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INCLUSIONS AND COINCIDENCES FOR MULTIPLE COHEN POSITIVE STRONGLY p-SUMMING m-LINEAR OPERATORS

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Abstract. We compare a new class of multiple Cohen positive strongly p-summing multilinear operators along with different classes of positive multilinear p-summability and investigate a duality relationship in terms of the tensor norm.

1. Introduction and preliminaries

The well-known p-summing linear operators, introduced by Pietsch, knew many generalizations to the multilinear case [9–13]. In parallel, Bu and Labuschagne [5] generalized this notion for the positive multilinear operators. As a prototype of those generalizations, we attempt to set out a new generalization of the concept of Cohen positive strongly p-summing [1, 3, 6]. In this work, we introduce the new class of multiple Cohen positive strongly p-summing operators and compare it with the class of Cohen positive strongly p-summing m-linear operators [3] and positive multiple p-summing m-linear operators [5], by giving a generalization to Cohen's theorems [7], as well as, investigating a relationship with the class multiple Cohen positive p-nuclear operators.

Starting by fixing notations, throughout this paper, $X, X_1, \ldots, X_m, Y$ will be Banach spaces and $E, E_1, \ldots, E_m, F, G$ will be Banach lattices $m \in \mathbb{N}^*$. Let $\mathcal{L}(X_1, \ldots, X_m; Y)$ denote the Banach space of all continuous m-linear operators from $X_1, \cdots, X_m$ to Y. If $Y = \mathbb{K}$, then we write $\mathcal{L}(X_1, \ldots, X_m)$. In the case when $X_1 = \cdots = X_m = X$, we simply write $\mathcal{L}(^m X; Y)$. For a Banach space X,

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X^* will denote its topological dual, and B_X will denote its closed unit ball. For $1 \leq p \leq \infty$, let p^* be its conjugate; that is, $\frac{1}{p} + \frac{1}{p^*} = 1$.

Let E be a Banach lattice. We denote by E^+ the positive cone $\{x \in E,\ x \geq 0\}$. For $x \in E$, let $x^+ := \sup\{x, 0\}$ and $x^- := \sup\{-x, 0\}$ be the positive part and the negative part of x, respectively.
For any $x \in E$, we have

$$x = x^+ - x^- \quad \textit{and} \quad |x| = x^+ + x^-.$$

We denote by $\ell_p^n(X)$ the space of all finite sequences $(x_i)_{i=1}^n$ in X with the norm

$$\|(x_i)_{i=1}^n\|_p = \left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}},$$

and by $\ell_{p,weak}^n(X)$, the space of all finite sequences $(x_i)_{i=1}^n$ in X with the norm

$$w_p((x_i)_{i=1}^n) = \|(x_i)_{i=1}^n\|_{\ell_{p,weak}^n(X)} = \sup_{\varphi \in B_{X^*}} \left(\sum_{i=1}^n |\langle x_i, \varphi\rangle|^p\right)^{\frac{1}{p}}.$$

Then $\ell_{p,weak}^n(X)$ is a Banach space with respect to the norm w_p. Consider the case where X is substituted by a Banach lattice E, and define

$$\ell_{p,|weak|}^n(E) := \left\{(x_i)_{i=1}^n \ :\ (|x_i|)_{i=1}^n \in \ell_{p,weak}^n(E)\right\},$$

and

$$\|(x_i)_{i=1}^n\|_{\ell_{p,|weak|}^n(E)} = w_p((|x_i|)_{i=1}^n).$$

Let $B_{E^*}^+ := B_{E^*} \cap E^{*+}$. If $x_1, \dots, x_n \geq 0$, then

$$\|(x_i)_{i=1}^n\|_{\ell_{p,|weak|}^n(E)} = \sup_{\xi \in B_{E^*}^+} \left(\sum_{i=1}^n \langle x_i, \xi\rangle^p\right)^{\frac{1}{p}} = \|(x_i)_{i=1}^n\|_{\ell_{p,weak}^n(E)}. \tag{1.1}$$

If $1 < p < \infty$, then we denote by $\ell_p(F, \mathbb{N}^m)$ the vector space of all families $(y_j)_{j \in \mathbb{N}^m}$ of elements such that

$$\|(y_j)_{j\in\mathbb{N}^m}\|_p = \left(\sum_{j\in\mathbb{N}^m} \|y_j\|^p\right)^{\frac{1}{p}} < +\infty,$$

and by $\ell_{p,weak}(F, \mathbb{N}^m)$ the vector space of all families $(y_j)_{j\in\mathbb{N}^m}$ of elements such that

$$\|(y_j)_{j\in\mathbb{N}^m}\|_{\ell_{p,weak}(F,\mathbb{N}^m)} = \sup_{\phi \in B_{F^*}} \left(\sum_{j\in\mathbb{N}^m} |\langle \phi, y_j\rangle|^p\right)^{\frac{1}{p}} < +\infty.$$

We observe that $\|\cdot\|_p$ and $\|\cdot\|_{\ell_{p,weak}(F,\mathbb{N}^m)}$ are norms on $\ell_p(F, \mathbb{N}^m)$ and $\ell_{p,weak}(F, \mathbb{N}^m)$, respectively, and an element of $\mathbb{N}^m$ is represented by $(j_1, \dots, j_m)$. From now on, to avoid encumbered notations we denote $\ell_p(F)$ instead of $\ell_p(F, \mathbb{N}^m)$ and by $\ell_{p,weak}(F)$ instead of $\ell_{p,weak}(F, \mathbb{N}^m)$.

