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A NOTE ON AN ALGORITHM FOR SOLUTION OF THE LYAPUNOV MATRIX EQUATION

NAWAL HELAL AL-HARBI[1,3], MUSTAPHA RAÏSSOULI[2*] AND DAOUD SULEIMAN MASHAT[1]

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Abstract. In this paper, a matrix iterative algorithm is resorted to in the aim to solve numerically the well-known Lyapunov matrix equation $AX + XA^T = C$, which arises in some areas of applied science. Numerical examples illustrating the significance of our approach are provided as well.

1. Introduction and preliminaries

The so-called Sylvester matrix equation is given by

$$AX + XB = C, \tag{1.1}$$

where $A, B \in \mathcal{M}_n$ and $C \in \mathcal{M}_n$ are given real $n \times n$ matrices and $X \in \mathcal{M}_n$ is an unknown real $n \times n$ matrix. The Sylvester equation is the most studied in the literature by virtue of its numerous applications in various scientific fields. Indeed, Sylvester equation is a good tool for finding the direct solution of the discrete Poisson equation in partial differential equations; see [4,7]. In many control theory issues, like those involving dynamical systems, the Sylvester equation is used for completing the resolution of such systems [3,9]. Furthermore, Sylvester equation arises in the restoration of two-dimensional images by Wiener's minimum mean square error filter when the noise is white and Gaussian [5,9]. Finally, the study of stability of systems in Clifford's geometric algebra needs to solve a linear quaternion equation of Sylvester type; see [6,13].

It is well known that (1.1) has one and only one solution if and only if

$$\lambda_i + \mu_j \neq 0 \qquad \text{for any } \; i, j = 1, \dots, n, \tag{1.2}$$

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where $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_n$ are the eigenvalues of A and B, respectively.

If $B = A^T$, where A^T is the transpose matrix of A, then we have a special case of the Sylvester equation,

$$AX + XA^T = C, \tag{1.3}$$

known as the (symmetric) Lyapunov matrix equation [10, 11]. If A is symmetric, then we have a particular case of (1.3), namely,

$$AX + XA = C. \tag{1.4}$$

We need more basic notions and notations. A real matrix $P = (p_{ij}) \in \mathcal{M}_n$ is called positive, in short $P \geq 0$, if P is symmetric and $\sum_{i,j=1}^{n} p_{ij} x_i x_j \geq 0$ for all $(x_1, \dots, x_n) \in \mathbb{R}^n$. For $P \geq 0$, the notation $P^{1/2}$ refers to the positive square root of P; that is, $X := P^{1/2}$ is the unique positive matrix solution of $X^2 = P$. For $P, Q \in \mathcal{M}_n$, we write $P \leq Q$ if P and Q are both symmetric and $Q - P \geq 0$. Since $\mathcal{M}_n$ is a finite-dimensional Banach algebra, then the limit and convergence in $\mathcal{M}_n$, considered throughout this paper, are defined in the matrix topology sense when $\mathcal{M}_n$ is endowed with any norm $\|\cdot\|$ among its equivalent norms.

Let $P \in \mathcal{M}_n$ be positive invertible and let $Q \in \mathcal{M}_n$ be symmetric. Then we set

$$\big[P, Q\big] := \lim_{t \downarrow 0} \frac{\big(P + tQ\big)^{1/2} - P^{1/2}}{t}. \tag{1.5}$$

Following [12], $\big[P, Q\big]$ exists. In particular, $\big[P^2, Q\big]$ exists for any symmetric invertible matrix P. As pointed out in [12], $Z := \big[P^2, Q\big]$ is the unique solution of the particular Lyapunov matrix equation $PZ + ZP = Q$. An explicit integral expression of $\big[P, Q\big]$ is given by (see [12])

$$\big[P, Q\big] = QP^{-1/2} - \frac{1}{\pi} P \int_0^\infty \frac{1}{\sqrt{s}} \big(P + sI\big)^{-1} Q \big(P + sI\big)^{-1} \, ds, \tag{1.6}$$

where I denotes the $n \times n$-matrix identity.

