 0$ for every $f \in H$. □

Remark 2.2. If $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence in H and S is defined by $Sf := \sum_k \langle f, f_k\rangle f_k$ for every $f \in H$, then for every $c \in \mathbb{R}$, one can find an operator T for which (1.1) holds. To see this, we just need to let $T = S - cI$. Indeed, what matters is the way c affects T. For example, Proposition 2.1 (3) shows that when $\{f_k\}_{k=1}^{\infty}$ is a frame, for certain values of c, the resulting operators T are positive.

According to Proposition 2.1, the positivity of both T and c implies that $\{f_k\}_{k=1}^{\infty}$ is a frame with lower bound c. In the following theorem, we use a reverse Schwarz inequality to provide some other conditions that make $\{f_k\}_{k=1}^{\infty}$ into a frame. We will see shortly (in Example 2.4) that in some cases, the proposed lower bound can be a better substitute for the value c itself as a lower bound for $\{f_k\}_{k=1}^{\infty}$. Moreover, the positivity of c is not assumed in Theorem 2.3 below.

Theorem 2.3. *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in H such that for every $f \in H$,*

$$\sum_{k=1}^{\infty} \langle f, f_k\rangle f_k = cf + Tf, \tag{2.1}$$

where c is a real scalar and T is a bounded operator.

(1) *The sequence $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence with bound $|c| + ||T||$.*
(2) *Assume that $\lambda \in \mathbb{F}\backslash\{0\}$ and $\tau \geq 0$ are such that*

$$||\lambda f - Tf|| \leq \frac{\tau}{\sqrt{1+\tau^2}} ||Tf||, \tag{2.2}$$

for every $f \in H$. If T is also bounded from below by D and

$$\frac{D}{\sqrt{1+\tau^2}} > |c|, \tag{2.3}$$

then $\{f_k\}_{k=1}^{\infty}$ is a frame with lower frame bound

$$\frac{D}{\sqrt{1+\tau^2}} - |c|.$$

(3) *With the assumptions of (2), suppose also that $\{g_k\}_{k=1}^{\infty}$ is a μ-perturbation of $\{f_k\}_{k=1}^{\infty}$. If $\mu < \sqrt{\frac{D}{\sqrt{1+\tau^2}} - |c|}$, then $\{g_k\}_{k=1}^{\infty}$ is a frame for H with bounds $\left(\sqrt{\frac{D}{\sqrt{1+\tau^2}} - |c|} - \mu\right)^2$ and $\left(\sqrt{|c| + \|T\|} - \mu\right)^2$.*

Proof. (1) It follows from (2.1) that

$$\begin{aligned}
\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 &= |\langle cf, f\rangle + \langle Tf, f\rangle| \\
&\leq |c| \ \|f\|^2 + |\langle Tf, f\rangle| \\
&\leq |c| \ \|f\|^2 + \|Tf\|.\|f\| \\
&\leq |c| \ \|f\|^2 + \|T\|.\|f\|.\|f\| \\
&= (|c| + \|T\|)\|f\|^2.
\end{aligned}$$

(2) In view of Theorem 1.5 and the assumption that T is bounded from below by D, we can write

$$\begin{aligned}\sum_{k=1}^{\infty}|\langle f,f_k\rangle|^2 &= |\langle cf,f\rangle+\langle Tf,f\rangle| \\ &\geq |\langle Tf,f\rangle|-|c|\ ||f||^2 \\ &= |\langle f,Tf\rangle|-|c|\ ||f||^2 \\ &\geq \frac{1}{\sqrt{1+\tau^2}}||f||\ ||Tf||-|c|\ ||f||^2 \qquad (2.4)\\ &\geq \frac{D}{\sqrt{1+\tau^2}}||f||^2-|c|\ ||f||^2 \qquad (2.5)\\ &= \left(\frac{D}{\sqrt{1+\tau^2}}-|c|\right)||f||^2.\end{aligned}$$

In fact, (2.4) follows from (2.2) and Theorem 1.5, and (2.5) is an outgrowth of the assumption that T is bounded from below by D.