Let $1 \leq p \leq \infty$ and let $\lambda > 1$. The Banach space X is said to be an $\mathcal{L}_{p,\lambda}$-space if every finite-dimensional subspace Y of X is contained in a finite-dimensional subspace Z of X for which there is an isomorphism $v : Z \to \ell_p^{dimZ}$ with $\|v\|\|v^{-1}\| < \lambda$. We say that X is an $\mathcal{L}_p$-space if it is an $\mathcal{L}_{p,\lambda}$-space for some $\lambda > 1$ (see [8]).

We recall the definition of finite type operators. [[9]] An m-linear operator $T \in \mathcal{L}(X_1, \dots, X_m; Y)$ is said to be finite type if it is generated by mappings of the form

$$T_{y\otimes_{j=1}^m x_j^*} = x_1^* \otimes \cdots \otimes x_m^* \otimes y : (x^1, \dots, x^m) \to x_1^*(x^1) \dots x_m^*(x^m)y$$

for some nonzero $x_j^* \in X_j^*$ $(1 \leq j \leq m)$ and $y \in Y$. The vector space of all m-linear operators of finite type is denoted by $\mathcal{L}_f(X_1, \dots, X_m; Y)$. We will also need in what follows some definitions of positive summing linear and multilinear operators. [[2]] Let $1 \leq p < \infty$. An operator $T : E \longrightarrow X$ is said to be positive p-summing, if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}, x_1, \dots, x_n \in E$, the following inequality holds:

$$\|T\,(x_i)_{i=1}^n\|_p \leq C\,\|(x_i)_{i=1}^n\|_{\ell_{p,|weak|}^n(E)}\,. \tag{1.2}$$

Also, for $p = \infty$,

$$\sup_{1\leq i\leq n} \|T\,(x_i)\| \leq C\,\|(x_i)_{i=1}^n\|_{\ell_{\infty,|weak|}^n(E)}\,.$$

We denote by $\Pi_p^+(E; X)$, the space of positive p-summing operators from E into X. Moreover, $\Pi_p^+(E, X)$ becomes a Banach space with norm $\pi_p^+(\cdot)$ given by the infimum of the constants $C > 0$ that verify the inequality (1.2). We have $\Pi_\infty^+(E; X) = \mathcal{L}(E; X)$. [[3]] Let $1 \leq p \leq \infty$. An m-linear operator $T : X_1 \times \cdots \times X_m \longrightarrow F$ is a Cohen positive strongly p-summing multilinear operator, if there is a constant $C > 0$ such that for any $x_1^j, \dots, x_n^j \in X_j$, $1 \leq j \leq m$, and any $y_1^*, \dots, y_n^* \in F^*$

$$\left\|\left\langle T(x_i^1, \dots, x_i^m), y_i^*\right\rangle\right\|_{\ell_1^n} \leq C\left(\sum_{i=1}^n \prod_{j=1}^m \left\|x_i^j\right\|_{X_j}^p\right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|weak|}^n(F^*)}\,. \tag{1.3}$$

Moreover, the class of all Cohen positive strongly p-summing m-linear operators from $X_1 \times \cdots \times X_m$ into F is denoted by $\mathcal{D}_p^{m+}(X_1, \dots, X_m; F)$. This space is a Banach space with the norm $d_p^{m+}(\cdot)$, which is the smallest constant C such that the inequality (1.3) holds. [[5]] An m-linear operator $T : E_1 \times \cdots \times E_m \to Y$ is called positive multiple p-summing if there exists a constant $K > 0$ such that for every choice of finite sequences $(x_i^j)_{i=1}^{n_j} \subseteq E_j^+$; $1 \leq j \leq m$, we have

$$\left(\sum_{i_1,\dots,i_m=1}^{n_1,\dots,n_m} \|T(x_{i_1}^1, \dots, x_{i_m}^m)\|^p\right)^{\frac{1}{p}} \leq K \prod_{j=1}^m \|(x_i^j)_{i=1}^{n_j}\|_{\ell_{p,weak}^n(E_j)}. \tag{1.4}$$

In this case, we define the positive multiple p-summing norm of T by

$$\Lambda_p(T) = \inf\{K : K \text{ verifies the inequality } 1.4\}\,.$$

It is easily verified that the class $\Lambda_p^{mult}(E_1, \dots, E_m; Y)$ of positive multiple p-summing m-linear operators, with its associated norm Λ_p, is a Banach space.

Taking the advantage of the definition of Cohen positive p-nuclear m-linear operators initiated by authors in [4], we define similarly the multiple Cohen positive p-nuclear operators as follows. For $1 \leq p < \infty$, an m-linear operator $T : E_1 \times \cdots \times E_m \longrightarrow F$ is called multiple Cohen positive p-nuclear if there exists a constant $C > 0$ such that for any $(x^j_{i_j})^n_{i_j=1} \subset E_j$ $(1 \leq j \leq m)$ and any $y^*_{i_1,\ldots,i_m} \in F^*$, we have

$$\|(\langle T(x^1_{i_1}, \ldots, x^m_{i_m}), y^*_{i_1,\ldots,i_m}\rangle)^n_{i_1,\ldots,i_m=1}\|_{\ell^n_1} \\ \leq C \prod_{j=1}^{m} \|(x^i_i)^n_{i=1}\|_{\ell^n_{p,|weak|}(E_j)} \|(y^*_{i_1,\ldots,i_m})^n_{i_1,\ldots,i_m=1}\|_{\ell^n_{p^*,|weak|}(F^*)}.$$

Moreover, the class of all multiple Cohen positive p-nuclear operators from $E_1 \times \cdots \times E_m$ into F, is denoted $\mathcal{N}^{mult+}_p(E_1, \ldots, E_m; F)$. It is a Banach space with the norm $n^{mult+}_p(\cdot)$, which is the smallest constant C such that the above inequality holds.