The purpose of this paper is to solve numerically (1.4), for symmetric X, when A is positive invertible and C is symmetric. An algorithm is resorted to approach a symmetric solution of (1.1). Afterwards, we derive the algorithm that corresponds to the numerical resolution of (1.3). Some numerical examples illustrating the significance of the considered algorithms are presented.

2. An algorithm approximating $[P, Q]$

In the practical context, (1.6) is not useful by virtue of the involved complicated matrix integral. In order to solve numerically (1.1), (1.3), and (1.4), we will first construct an iterative scheme approaching $[P, Q]$. We recall that if P is positive invertible, then the matrix sequence $X_k := X_k(P)$ defined by

$$X_0 = I, \quad X_{k+1} = \frac{1}{2} X_k + \frac{1}{2} P X_k^{-1} \tag{2.1}$$

converges quadratically to $P^{1/2}$. That is, for all $k \geq 0$, the following inequality

$$\|X_{k+1} - P^{1/2}\| \leq c \|X_k - P^{1/2}\|^2 \tag{2.2}$$

holds for some fixed real number $c > 0$. The sequence (X_k) corresponds to the (modified) Newton algorithm associated to the positive solution of the matrix equation $X^2 = P$; see [1, 8], for instance. Note that (2.2) means that (X_k) converges to $P^{1/2}$ with a speed of convergence of order equal to 2. Otherwise, as investigated out in [2], the convergence of (X_k) to $P^{1/2}$ can be successively accelerated into orders equal to $2^2, 2^3, \ldots$.

Inspired by the previous process and by virtue of (1.5) defining $[P, Q]$, we then consider the matrix sequence $Y_k := Y_k(P, Q, t)$ that converges to $(P + tQ)^{1/2}$, when $t > 0$ is fixed. According to the algorithm (2.1), the matrix sequence (Y_k) can be defined as follows:

$$Y_0 = I, \quad Y_{k+1} = \frac{1}{2}Y_k + \frac{1}{2}(P + tQ)Y_k^{-1}. \tag{2.3}$$

Because of (1.5) we have

$$(P + tQ)^{1/2} = P^{1/2} + t[P, Q] + t\, o(t), \quad o(t) = o(P, Q, t) \to 0 \text{ as } t \to 0, \tag{2.4}$$

and it is therefore natural to try searching Y_k in the following form:

$$Y_k = X_k + t\, Z_k + t\, o_k(t), \quad o_k(t) \to 0, \ t \to 0, \tag{2.5}$$

where $Z_k = Z_k(P, Q)$ is the researched matrix sequence, which is intended to tend towards $[P, Q]$. Hence (2.3) with (2.5) yields

$$Y_{k+1} = \frac{1}{2}X_k + \frac{1}{2}tZ_k + t\, o_k(t) + \frac{1}{2}(P + tQ)\Big(X_k + tZ_k + t\, o_k(t)\Big)^{-1},$$

or, equivalently,

$$Y_{k+1} = \frac{1}{2}X_k + \frac{1}{2}tZ_k + t\, o_k(t) + \frac{1}{2}(P + tQ)\Big(X_k^{-1} - tX_k^{-1}\big(Z_k + o_k(t)\big)X_k^{-1}\Big). \tag{2.6}$$

Since $X_0 = I$ and $Y_0 = I$, we then deduce from (2.5) that the initial data for the matrix sequence (Z_k) can be chosen as $Z_0 = 0$. In another part, (2.5) implies that

$$Y_{k+1} = X_{k+1} + t\, Z_{k+1} + t\, o_{k+1}(t), \quad o_{k+1}(t) \to 0, \ t \to 0. \tag{2.7}$$

If we proceed by identification, (2.6) and (2.7) allow us to deduce that our desired matrix sequence $Z_k = Z_k(P, Q)$ may be defined as follows:

$$Z_0 = 0, \quad Z_{k+1} = \frac{1}{2}Z_k + \frac{1}{2}QX_k^{-1} - \frac{1}{2}PX_k^{-1}Z_kX_k^{-1}, \tag{2.8}$$

where $X_k := X_k(P)$ is defined by (2.1).