(3) Since $\{f_k\}_{k=1}^{\infty}$ is a frame with bounds $|c|+||T||$ and $\frac{D}{\sqrt{1+\tau^2}}-|c|$, for every $f\in H$,

$$\left(\sqrt{\frac{D}{\sqrt{1+\tau^2}}-|c|}\right)\|f\|\leq\|U_{\mathcal{F}}^*f\|\leq\left(\sqrt{|c|+||T||}\right)\|f\|.$$

Therefore,

$$\|U_{\mathcal{G}}^*(f)\|\geq\|U_{\mathcal{F}}^*(f)\|-\|\left(U_{\mathcal{F}}(f)-U_{\mathcal{G}}(f)\right)^*\|\geq\left(\sqrt{\frac{D}{\sqrt{1+\tau^2}}-|c|}-\mu\right)\|f\|.$$

Furthermore,

$$\|U_{\mathcal{G}}^*(f)\|\leq\|\left(U_{\mathcal{F}}(f)-U_{\mathcal{G}}(f)\right)^*\|+\|U_{\mathcal{F}}^*(f)\|\leq(\mu+\sqrt{|c|+||T||})\|f\|.$$

Hence,

$$\left(\sqrt{\frac{D}{\sqrt{1+\tau^2}}-|c|}-\mu\right)^2\|f\|^2\leq\sum_{i=1}^{\infty}|\langle f,g_k\rangle|^2\leq\left(\mu+\sqrt{|c|+||T||}\right)^2\|f\|^2.$$

□

In the following example, we present an operator T and constants c for which (2.2) and (2.3) are true.

Example 2.4. Define $T:\ell_2\to\ell_2$ by

$$f:=(f_1,f_2,f_3,\ldots)\mapsto(0.99f_1,0.98f_2,0.97f_3,f_4,f_5,\ldots).$$

Then, T is linear and

$$\frac{1}{\sqrt{2}}||f||_2\leq||Tf||_2\leq||f||_2, \qquad (2.6)$$

for every $f \in \ell_2$. So, we may let $D = \frac{1}{\sqrt{2}}$ in the context of Theorem 2.3. Also, by letting $\lambda = 0.99$, we see that

$$\begin{aligned} ||\lambda f - Tf||_2^2 &\leq \left((0.02)^2 + (0.01)^2\right) ||f||_2^2 \\ &= (5)(10^{-4})||f||_2^2. \end{aligned}$$

Therefore, choosing $\tau = 1$ and having (2.6) in mind, we can write

$$\begin{aligned} ||\lambda f - Tf||_2 &\leq \frac{\sqrt{5}}{100}||f||_2 \\ &\leq \frac{\sqrt{10}}{100}||Tf||_2 \\ &\leq \frac{1}{\sqrt{2}}||Tf||_2 \\ &= \frac{\tau}{\sqrt{1+\tau^2}}||Tf||_2, \end{aligned}$$

for every $f \in H$. Therefore, (2.2) holds for our choices of T, λ, and τ. If $0 < c < \frac{1}{4}$, for example $c = \frac{1}{5}$, then

$$\frac{D}{\sqrt{1+\tau^2}} = \frac{1}{2} > c,$$

so that (2.3) also holds. Moreover,

$$\frac{D}{\sqrt{1+\tau^2}} - c = \frac{1}{2} - c > c.$$

This shows that for $0 < c < \frac{1}{4}$, the lower bound proposed in Theorem 2.3 is better than the lower bound c presented in Proposition 2.1.

In the next section, we focus on Bessel sequences for which the operator S defined by (1.1) is of the form $S = cI + K$ with K a compact or finite-rank operator.