2. Multiple Cohen positive strongly p-summing operators

In this section, we give a new notion of multiple Cohen positive strongly p-summing operators, as a prototype of the multiple Cohen strongly summing operators initiated by Campos in [6] and motivated by Matos in his famous paper "Fully absolutely summing and Hilbert-Schmidt multilinear mappings" [9], as well as studying inclusions and coincidences with some known spaces.

All along this section, the Banach lattice F will be finite-dimensional. Let $1 \leq p \leq \infty$. An m-linear operator $T : X_1 \times \cdots \times X_m \longrightarrow F$ is a multiple Cohen positive strongly p-summing m-linear operator, if there is a constant $C > 0$ such that for any $n \in \mathbb{N}^*$, $y^*_{i_1,\ldots,i_m} \in F^{*+}$ and any $x^j_{i_j} \in X_j$ such that $1 \leq j \leq m$, $1 \leq i \leq n$, $1 \leq i_j \leq n$, and

$$\sum_{i_1,\ldots,i_m=1}^{n} |\langle T(x^1_{i_1}, \ldots, x^m_{i_m}), y^*_{i_1,\ldots,i_m}\rangle| \\ \leq C \left(\prod_{j=1}^{m} \|(x^j_i)^n_{i=1}\|_p \right) \|(y^*_{i_1,\ldots,i_m})^n_{i_1,\ldots,i_m=1}\|_{\ell^n_{p^*,weak}(F^*)}.$$

The class of all multiple Cohen positive strongly p-summing m-linear operators from $X_1 \times \cdots \times X_m$ into F is a Banach space denoted by $\mathcal{D}^{mult+}_p(X_1, \ldots, X_m; F)$, with the norm $d^{mult+}_p(\cdot)$ given by the infimum of constants C verifying the above inequality. The next result is a characterization to the class of multiple Cohen positive strongly p-summing operators, which we will use mostly in Section 4. Let $T : X_1 \times \cdots \times X_m \to F$. Then T is a multiple Cohen positive strongly p-summing m-linear operator if and only if there exists a constant $K > 0$ such that the following inequality holds:

$$\sum_{i_1,\ldots,i_m=1}^{n} |\langle T(x_{i_1}^1,\ldots,x_{i_m}^m), y_{i_1,\ldots,i_m}^*\rangle| \leq K \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p\right) \|(y_{i_1,\ldots,i_m}^*)_{i_1,\ldots,i_m=1}^n\|_{\ell_{p^*,|weak|}^n(F^*)},$$

for any $n \in \mathbb{N}^*$, $y_{i_1,\ldots,i_m}^* \in F^*$ and any $x_{i_j}^j \in X_j$ such that $1 \leq j \leq m$, $1 \leq i \leq n$, and $1 \leq i_j \leq n$.

Proof. For the sufficiency, letting $n \in \mathbb{N}$, $y_{i_1,\ldots,i_m}^* \in F^{*+}$, and $x_{i_j}^j \in X_j$ for $1 \leq j \leq m$, $1 \leq i \leq n$, and $1 \leq i_j \leq n$, we have

$$\begin{aligned}
&\sum_{i_1,\ldots,i_m=1}^{n} |\langle T(x_{i_1}^1,\ldots,x_{i_m}^m), y_{i_1,\ldots,i_m}^*\rangle| \\
&= \sum_{i_1,\ldots,i_m=1}^{n} |\langle T(x_{i_1}^1,\ldots,x_{i_m}^m), y_{i_1,\ldots,i_m}^{*+} - y_{i_1,\ldots,i_m}^{*-}\rangle| \\
&\leq \sum_{i_1,\ldots,i_m=1}^{n} |\langle T(x_{i_1}^1,\ldots,x_{i_m}^m), y_{i_1,\ldots,i_m}^{*+}\rangle| \\
&\quad + \sum_{i_1,\ldots,i_m=1}^{n} |\langle T(x_{i_1}^1,\ldots,x_{i_m}^m), y_{i_1,\ldots,i_m}^{*-}\rangle| \\
&\leq K \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p\right) \|(y_{i_1,\ldots,i_m}^{*+})_{i_1,\ldots,i_m=1}^n\|_{\ell_{p^*,weak}^n(F^*)} \\
&\quad + K \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p\right) \|(y_{i_1,\ldots,i_m}^{*-})_{i_1,\ldots,i_m=1}^n\|_{\ell_{p^*,weak}^n(F^*)} \\
&\leq 2K \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p\right) \|(|y_{i_1,\ldots,i_m}^*|)_{i_1,\ldots,i_m=1}^n\|_{\ell_{p^*,weak}^n(F^*)} \\
&\leq 2K \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p\right) \|(y_{i_1,\ldots,i_m}^*)_{i_1,\ldots,i_m=1}^n\|_{\ell_{p^*,|weak|}^n(F^*)} \\
&= C \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p\right) \|(y_{i_1,\ldots,i_m}^*)_{i_1,\ldots,i_m=1}^n\|_{\ell_{p^*,|weak|}^n(F^*)}.
\end{aligned}$$