Since (X_k) converges to $P^{1/2}$ and (Y_k) converges to $(P+tQ)^{1/2}$, then (2.5) with (2.4) gives us a feeling that (Z_k) converges to the suitable limit $[P, Q]$. Note that if we put $Z := \lim Z_k$, then letting $k \uparrow \infty$ in (1.2) and using $\lim X_k = P^{1/2}$, we get $Z = QP^{-1/2} - P^{1/2}ZP^{-1/2}$, or, equivalently, $P^{1/2}Z + ZP^{1/2} = Q$; that is, $Z = [P, Q]$.

3. Application for (1.4) and (1.1)

We preserve the same notations as in the previous section. Under convenient hypothesis, we will apply in this section the algorithm (2.8) to resort to an algorithm approximating the solution of (1.4) and then that of (1.1).

3.1. **Algorithm for the solution of** (1.4)**.** Following (1.2), if A is positive (resp. negative) and invertible, then (1.4) has one and only one symmetric solution. Using the generalized matrix product, this solution is given by $X = [A^2, C]$, which is symmetric when C is. Following (1.6), the integral expression of X is given by

$$X = \left[A^2, C\right] = CA^{-1} - \frac{1}{\pi}A^2 \int_0^\infty \frac{1}{\sqrt{s}}\left(A^2 + sI\right)^{-1} C \left(A^2 + sI\right)^{-1} ds.$$

If moreover C is positive, then for $t > 0$ the Löwner implication allows us to write

$$A^2 + tC \geq A^2 \geq 0 \implies \left(A^2 + tC\right)^{1/2} \geq A,$$

which, with (1.5), implies that $X = \left[A^2, C\right]$ is positive.

We are in the position to resort to a sequence (T_k) approximating the solution of (1.4). Let us define via (2.1) the sequence $V_k = V_k(A)$ as follows:

$$V_0 = I, \quad V_{k+1} = \frac{1}{2}V_k + \frac{1}{2}A^2 V_k^{-1}. \tag{3.1}$$

Following the construction of (Z_k), we may define $T_k = T_k(A; C)$ by demanding

$$T_0 = 0, \quad T_{k+1} = \frac{1}{2}T_k + \frac{1}{2}CV_k^{-1} - \frac{1}{2}A^2 V_k^{-1} T_k V_k^{-1}, \tag{3.2}$$

where A is positive invertible and C is symmetric.

Now, we state the following example, which illustrates numerically the significance of the algorithm (3.2).

Example 3.1. Let us consider the following matrices:

$$A = \begin{bmatrix} 17 & 2 & -5 \\ 2 & 7 & -2 \\ -5 & -2 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 24 & 10 & -5 \\ 10 & 30 & -8 \\ -5 & -8 & 55 \end{bmatrix}.$$

Executing MATLAB 2021a with 3500 digits floating arithmetic and using (3.2) with (3.1), we obtain the estimations as given in Table 1.

Clearly, the convergence of algorithms (3.2) and (3.1) turns out to be of order 4. It needs 18 iterations to satisfy (1.4). As a positive solution of (1.4), we can adopt the approximative matrix X_a given by

$$X \approx X_a := T_{18} = \begin{bmatrix} 1.39684253186054 & 0.629913567053681 & 2.60123003514731 \\ 0.629913567053681 & 2.39094082370622 & 1.49820645002545 \\ 2.60123003514731 & 1.49820645002545 & 10.8756407689469 \end{bmatrix},$$

with

$$\|AX_a + X_a A - C\| = 4.0565e - 3495.$$

3.2. **Algorithm for the solution of** (1.1)**.** We mention that, if A and B are both positive (or both negative) with A or B invertible, then (1.1) has one and only one solution. Let $A, B \in \mathcal{M}_n$ be symmetric. Assume that there is a symmetric solution $X \in \mathcal{M}_n$ of $AX + XB = C$. By transposing, we get $XA + BX = C^T$. Adding side by side these two latter equations, we obtain