3. Compact-tight and finite-rank-tight frames

Let T be a compact self-adjoint operator. By the spectral theorem (Theorem 1.1), the operator T has the form

$$T = \sum_{k=1}^{\infty} \lambda_k \langle \cdot, e_k \rangle e_k, \tag{3.1}$$

where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H consisting of the eigenvectors of T. In the following theorem, we use this fact to present conditions that allow us to conclude that a Bessel sequence is a frame.

Theorem 3.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence such that $S = cI + T$, for some $c > 0$ and a compact operator T. Assume that T has a spectral decomposition like (3.1) such that*

$$\lambda := c + \inf_k \lambda_k > 0.$$

Then, $\{f_k\}_{k=1}^{\infty}$ is a frame with lower bound λ.

Proof. According to Proposition 2.1 (1), the operator T is self-adjoint. So, we can use the spectral theorem. For every $f \in H$,

$$\begin{aligned} Sf &= cf + \sum_{k=1}^{\infty} \lambda_k \langle f, e_k \rangle e_k \\ &= c\sum_{k=1}^{\infty} \langle f, e_k \rangle e_k + \sum_{k=1}^{\infty} \lambda_k \langle f, e_k \rangle e_k \\ &= \sum_{k=1}^{\infty} (c + \lambda_k) \langle f, e_k \rangle e_k. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 &= \langle Sf, f \rangle \\ &= \sum_{k=1}^{\infty} (c + \lambda_k) |\langle f, e_k \rangle|^2 \\ &\geq \lambda ||f||^2. \end{aligned}$$

□

The following definition facilitates our discussion.

Definition 3.2. We say that a frame is compact-tight (respectively, finite-rank-tight) if its frame operator S is a compact (respectively, finite-rank) perturbation of a constant multiple of the identity, that is, $S = cI + K$ with K being a compact (respectively, finite-rank) operator.

Let $\{f_k\}_{k=1}^{\infty}$ be a compact-tight frame whose frame operator S can be written as $c_1 I + K_1$ and $c_2 I + K_2$, with K_1 and K_2 being compact operators. Then, it follows from the equality

$$(c_1 - c_2)I = K_2 - K_1 \tag{3.2}$$

that $c_1 = c_2$. In fact, the operator on the right side of (3.2) is a compact operator, while the operator on the left cannot be compact unless $c_1 - c_2 = 0$. Indeed, in the latter case, (3.2) shows that $K_1 = K_2$. We summarize our above discussion in the following proposition.

Proposition 3.3. *Let $\{f_k\}_{k=1}^{\infty}$ be a compact-tight frame with frame operator S. Then, the representation of S as a compact perturbation of a constant multiple of the identity is unique.*

A similar reasoning allows us to formulate a version of Proposition 3.3 for finite-rank-tight frames.

If $\{f_k\}_{k=1}^{\infty}$ is a compact-tight frame whose frame operator is $S = cI + K$, we say that $\{f_k\}_{k=1}^{\infty}$ is a (c, K)-compact-tight frame.

Remark 3.4. In general, a frame is compact-tight if and only if its frame operator S can be written as

$$Sf = \sum_{k=1}^{\infty}(c+\lambda_k)\langle f, e_k\rangle e_k,$$

for some nonzero real number c, a sequence $\{\lambda_k\}_{k=1}^{\infty}$ of nonzero real numbers converging to 0, and an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of H.

In our next theorem, we present a procedure for constructing compact-tight frames from a given orthonormal basis.