The necessity follows from formula (1.1). □

Every finite type m-linear operator from $X_1 \times \cdots \times X_m$ into the finite-dimensional Banach lattice F, is a multiple Cohen positive strongly p-summing m-linear operator. Indeed, letting $T = \phi_1 \otimes \cdots \otimes \phi_m \otimes b$ with $\phi_1 \in X_1^*, \ldots, \phi_m \in X_m^*$ and $b \in F$ and letting $n \in \mathbb{N}$, $y_{i_1,\ldots,i_m}^* \in F^{*+}$ and $x_{i_j}^j \in X_j$ for $1 \leq j \leq m$, $\;1 \leq i \leq n$

and $1 \leq i_j \leq n$, we have

$$\begin{aligned}
&\sum_{i_1,\ldots,i_m=1}^{n} |\langle \phi_1 \otimes \cdots \otimes \phi_m \otimes b(x^1_{i_1}, \ldots, x^m_{i_m}), y^*_{i_1,\ldots,i_m}\rangle| \\
&\leq \|\phi_1 \otimes \cdots \otimes \phi_m \otimes b(x^1_{i_1}, \ldots, x^m_{i_m})\| \|(y^*_{i_1,\ldots,i_m})^n_{i_1,\ldots,i_m=1}\| \\
&\leq \|(\phi_1 \otimes \cdots \otimes \phi_m \otimes b(x^1_{i_1}, \ldots, x^m_{i_m}))^n_{i_1,\ldots,i_m=1}\|_p \cdot \|(y^*_{i_1,\ldots,i_m})^n_{i_1,\ldots,i_m=1}\|_{p^*} \\
&\leq \|b\| \prod_{j=1}^{m} \|\phi_j(x^j_i)^n_{i=1}\|_p \|(y^*_{i_1,\ldots,i_m})^n_{i_1,\ldots,i_m=1}\|_{p^*} \\
&\leq \|b\| \prod_{j=1}^{m} \|\phi_j(x^j_i)^n_{i=1}\|_p \|(y^*_{i_1,\ldots,i_m})^n_{i_1,\ldots,i_m=1}\|_{\ell^n_{p^*,weak}(F^*)}.
\end{aligned}$$

Hence, $d_p^{mult+}(\phi_1 \otimes \cdots \otimes \phi_m \otimes b) \leq \|b\| \|\phi_1\| \cdots \|\phi_m\|$. It was proved in [6, Proposition 4.4] that every Cohen strongly p-summing multilinear operator is multiple Cohen strongly p-summing. By using [3, Theorem 2.5] instead of [6, Theorem 3.8] in the proof giving in [6] and making the necessary adaptations, we obtain the following result. Every Cohen positive strongly p-summing m-linear operator is multiple Cohen positive strongly p-summing m-linear operator and

$$d_p^{mult+}(\cdot) \leq d_p^{m+}(\cdot).$$

Next, we investigate a composition relationship for our class of operators. Let $1 \leq p, q, r \leq \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and let $D_1, \ldots, D_m$ be Banach lattices. If $S \in \mathcal{D}_p^{mult+}(E_1, \ldots, E_m; F)$ and $T_j \in \Pi_q^+(D_j, E_j)$ with $1 \leq j \leq m$, then $S \circ (T_1, \ldots, T_m) \in \mathcal{N}_r^{mult+}(D_1, \ldots, D_m; F)$.

Proof. Take $T_j \in \Pi_q^+(D_j, E_j)$ with $1 \leq j \leq m$. From the domination theorem for positive summing operators [1, Theorem 3.3], there is $\mu_j \in B^+_{D_j^*}$ such that for every $x \in D_j$, we have

$$\|T_j(x)\| \leq \pi_q^+(T_j) \left(\int_{B^+_{D_j^*}} |\phi(x)|^q d\mu_j(\phi) \right)^{\frac{1}{q}}.$$

We take

$$\rho_i^j = \left(\int_{B^+_{D_j^*}} |\phi(x_i^j)|^r d\mu_j(\phi) \right)^{\frac{1}{q}}$$

for $1 \leq j \leq m$ and $1 \leq i \leq n$. Let x_i^j in D_j. Without loss of generality, we may consider $T_j(x_i^j) \neq 0$, for all $1 \leq j \leq m$ and $1 \leq i \leq n$. Hence $\rho_i^j > 0$, and we can define $z_i^j = \frac{x_i^j}{\rho_i^j}$. Now, for $a_1, \ldots, a_n \in \mathbb{K}$ with $\sum_{i=1}^{n} |a_i|^{p^*} < 1$, since $\frac{1}{r^*} + \frac{1}{p} + \frac{1}{q} = 1$, we can use the Hölder's inequality in order to write