$$(A + B)X + X(A + B) = C + C^T, \tag{3.3}$$

TABLE 1. Estimations of Example 3.1

k	$\|V_{k+1}-V_k\|_2$	$\|T_{k+1}-T_k\|_2$
	(3.1)	(3.2)
1	184.42	29.548
2	91.711	13.929
3	44.879	6.2594
4	20.626	2.8645
5	7.5429	1.1687
6	1.3771	0.30504
7	0.049181	0.018487
8	0.000062888	0.000043567
9	$1.0283e-10$	$1.3691e-10$
10	$2.749e-22$	$7.1798e-22$
11	$1.9648e-45$	$1.0166e-44$
12	$1.0037e-91$	$1.0337e-90$
13	$2.6191e-184$	$5.3822e-183$
14	$1.7835e-369$	$7.3215e-368$
15	$8.2701e-740$	$6.786e-738$
16	$1.7782e-1480$	$2.9174e-1478$
17	$8.2213e-2962$	$2.6972e-2959$
18	$2.3605e-3498$	$2.3204e-3496$

which is a Lyapunov equation as (1.4). If $A+B$ is positive (resp. negative) and invertible, then (3.3) has a unique solution given by $X=\left[(A+B)^2, C+C^T\right]$, which is obviously symmetric. Following (1.6), the integral expression of X is given by

$$\begin{aligned} X = \Big[(A+B)^2, C+C^T\Big] &= (C+C^T)(A+B)^{-1} \\ &- \frac{1}{\pi}(A+B)^2\int_0^\infty \frac{1}{\sqrt{s}}\Big((A+B)^2+sI\Big)^{-1}(C+C^T)\Big((A+B)^2+sI\Big)^{-1}\,ds. \end{aligned}$$

If moreover C is accretive ($C+C^T$ is positive), then $X=\left[(A+B)^2, C+C^T\right]$ is positive. In particular, if C is positive, then so is X, with $X=2\left[(A+B)^2, C\right]$.

To resort to a sequence (U_k) approximating the symmetric solution of (1.1) if any, let us define via (2.1) the sequence $W_k=W_k(A,B)$ as follows:

$$W_0=I, \quad W_{k+1}=\frac{1}{2}W_k+\frac{1}{2}(A+B)^2W_k^{-1}. \tag{3.4}$$

Following the construction of (Z_k), we may define $U_k = U_k(A, B; C)$ by demanding

$$U_0 = 0, \quad U_{k+1} = \frac{1}{2}U_k + \frac{1}{2}\big(C + C^T\big)W_k^{-1} - \frac{1}{2}\big(A + B\big)^2 W_k^{-1} U_k W_k^{-1}, \tag{3.5}$$

where A and B are symmetric such that $A + B$ is positive (resp., negative) and invertible, and $C \in \mathcal{M}_n$ is an arbitrary matrix.

We end this section by presenting the following example, which illustrates numerically the convergence of the algorithm (3.5).

Example 3.2. Let $A, B, C \in \mathcal{M}_2$ be given by

$$A = \begin{bmatrix} 44 & -5 \\ -5 & 75 \end{bmatrix}, \quad B = \begin{bmatrix} -18 & -2 \\ -2 & -15 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 85 & -45 \\ -76 & 127 \end{bmatrix}.$$

As in Example 3.1, by using (3.5) with (3.4), we find estimations concerning the solution of (1.1) in Table 2.

TABLE 2. Estimations of Example 3.2

k	$\Vert W_{k+1} - W_k\Vert_2$	$\Vert U_{k+1} - U_k\Vert_2$
	(3.4)	(3.5)
1	1883.5	170.04
2	941.27	85.02
3	469.64	42.509
4	232.84	21.247
5	112.57	10.575
6	49.417	5.0768
7	15.494	2.0381
8	1.896	0.39485
9	0.029268	0.010846
10	$6.9774e-6$	$4.871e-6$
11	$3.9655e-13$	$5.3727e-13$
12	$1.2809e-27$	$3.4191e-27$
13	$1.3364e-56$	$7.0812e-56$
14	$1.4547e-114$	$1.5359e-113$
15	$1.7238e-230$	$3.633e-229$
16	$2.4203e-462$	$1.0192e-460$
17	$4.7714e-926$	$4.0168e-924$
18	$1.8544e-1853$	$3.1215e-1851$
19	$2.0025e-3507$	$5.0063e-3508$

In this example, we use Table 2 to show the error analysis. It is obvious that the effect of algorithms (3.5) with (3.4) is optimal compared with other algorithms. It can be seen that the variant versions of the algorithms perform better.