Theorem 3.5. *Let c be a positive real number, and let $\{t_k\}_{k=1}^{\infty}$ be a sequence of real numbers greater than c such that $\lim_{k\longrightarrow\infty} t_k = c$. If $\{e_k\}_{k=1}^{\infty}$ is any orthonormal basis of H, then $\{\sqrt{t_k}e_k\}_{k=1}^{\infty}$ is a compact-tight frame for H with frame bounds c and $\sup_k t_k$.*

Proof. Given $f \in H$, the equality

$$\sum_{k=1}^{\infty}|\langle f, \sqrt{t_k}e_k\rangle|^2 = \sum_{k=1}^{\infty} t_k|\langle f, e_k\rangle|^2$$

implies that

$$c\|f\|^2 \leq \sum|\langle f, \sqrt{t_k}e_k\rangle|^2 \leq (\sup_k t_k)\|f\|^2.$$

Thus, $\{\sqrt{t_k}e_k\}_{k=1}^{\infty}$ is a frame for H with frame bounds c and $\sup_k t_k$. To prove that this frame is compact-tight, note that

$$\begin{aligned} Sf &= \sum_{k=1}^{\infty}\langle f, \sqrt{t_k}e_k\rangle\sqrt{t_k}e_k \\ &= \sum_{k=1}^{\infty} t_k\langle f, e_k\rangle e_k \\ &= c\sum_{k=1}^{\infty}\langle f, e_k\rangle e_k + \sum_{k=1}^{\infty}(t_k - c)\langle f, e_k\rangle e_k \\ &= cf + Kf. \end{aligned}$$

Here,

$$K := \sum_{k=1}^{\infty}(t_k - c)\langle ., e_k\rangle e_k$$

is a compact operator because $(t_k - c) \to 0$ as $k \to \infty$. □

In what follows, we present a simple example for Theorem 3.5.

Example 3.6. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. Since

$$t_k := 1 + \frac{1}{k^2} > 1$$

for every k and

$$\lim_{k\to\infty} t_k = 1,$$

Theorem 3.5 shows that the sequence

$$\{g_k\}_{k=1}^{\infty} := \{\sqrt{t_k} e_k\}_{k=1}^{\infty}$$

is a compact-tight frame with bounds 1 and 2.

Of course, the procedure proposed in Theorem 3.5 is not the only way for the construction of compact-tight frames. This is what we show in the following example.

Example 3.7. Let $\{g_k\}_{k=1}^{\infty}$ be as in Example 3.6, and let $\{f_k\}_{k=1}^{\infty}$ be the sequence in which $\sqrt{\frac{1}{k}(1+\frac{1}{k^2})e_k}$ is repeated k times, for each k. Then, $\{f_k\}_{k=1}^{\infty}$ is also a frame with the same frame operator as $\{g_k\}_{k=1}^{\infty}$. This is because

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 = \sum_{k=1}^{\infty} k \left|\left\langle f, \sqrt{\frac{1}{k}\left(1+\frac{1}{k^2}\right) e_k}\right\rangle\right|^2 = \sum_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right)|\langle f, e_k\rangle|^2$$

and

$$\sum_{k=1}^{\infty} |\langle f, g_k\rangle|^2 = \sum_{k=1}^{\infty} \left|\left\langle f, \sqrt{1+\frac{1}{k^2} e_k}\right\rangle\right|^2 = \sum_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right)|\langle f, e_k\rangle|^2.$$

Therefore, $\{f_k\}_{k=1}^{\infty}$ is also a compact-tight frame. However, it is clear that this frame is not constructed in the way proposed in Theorem 3.5.

The following example presents a frame that is not compact-tight.

Example 3.8. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H, and consider $f_k = \sqrt{2+\frac{1}{k^2} e_k}$ when k is even, and $f_k = \sqrt{3+\frac{1}{k^2} e_k}$ otherwise. Then

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 = \sum_{k\in\mathbb{N}_e}\left(2+\frac{1}{k^2}\right)|\langle f, e_k\rangle|^2 + \sum_{k\in\mathbb{N}_o}\left(3+\frac{1}{k^2}\right)|\langle f, e_k\rangle|^2,$$

where $\mathbb{N}_e$ and $\mathbb{N}_o$ denote the sets of even and odd natural numbers, respectively. Hence

$$2||f||^2 \le \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \le 4||f||^2,$$

showing that $\{f_k\}_{k=1}^{\infty}$ is a frame with bounds 2 and 4. If $S = cI + T$ for some scalar c and an operator T, then

$$\begin{aligned} Tf &= (S-cI)f \\ &= \sum_{k\in\mathbb{N}_e}\left(2-c+\frac{1}{k^2}\right)\langle f, e_k\rangle e_k + \sum_{k\in\mathbb{N}_o}\left(3-c+\frac{1}{k^2}\right)\langle f, e_k\rangle e_k. \end{aligned}$$