$$\begin{aligned}
\Big|\sum_{i=1}^{n}\phi(a_i z_i^j)\Big| &\le \sum_{i=1}^{n}|a_i|^{\frac{p^*}{r^*}}|a_i|^{\frac{p^*}{q}}\frac{1}{\rho_i^j}|\phi(x_i^j)|^{\frac{r}{q}}|\phi(x_i^j)|^{\frac{r}{p}} \\
&\le \left(\sum_{i=1}^{n}|a_i|^{p^*}\right)^{\frac{1}{r^*}}\left(\sum_{i=1}^{n}|a_i|^{p^*}\frac{1}{(\rho_i^j)^q}|\phi(x_i^j)|^r\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n}|\phi(x_i^j)|^r\right)^{\frac{1}{p}} \\
&\le \left(\sum_{i=1}^{n}|a_i|^{p^*}\frac{1}{(\rho_i^j)^q}|\phi(x_i^j)|^r\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n}|\phi(x_i^j)|^r\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\Big\|\sum_{i=1}^{n}a_i T_j(z_i^j)\Big\| &= \Big\|\sum_{i=1}^{n}T_j(a_i z_i^j)\Big\| \\
&\le \pi_q^+(T_j)\left(\int_{B_{D_j}^+}\Big|\sum_{i=1}^{n}\phi(a_i z_i^j)\Big|^q d\mu_j(\phi)\right)^{\frac{1}{q}} \\
&\le \pi_q^+(T_j)\left(\sum_{i=1}^{n}|a_i|^{p^*}\frac{1}{(\rho_i^j)^q}\int_{B_{D_j}^+}|\phi(x_i^j)|^r d\mu_j(\phi)\right)^{\frac{1}{q}}\left(\|(x_i^j)_{i=1}^n\|_{\ell^n_{r,weak}}\right)^{\frac{r}{p}} \\
&\le \pi_q^+(T_j)\left(\|(x_i^j)_{i=1}^n\|_{\ell^n_{r,weak}}\right)^{\frac{r}{p}}.
\end{aligned}$$

Hence, Krivine's calculus implies

$$\|(T_j(z_i^j))_{i=1}^n\|_p \le \pi_q^+(T_j)\left(\|(x_i^j)_{i=1}^n\|_{\ell^n_{r,weak}}\right)^{\frac{r}{p}}.$$

Now, for $x_{i_j}^j$ in D_j, $1\le j\le m$ and $y^*_{i_1,\dots,i_m}$ in F^{*+}, we have

$$\begin{aligned}
&\sum_{i_1,\dots,i_m=1}^{n}|\langle S(T_1(x_{i_1}^1),\dots,T_m(x_{i_m}^m)),y^*_{i_1,\dots,i_m}\rangle| \\
&\quad\le \|S(T_1(x_{i_1}^1),\dots,T_m(x_{i_m}^m))\|\|(y^*_{i_1,\dots,i_m})_{i_1,\dots,i_m=1}^n\| \\
&\quad\le \|(\rho_{i_1}^1\dots\rho_{i_m}^m)S(T_1(z_{i_1}^1),\dots,T_m(z_{i_m}^m))\|\|(y^*_{i_1,\dots,i_m})_{i_1,\dots,i_m=1}^n\| \\
&\quad\le \|(\rho_{i_1}^1\dots\rho_{i_m}^m)S(T_1(z_{i_1}^1),\dots,T_m(z_{i_m}^m))\|_r\|(y^*_{i_1,\dots,i_m})_{i_1,\dots,i_m=1}^n\|_{r^*} \\
&\quad\le \|(\rho_{i_1}^1\dots\rho_{i_m}^m)S(T_1(z_{i_1}^1),\dots,T_m(z_{i_m}^m))\|_r\|(y^*_{i_1,\dots,i_m})_{i_1,\dots,i_m=1}^n\|_{\ell^n_{r^*,weak}(F^*)}.
\end{aligned}$$

Applying Hölder's inequality and the fact that the operator S is multiple Cohen positive strongly p-summing, we obtain

$$\begin{aligned}
&\sum_{i_1,\ldots,i_m=1}^{n} |\langle S(T_1(x^1_{i_1}),\ldots,T_m(x^m_{i_m})), y^*_{i_1,\ldots,i_m}\rangle| \\
&\le d_p^{mult+}(S)\left(\sum_{i_1,\ldots,i_m=1}^{n}(\rho^1_{i_1}\ldots\rho^m_{i_m})^q\right)^{\frac{1}{q}}\prod_{j=1}^{m}\|(T_j(z^j_i))_{i=1}^n\|_p\|(y^*_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^n\|_{\ell^n_{r^*,weak}(F^*)} \\
&\le d_p^{mult+}(S)\prod_{j=1}^{m}\|(\rho^j_i)_{i=1}^n\|_q\prod_{j=1}^{m}\pi_q(T_j)\left(\|(x^j_i)_{i=1}^n\|_{\ell^n_{r,weak}}\right)^{\frac{r}{p}}\|(y^*_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^n\|_{\ell^n_{r^*,weak}(F^*)} \\
&\le d_p^{mult+}(S)\prod_{j=1}^{m}\pi_q(T_j)\left(\|(x^j_i)_{i=1}^n\|_{\ell^n_{r,weak}}\right)^{\frac{r}{p}+\frac{r}{q}}\|(y^*_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^n\|_{\ell^n_{r^*,weak}(F^*)} \\
&\le d_p^{mult+}(S)\prod_{j=1}^{m}\pi_q(T_j)\|(x^j_i)_{i=1}^n\|_{\ell^n_{r,weak}}\|(y^*_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^n\|_{\ell^n_{r^*,weak}(F^*)}.
\end{aligned}$$

Therefore, $S\circ(T_1,\ldots,T_m)$ is a multiple Cohen positive r-nuclear operator. □