4. An algorithm for the solution of (1.3)

In this section, we shall be dealing with an algorithm derived from (2.8) for approximating the solution of the matrix equation (1.3). Let $A, C \in \mathcal{M}_n$ with C be symmetric. It is often of interest to search $X \in \mathcal{M}_n$ positive invertible satisfying (1.3). Assume that further A satisfies the following condition:

$$A = MP \text{ for some (symmetric) positive invertible } M, P \in \mathcal{M}_n. \tag{4.1}$$

Following [12], (1.3) has a unique positive invertible solution X given by

$$X = P^{-1/2}\Big[P^{1/2}A^2P^{-1/2}, P^{1/2}CP^{1/2}\Big]P^{-1/2}. \tag{4.2}$$

We note that $P^{1/2}A^2P^{-1/2}$ is positive invertible, since

$$P^{1/2}A^2P^{-1/2} = P^{1/2}MPMPP^{-1/2} = P^{1/2}(MPM)P^{1/2}.$$

Inspired by (2.8) and using (4.2), we will be interest to resort to an algorithm approximating the solution X of (1.3). We can proceed in two different ways:

First way: Following (2.1), the matrix sequence (E_k) defined by

$$E_0 = I, \quad E_{k+1} = \frac{1}{2}E_k + \frac{1}{2}P^{1/2}A^2P^{-1/2}E_k^{-1} \tag{4.3}$$

is an approximation of $\Big(P^{1/2}A^2P^{-1/2}\Big)^{1/2}$ when $k \uparrow \infty$. According to (2.8), the matrix sequence (F_k) defined by

$$F_0 = 0, \quad F_{k+1} = \frac{1}{2}F_k + \frac{1}{2}P^{1/2}CP^{1/2}E_k^{-1} - \frac{1}{2}P^{1/2}A^2P^{-1/2}E_k^{-1}F_kE_k^{-1} \tag{4.4}$$

is an approximation of $F := \Big[P^{1/2}A^2P^{-1/2}, P^{1/2}CP^{1/2}\Big]$ when $k \uparrow \infty$. Multiplying each side of (4.4) at left and at right by $P^{-1/2}$ and setting $G_k := P^{-1/2}F_kP^{-1/2}$, the matrix sequence (G_k) is defined through

$$G_0 = 0,\ G_{k+1} = \frac{1}{2}G_k + \frac{1}{2}CP^{1/2}E_k^{-1}P^{-1/2} - \frac{1}{2}A^2P^{-1/2}E_k^{-1}P^{1/2}G_kP^{1/2}E_k^{-1}P^{-1/2}. \tag{4.5}$$

This with (4.2) asserts that (G_k) is an approximation for the solution X of (1.3) when $k \uparrow \infty$. Now, we will try to escape the term $P^{1/2}$ from (4.3) and (4.5) for obtaining representative algorithms involving only elementary matrix operations. Multiplying each side of (4.3) at left and at right by $P^{-1/2}$, setting $K_k := P^{-1/2}E_kP^{-1/2}$ and remarking that K_k is positive invertible (since E_k is) for each $k \geq 0$, we get after a simple algebraic manipulation that

$$K_0 = P^{-1}, \quad K_{k+1} = \frac{1}{2}K_k + \frac{1}{2}A^2P^{-1}K_k^{-1}P^{-1}, \tag{4.6}$$

which does not involve $P^{1/2}$ and represents equivalently (4.3). With this, if we set

$$H_k = \big(PK_k\big)^{-1}, \tag{4.7}$$

then (4.5) becomes (after simple algebraic operations)

$$G_0 = 0, \quad G_{k+1} = \frac{1}{2}G_k + \frac{1}{2}CH_k - \frac{1}{2}A^2 H_k^T G_k H_k, \tag{4.8}$$

which also does not contain $P^{1/2}$ and is an equivalent definition for (G_k).

Summarizing, the matrix sequence (G_k) is an approximation of X when $k \uparrow \infty$.