Thus

$$Tf = \sum_{k=1}^{\infty} \lambda_k \langle f, e_k\rangle e_k,$$

where the sequence $\{\lambda_k\}_{k=1}^{\infty}$ does not converge. The operator T cannot be, accordingly, compact.

In the following proposition, we present some conditions under which a frame obtained from an orthonormal basis is finite-rank-tight.

Proposition 3.9. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame in H that is obtained from an orthonormal basis by repeating a finite number of the basis elements, each a finite number of times. Then, $\{f_k\}_{k=1}^{\infty}$ is a finite-rank-tight frame.*

Proof. Let $\{e_1, e_2, \dots\}$ be an orthonormal basis for H. Suppose that $\{f_k\}_{k=1}^{\infty}$ is obtained from the basis by repeating the basis elements $e_{j_1}, \dots, e_{j_n}$, such that e_{j_m} is repeated $l_m > 1$ times for each $m \in \{1, \dots, n\}$. Then, for every $f \in H$,

$$\begin{aligned} Sf &= \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k \\ &= \sum_{i=1}^{\infty} \langle f, e_i \rangle e_i + \sum_{m=1}^{n} (l_m - 1) \langle f, e_{j_m} \rangle e_{j_m} \\ &= f + Kf. \end{aligned}$$

Here,

$$K = \sum_{m=1}^{n} (l_m - 1) \langle \cdot, e_{j_m} \rangle e_{j_m}$$

is a finite-rank operator. □

Now, we consider a quite natural question. Is the canonical dual of a compact-tight frame, compact-tight? The affirmative answer is given in our next theorem.

Theorem 3.10. *The canonical dual of a compact-tight frame, is compact-tight.*

Proof. Let $\{f_k\}_{k=1}^{\infty}$ be a (c, K)-compact-tight frame such that the frame operator S can be written as $S = cI + K$, in which $c \neq 0$ and K is a compact operator. Then, the operator S^{-1} can be written in the form

$$S^{-1} = c^{-1} I + T, \tag{3.3}$$

where T is a compact operator. To see this, we just need to choose T in such a way that

$$TS = -c^{-1} K.$$

Then, it is clear that T is a compact operator. Also,

$$\begin{aligned} (c^{-1} I + T) S &= c^{-1} S + TS \\ &= c^{-1} (cI + K) - c^{-1} K \\ &= I. \end{aligned}$$

This completes the proof by obtaining the desired representation (3.3). □

Next, we present a theorem on the weaving of certain compact-tight frames.

Theorem 3.11. *Let $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$ and $\mathcal{G} = \{g_k\}_{k=1}^{\infty}$ be $(1, K_1)$- and $(1, K_2)$-compact-tight frames, respectively. Also, assume that there exists an infinite subset σ of $\mathbb{N}$ such that $\{f_k\}_{k \in \sigma}$ and $\{g_k\}_{k \in \sigma^c}$ are orthonormal bases. Then, $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ cannot be woven.*