3. Cohen's-type theorems for multiple positive strongly p-summing operators

In this section, we investigate inclusions between the class of multiple positive p-summing and multiple positive strongly p-summing operators. By giving the Cohen's-type theorems [7] in the positive multilinear situation. Let $r_1,\ldots,r_m\in\mathbb{N}^*$ and $1<p\le\infty$. Let T be a multilinear operator form $\ell_p^{r_1}\times\cdots\times\ell_p^{r_m}$ into the finite-dimensional Banach lattice F. Then, T belongs to $\Lambda_{p^*}^{mult}(\ell_p^{r_1},\ldots,\ell_p^{r_m};F)$ and $\mathcal{D}_p^{mult+}(\ell_p^{r_1},\ldots,\ell_p^{r_m};F)$ with $d_p^{mult+}(T)\le\Lambda_{p^*}(T)$.

Proof. Let the multilinear operator $T:\ell_p^{r_1}\times\cdots\times\ell_p^{r_m}\to F$. Then T is finite type. Thus obviously T is in $\Lambda_{p^*}^{mult}(\ell_p^{r_1},\ldots,\ell_p^{r_m};F)$ and from Example 2, T is in $\mathcal{D}_p^{mult+}(\ell_p^{r_1},\ldots,\ell_p^{r_m};F)$. Let now $(e_{k_j})_{k_j=1}^{r_j}$ be the standard basis for $\ell_p^{r_j}$ $1\le j\le m$.

Since T is positive multiple p-summing, then

$$\begin{aligned}
\left(\sum_{k_1,\ldots,k_m=1}^{r_1,\ldots,r_m}\|T(e_{k_1},\ldots,e_{k_m})\|^{p^*}\right)^{\frac{1}{p^*}} &\le \Lambda_{p^*}(T)\prod_{j=1}^{m}\|(e_{k_j})_{k_j=1}^{r_j}\|_{\ell^n_{p^*,weak}} \\
&\le \Lambda_{p^*}(T).
\end{aligned}$$

Let $(x^1_{i_1},\dots,x^m_{i_m}) \in \ell_p^{r_1}\times\cdots\times\ell_p^{r_m}$ such that $x^j_{i_j} = \sum\limits_{k_j=1}^{r_j} a^j_{k_j,i_j} e_{k_j}$, $y^*_{i_1,\dots,i_m} \in F^{*+}$, and

$$\begin{aligned}\sum_{i_1,\dots,i_m=1}^{n} & |\langle T(x^1_{i_1},\dots,x^m_{i_m}), y^*_{i_1,\dots,i_m}\rangle| \\ & \le \sum_{i_1,\dots,i_m=1}^{n}\left(\sum_{k_1,\dots,k_m=1}^{r_1,\dots,r_m} |\langle T(a^1_{k_1,i_1}e_{k_1},\dots,a^m_{k_m i_m}e_{k_m}), y^*_{i_1,\dots,i_m}\rangle|\right) \\ & \le \sum_{i_1,\dots,i_m=1}^{n}\sum_{k_1,\dots,k_m=1}^{r_1,\dots,r_m}\left(|a^1_{k_1,i_1}\dots a^m_{k_m,i_m}||\langle T(e_{k_1},\dots,e_{k_m}), y^*_{i_1,\dots,i_m}\rangle|\right).\end{aligned}$$

If $1<p<\infty$, then by the Hölder's inequality, we obtain

$$\begin{aligned}\sum_{i_1,\dots,i_m=1}^{n} & |\langle T(x^1_{i_1},\dots,x^m_{i_m}), y^*_{i_1,\dots,i_m}\rangle| \\ & \le \sum_{i_1,\dots,i_m=1}^{n}\left[\left(\sum_{k_1,\dots,k_m=1}^{r_1,\dots,r_m}|a^1_{k_1,i_1}\dots a^m_{k_m,i_m}|^p\right)^{\frac{1}{p}}\left(\sum_{k_1,\dots,k_m=1}^{r_1,\dots,r_m}|\langle T(e_{k_1},\dots,e_{k_m}), y^*_{i_1,\dots,i_m}\rangle|^{p^*}\right)^{\frac{1}{p^*}}\right] \\ & \le \sum_{i_1,\dots,i_m=1}^{n}\left[\|x^1_{i_1}\|\cdots\|x^m_{i_m}\|\left(\sum_{k_1,\dots,k_m=1}^{r_1,\dots,r_m}|\langle T(e_{k_1},\dots,e_{k_m}), y^*_{i_1,\dots,i_m}\rangle|^{p^*}\right)^{\frac{1}{p^*}}\right] \\ & \le \left(\sum_{i_1,\dots,i_m=1}^{n}\|x^1_{i_1}\|\cdots\|x^m_{i_m}\|\right)\left(\sum_{k_1,\dots,k_m=1}^{r_1,\dots,r_m}\|T(e_{k_1},\dots,e_{k_m})\|^{p^*}\right)^{\frac{1}{p^*}}\|(y^*_{i_1,\dots,i_m})_{i=1}^n\|_{\ell^n_{p^*,weak}(F^*)} \\ & \le \Lambda_{p^*}(T)\left(\sum_{i=1}^{n}\prod_{j=1}^{m}\|x^j_i\|^p\right)^{\frac{1}{p}}\|(y^*_{i_1,\dots,i_m})_{i=1}^n\|_{\ell^n_{p^*,weak}(F^*)}.\end{aligned}$$

This implies that $d_p^{mult+}(T)\le\Lambda_{p^*}(T)$.