Second way: Here, we first approximate $P^{1/2}$ by X_k given by (2.1). By (4.2) with an argument of continuity, the matrix sequence

$$E_k = X_k^{-1}\Big[X_k A^2 X_k^{-1}, X_k C X_k\Big] X_k^{-1} \tag{4.9}$$

is an approximation of X when $k \uparrow \infty$. For $k \geq 0$ fixed, let $(F_{k,m})_{m\geq 0}$ be defined as

$$F_{k,0} = I, \quad F_{k,m+1} = \frac{1}{2}F_{k,m} + \frac{1}{2}X_k A^2 X_k^{-1} F_{k,m}^{-1}. \tag{4.10}$$

According to (2.1) again, $F_{k,m}$ approximates $\big(X_k A^2 X_k^{-1}\big)^{1/2}$ when $m \uparrow \infty$. Following (2.8), the matrix double-sequence

$$G_{k,0} = 0, \quad G_{k,m+1} = \frac{1}{2}G_{k,m} + \frac{1}{2}X_k C X_k F_{k,m}^{-1} - \frac{1}{2}X_k A^2 X_k^{-1} F_{k,m}^{-1} G_{k,m} F_{k,m}^{-1} \tag{4.11}$$

is an approximation of $\Big[X_k A^2 X_k^{-1}, X_k C X_k\Big]$ when $m \uparrow \infty$. Multiplying all sides of (4.11) at left and at right by X_k^{-1} and then setting

$$H_{k,m} := X_k^{-1} G_{k,m} X_k^{-1} \ \text{ and } \ K_{k,m} := \big(X_k F_{k,m} X_k^{-1}\big)^{-1}, \tag{4.12}$$

we infer that the matrix double-sequence given by

$$H_{k,0} = 0, \qquad H_{k,m+1} = \frac{1}{2}H_{k,m} + \frac{1}{2}CK_{k,m} - \frac{1}{2}A^2\big(K_{k,m}\big)^T H_{k,m} K_{k,m} \tag{4.13}$$

is an approximation of E_k when $m \uparrow \infty$. This, when combined with (4.9) and the fact that E_k approximates X as $k \uparrow \infty$, allows us to conclude that the matrix double-sequence $(H_{k,m})_{k,m}$ is an approximation of X, the solution of (1.3), when $k, m \uparrow \infty$.

The following remark may be of interest for the reader.

Remark 4.1. (i) If A is diagonalizable with spectra in $(0, \infty)$, then A satisfies the condition (4.1). In fact, let $A = Q^{-1}DQ$ with D diagonal positive. Then we have $A = MP$ with $M = Q^{-1}DQ^{-T}$ and $P = Q^T Q$.
(ii) The decomposition $A = MP$ in (4.1) is not unique. However, when (1.3) has a unique solution X given by (4.2) then X does not depend on the choice of P.

Using the two previous ways, we will present the following example in order to illustrate numerically the convergence of (G_k) and $(H_{k,m})$ to the solution X of (1.3).

Example 4.2. Consider the following matrices:

$$A = \begin{bmatrix} 10 & -10 & 9 \\ -11 & 16 & -11 \\ 9 & -10 & 10 \end{bmatrix}, \quad C = \begin{bmatrix} 98 & -81 & 65 \\ -81 & 64 & -36 \\ 65 & -36 & 38 \end{bmatrix}.$$

One can check that A can be decomposed as $A = MP$, with

$$M = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 3 & -2 & 2 \\ -2 & 4 & -2 \\ 2 & -2 & 3 \end{bmatrix}.$$

According to the two previous ways, we get the estimations about the solution of (1.3) in Tables 3 and 4, respectively.