Proof. Assume on the contrary that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are woven. Then, for the partition $\{\sigma, \sigma^c\}$, we obtain the frame $\{f_k\}_{k\in\sigma^c} \cup \{g_k\}_{k\in\sigma}$. According to the definition of compact-tight frames, the frame operators for $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ can be written as

$$\begin{aligned} S_{\mathcal{F}}f =&\sum_{k=1}^{\infty}\langle f, f_k\rangle f_k \\ =&\sum_{k\in\sigma}\langle f, f_k\rangle f_k + \sum_{k\in\sigma^c}\langle f, f_k\rangle f_k \\ =&f + K_1 f \end{aligned}$$

and

$$\begin{aligned} S_{\mathcal{G}}f =&\sum_{k=1}^{\infty}\langle f, g_k\rangle g_k \\ =&\sum_{k\in\sigma^c}\langle f, g_k\rangle g_k + \sum_{k\in\sigma}\langle f, g_k\rangle g_k \\ =&f + K_2 f. \end{aligned}$$

Denote the frame operator of $\{f_k\}_{k\in\sigma^c} \cup \{g_k\}_{k\in\sigma}$ by $S_{\mathcal{W}}$. Then for every $f \in H$,

$$\begin{aligned} S_{\mathcal{W}}f =&\sum_{k\in\sigma^c}\langle f, f_k\rangle f_k + \sum_{k\in\sigma}\langle f, g_k\rangle g_k \\ =&K_1 f + K_2 f \\ =&(K_1 + K_2) f. \end{aligned}$$

Thus, $S_{\mathcal{W}} = K_1 + K_2$. Since K_1 and K_2 are compact operators, $S_{\mathcal{W}}$ is a compact operator, which is a contradiction. $\square$

Of course, it is necessary to present a concrete example of frames that satisfy the hypotheses of Theorem 3.11. We conclude the paper with such an example.

Example 3.12. Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be the sequences defined by

$$\{f_k\}_{k=1}^{\infty} = \left\{e_1, e_1, \frac{1}{2}e_2, e_2, \frac{1}{3}e_3, e_3, \frac{1}{4}e_4, e_4, \ldots\right\}$$

and

$$\{g_k\}_{k=1}^{\infty} = \left\{e_1, \frac{1}{2}e_1, e_2, \frac{1}{3}e_2, e_3, \frac{1}{4}e_3, \ldots\right\},$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for H. Then, it is clear that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are frames with bounds 1 and 2. Denote the frame operators of these frames by $S_{\mathcal{F}}$ and $S_{\mathcal{G}}$, respectively. Then,

$$S_{\mathcal{F}}f = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k = \sum_{j=1}^{\infty}\langle f, e_j\rangle e_j + \sum_{j=1}^{\infty}\frac{1}{j^2}\langle f, e_j\rangle e_j$$

and

$$S_{\mathcal{G}}f = \sum_{k=1}^{\infty} \langle f, g_k \rangle g_k = \sum_{j=1}^{\infty} \langle f, e_j \rangle e_j + \sum_{j=1}^{\infty} \frac{1}{(j+1)^2} \langle f, e_j \rangle e_j.$$

Thus, $\{f_k\}_{k=1}^{\infty}$ is a $(1, K_1)$-compact-tight frame, and $\{g_k\}_{k=1}^{\infty}$ is a $(1, K_2)$-compact-tight frame, where

$$K_1 f = \sum_{j=1}^{\infty} \frac{1}{j^2} \langle f, e_j \rangle e_j$$

and

$$K_2 f = \sum_{j=1}^{\infty} \frac{1}{(j+1)^2} \langle f, e_j \rangle e_j$$

are compact operators. Now, let $\sigma = \mathbb{N}_e$, so that $\sigma^c = \mathbb{N}_o$. Then, $\{f_k\}_{k\in\sigma}$ and $\{g_k\}_{k\in\sigma^c}$ are orthonormal bases, because these are both equal to $\{e_k\}_{k=1}^{\infty}$. Therefore, $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are frames that satisfy the hypotheses of Theorem 3.11.

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[1]Department of Mathematics, Faculty of Mathematics, Statistics and Computer Science, University of Sistan and Baluchestan, P. O. Box 98135-674, Zahedan, Iran.

Email address: smovahed97@gmail.com

Email address: giv@math.usb.ac.ir

Email address: ahmadi@hamoon.usb.ac.ir