If $p=\infty$, then

$$\begin{aligned}\sum_{i_1,\dots,i_m=1}^{n} & |\langle T(x^1_{i_1},\dots,x^m_{i_m}), y^*_{i_1,\dots,i_m}\rangle| \\ & \le \sup_{1\le i_j\le n}\sup_{1\le k_j\le r_j}|a^1_{k_1,i_1}\dots a^m_{k_m,i_m}|\left(\sum_{i_1,\dots,i_m=1}^{n}\sum_{k_1,\dots,k_m=1}^{r_1,\dots,r_m}|\langle T(e_{k_1},\dots,e_{k_m}), y^*_{i_1,\dots,i_m}\rangle|\right) \\ & \le \sup_{1\le i\le n}\prod_{j=1}^{m}\|(x^j_i)_{i=1}^n\|_{\ell_\infty^{r_j}}\sum_{k_1,\dots,k_m=1}^{r_1,\dots,r_m}\|T(e_{k_1},\dots,e_{k_m})\|\|(y^*_{i_1,\dots,i_m})_{i_1,\dots,i_m=1}^n\|_{\ell^n_{1,weak}(F^*)} \\ & \le \Lambda_1(T)\prod_{j=1}^{m}\|(x^j_i)_{i=1}^n\|_{\ell_\infty^{r_j}}\|(y^*_{i_1,\dots,i_m})_{i_1,\dots,i_m=1}^n\|_{\ell^n_{1,weak}(F^*)}.\end{aligned}$$

We obtain $d_\infty^{mult+}(T)\le\Lambda_1(T)$. This completes the proof. □

For $m \in \mathbb{N}^*$. Let $1 < p \leq \infty$ and let $E_j \quad (1 \leq j \leq m)$ be an $\mathcal{L}_{p,\lambda_j}$-space with $\lambda_j > 1$. Then

$$\Lambda_{p^*}^{mult}(E_1, \ldots, E_m; F) \subset \mathcal{D}_p^{mult+}(E_1, \ldots, E_m; F),$$

and $d_p^{mult+}(T) \leq \prod_{j=1}^{m} \lambda_j \Lambda_{p^*}(T)$.

Proof. Let $n \in \mathbb{N}^*$, $(x_{i_1}^1, \ldots, x_{i_m}^m)$ in $E_1 \times \cdots \times E_m$ and $T \in \Lambda_{p^*}^{mult}(E_1, \ldots, E_m; F)$.

Since E_j $(1 \leq j \leq m)$ is an $\mathcal{L}_{p,\lambda_j}$-space, there exists a finite-dimensional subspace $M_j \subset E_j$ containing a finite-dimensional subspace spanned by $x_{i_1}^1, \ldots, x_{i_m}^m$ and an invertible operator $S_j : \ell_p^{r_j} \to M_j$ $(Dim M_j = r_j)$ such that $\|S_j\| \|S_j^{-1}\| < \lambda_j$.

Consider the following diagram

$0.15in0.18in E_1 \times \cdots \times E_m [r]^T F M_1 [u]_{i_1} \times \cdots \times M_m [u]_{i_m} \ell_p^{r_1} [l]_{(S_1,\ldots,S_m)} \times \cdots \times [u]_{\tilde{T}} \ell_p^{r_m} span\left\{x_{1_1}^1, \ldots, x\right.$

where i_j and k_j for $(1 \leq j \leq m)$ are the canonical inclusion mappings and the operator $\tilde{T}$ is defined by $\tilde{T} = T(i_1 \circ S_1, \ldots, i_m \circ S_m)$. Since $T \in \Lambda_{p^*}^{mult}(E_1, \ldots, E_m; F)$ then

$$\Lambda_{p^*}(\tilde{T}) \leq \Lambda_{p^*}(T) \prod_{j=1}^{m} \|S_j\| \|i_j\|.$$

Therefore, using the previous theorem, we have $\tilde{T} \in \mathcal{D}_p^{mult+}(\ell_p^{r_1}, \ldots, \ell_p^{r_m}; F)$ and

$$d_p^{mult+}(\tilde{T}) \leq \Lambda_{p^*}(\tilde{T}) \leq \Lambda_{p^*}(T) \prod_{j=1}^{m} \|S_j\|.$$

If we let $z_{i_j}^j = S_j^{-1} x_{i_j}^j$ in $\ell_p^{r_j}$ and $y_{i_1,\ldots,i_m}^* \in F^{*+}$, then

$$\begin{aligned}
&\sum_{i_1,\ldots,i_m=1}^{n} |\langle T(x_{i_1}^1, \ldots, x_{i_m}^m), y_{i_1,\ldots,i_m}^* \rangle| \\
&\quad = \sum_{i_1,\ldots,i_m=1}^{n} |\langle \tilde{T}(z_{i_1}^1, \ldots, z_{i_m}^m), y_{i_1,\ldots,i_m}^* \rangle| \\
&\quad \leq d_p^{mult+}(\tilde{T}) \|(z_i^1)_{i=1}^n\|_p \cdots \|(z_i^m)_{i=1}^n\|_p \ \|(y_{i_1,\ldots,i_m}^*)_{i_1,\ldots,i_m=1}^n\|_{\ell_{p^*,weak}^n(F^*)} \\
&\quad \leq \Lambda_{p^*}(T) \prod_{j=1}^{m} \|S_j\| \|(z_i^1)_{i=1}^n\|_p \cdots \|(z_i^m)_{i=1}^n\|_p \ \|(y_{i_1,\ldots,i_m}^*)_{i_1,\ldots,i_m=1}^n\|_{\ell_{p^*,weak}^n(F^*)}.
\end{aligned}$$