TABLE 3. Estimations of Example 4.2 by the first way

k	$\|K_{k+1} - K_k\|_2$	$\|H_{k+1} - H_k\|_2$	$\|G_{k+1} - G_k\|_2$
	(4.6)	(4.7)	(4.8)
1	71.374	0	98.748
2	35.618	0.99826	49.335
3	17.674	0.098123	24.644
4	8.5741	0.027542	12.265
5	3.8126	0.006647	5.9809
6	1.2469	0.00924	2.5369
7	0.16969	0.0063593	0.55077
8	0.0032631	0.0011433	0.018913
9	$1.2076e-6$	0.000022848	0.000013195
10	$1.654e-13$	$8.4619e-9$	$3.5073e-12$
11	$3.1027e-27$	$1.159e-15$	$1.296e-25$
12	$1.0918e-54$	$2.174e-29$	$9.0515e-53$
13	$1.3519e-109$	$7.6502e-57$	$2.2331e-107$
14	$2.0729e-219$	$9.4729e-112$	$6.8346e-217$
15	$4.8731e-439$	$1.4524e-221$	$3.2104e-436$
16	$2.6932e-878$	$3.4145e-441$	$3.547e-875$
17	$8.2263e-1757$	$1.8871e-880$	$2.1663e-1753$
18	$7.6088e-3490$	$5.7642e-1759$	$2.5319e-3487$
19	$1.1949e-3488$	$1.7278e-3490$	$3.9841e-3486$

In the iterative algorithm (4.8), choosing the initial matrix $G_0 = 0$, we find that the error between G_{19} and G_{18} is $3.9841e-3486$ which is the same as that between G_{18} and G_{17}. We then compare the errors of theses iterative methods by tables. From Table 3, we can observe that the algorithms (4.6), (4.7), and (4.8) are accurate and that the number of iteration steps is the same.

TABLE 4. Estimations of Example 4.2 by the second way

k, m	$\|F_{k,m+1} - F_{k,m}\|_2$	$\|G_{k,m+1} - G_{k,m}\|_2$	$\|K_{k,m+1} - K_{k,m}\|_2$	$\|H_{k,m+1} - H_{k,m}\|_2$
	(4.10)	(4.11)	(4.12)	(4.13)
15, 1	524.64	692.21	524.64	49.414
15, 2	261.82	346.09	261.82	24.644
15, 3	129.92	173.0	129.92	12.265
15, 4	63.028	86.347	63.028	5.9809
15, 5	28.026	42.469	28.026	2.5369
15, 6	9.1663	18.313	9.1663	0.55077
15, 7	1.2474	4.0295	1.2474	0.018913
15, 8	0.023987	0.13904	0.023987	0.000013195
15, 9	$8.8774e-6$	0.000097088	$8.8774e-6$	$3.5073e-12$
15, 10	$1.2159e-12$	$2.58e-11$	$1.2159e-12$	$1.296e-25$
15, 11	$2.2808e-26$	$9.5309e-25$	$2.2808e-26$	$9.0515e-53$
15, 12	$8.0259e-54$	$6.6553e-52$	$8.0259e-54$	$2.2331e-107$
15, 13	$9.938e-109$	$1.6417e-106$	$9.938e-109$	$6.8346e-217$
15, 14	$1.5238e-218$	$5.0245e-216$	$1.5238e-218$	$3.2104e-436$
15, 15	$3.5822e-438$	$2.3601e-435$	$3.5822e-438$	$3.547e-875$
15, 16	$1.9798e-877$	$2.6074e-874$	$1.9798e-877$	$2.1663e-1753$
15, 17	$6.0472e-1756$	$1.5925e-1752$	$6.0472e-1756$	$1.9737e-3486$
15, 18	$1.0576e-3487$	$5.5141e-3486$	$1.0576e-3487$	$3.1533e-3485$

For the algorithm (4.13), the error between $H_{15,18}$ and $H_{15,17}$ is $3.1533e-3485$ and coincides with that between $H_{15,17}$ and $H_{15,16}$. As previous, Table 4 explains the accuracy of the algorithms (4.10), (4.11), (4.12), and (4.13) with the same number of iterations.

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[1]Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.
Email address: nhelielalharbi@stu.kau.edu.sa; dmashat@kau.edu.sa

[2]Department of Mathematics, University of Moulay Ismail, Meknes, Morocco.
Email address: raissouli.mustapha@gmail.com

[3]Department of Mathematics, College of Sciences, King Khalid University, Abha 61413, Saudi Arabia.
Email address: nhalharbi@kku.edu.sa