Since $z_{i_j}^j = S_j^{-1} x_{i_j}^j$, we obtain

$$\begin{aligned}
&\sum_{i_1,\ldots,i_m=1}^{n} |\langle T(x_{i_1}^1, \ldots, x_{i_m}^m), y_{i_1,\ldots,i_m}^* \rangle| \\
&\qquad \leq \prod_{j=1}^{m} \lambda_j \Lambda_{p^*}(T) \|(x_i^1)_{i=1}^n\|_p \cdots \|(x_i^m)_{i=1}^n\|_p \ \|(y_{i_1,\ldots,i_m}^*)_{i_1,\ldots,i_m=1}^n\|_{\ell_{p^*,weak}^n(F^*)}.
\end{aligned}$$

Therefore, T belongs to $\mathcal{D}_p^{mult+}(E_1,\ldots,E_m;F)$ and $d_p^{mult+}(T) \le \prod_{j=1}^{m} \lambda_j \Lambda_{p^*}(T)$. □

4. Connection with tensor product

In this section, we endow $X_1 \otimes \cdots \otimes X_m \otimes F^*$ with a norm in such way that its topological dual is isometric to the space of multiple Cohen positive strongly p-summing m-linear operators from $X_1 \times \cdots \times X_m$ into F.

For each $z \in X_1 \otimes \cdots \otimes X_m \otimes F$, we have

$$\delta_p^+(z) := \inf \left\{ \|(\lambda_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^n\|_{\ell_\infty^n} \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p \right) \ \|(y_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^n\|_{\ell^n_{p^*,|weak|}(F)} \right\},$$

where the infimum is taken over all representations $z = \sum_{i_1,\ldots,i_m=1}^{n} \lambda_{i_1,\ldots,i_m} x_{i_1}^1 \otimes \cdots \otimes x_{i_m}^m \otimes y_{i_1,\ldots,i_m}$. The application $z \mapsto \delta_p^+(z)$ is a norm on $X_1 \otimes \cdots \otimes X_m \otimes F$.

Proof. The proof of this proposition is similar to the proof of [14, Proposition 2.1]. □

The topological dual $\left(X_1 \otimes \cdots \otimes X_m \otimes F^*, \delta_p^+\right)^*$ of $\left(X_1 \otimes \cdots \otimes X_m \otimes F^*, \delta_p^+\right)$ is isometric to $\mathcal{D}_p^{mult+}(X_1,\ldots,X_m;F)$ through the mapping ϕ_T defined by

$$\begin{aligned} \phi_T : X_1 \otimes \cdots \otimes X_m \otimes F^* &\to \mathbb{R} \\ x^1 \otimes \cdots \otimes x^m \otimes y^* &\mapsto y^*(T(x^1,\ldots,x^m)). \end{aligned}$$

Proof. For any $z = \sum_{i_1,\ldots,i_m=1}^{n} \lambda_{i_1,\ldots,i_m} x_{i_1}^1 \otimes \cdots \otimes x_{i_m}^m \otimes y^*_{i_1,\ldots,i_m}$ in $X_1 \otimes \cdots \otimes X_m \otimes F^*$, we have

$$\begin{aligned} |\phi_T(z)| &= \Big| \sum_{i_1,\ldots,i_m=1}^{n} \lambda_{i_1,\ldots,i_m} y^*_{i_1,\ldots,i_m}(T(x_{i_1}^1,\ldots,x_{i_m}^m)) \Big| \\ &\le d_p^{mult+}(T) \|(\lambda_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^n\|_{\ell_\infty^n} \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p \right) \|(y^*_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)}. \end{aligned}$$

Hence for each z, we have $|\phi_T(z)| \le d_p^{mult+}(T)\delta_p^+(z)$, and so $\|\phi_T\| \le d_p^{mult+}(T)$.

For the other direction, letting $(\lambda_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^n \subset \mathbb{R}^*$, $(x_{i_j}^j)_{i_j=1}^n \subset X_j$ $(1 \le j \le m)$ and $(y^*_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^n \subset F^*$, we have

$$\begin{aligned} \Big| \sum_{i_1,\ldots,i_m=1}^{n} \lambda_{i_1,\ldots,i_m} y^*_{i_1,\ldots,i_m}(T(x_{i_1}^1,\ldots,x_{i_m}^m)) \Big| &= |\phi_T(z)| \\ &\le \|\phi_T\| \delta_p^+\Big(\sum_{i_1,\ldots,i_m=1}^{n} \lambda_{i_1,\ldots,i_m} x_{i_1}^1 \otimes \cdots \otimes x_{i_m}^m \otimes y^*_{i_1,\ldots,i_m} \Big) \\ &\le \|\phi_T\| \|(\lambda_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^n\|_{\ell_\infty^n} \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p \right) \ \|(y^*_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)}. \end{aligned}$$

It follows from Proposition 2.1 that $d_p^{mult+}(T) \leq \|\phi_T\|$. □

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