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SELECTION GAMES WITH MINIMAL USCO MAPS

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ABSTRACT. We establish relationships between various topological selection games involving the space of minimal usco maps with various topologies, including the topology of pointwise convergence and the topology of uniform convergence on compact sets, and the underlying domain using full- and limited-information strategies. We also tie these relationships to analogous results related to spaces of continuous functions. The primary games we consider include Rothberger-like games, generalized point-open games, strong fan-tightness games, Tkachuk's closed discrete selection game, and Gruenhage's W-games.

1. Introduction and preliminaries

Minimal upper-semicontinuous compact-valued functions have a rich history, apparently arising from the study of holomorphic functions and their so-called cluster sets; see [11]. The phrase *minimal usco* was coined by Christensen [7], where a topological game similar to the Banach–Mazur game was considered. In this paper, using some techniques similar to those of Holá and Holý [19], we tie connections between a space X and the space of minimal usco maps with the topology of uniform convergence on certain kinds of subspaces of X similar in spirit to those appearing in [2, 3, 9] (see also [10, 26]); in particular, most of the results come in the form of selection game equivalences or dualities, which rely on a variety of game-related results from [2, 3, 8, 35–37]. We also tie some of these results to existing results relating topological properties of a space X with the space of continuous real-valued functions on X with various topologies.

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Consequences of these results include Corollary 2.12, which captures [19, Corollary 4.5], that states that X is hemicompact if and only if $MU_k(X)$, the space of minimal usco maps into $\mathbb{R}$ on X with the topology of uniform convergence on compact subsets, is metrizable. Corollary 2.12 also shows that X is hemicompact if and only if $MU_k(X)$ is not discretely selective. Corollary 2.17 contains the assertion that X is k-Rothberger if and only if $MU_k(X)$ has strong countable fan-tightness at $\mathbf{0}$, the constant $\{0\}$ function. We end with Corollary 2.21, which characterizes a space having a countable so-called weak k-covering number in terms of strategic information in selection games.

We use the word *space* to mean *topological space*. When the parent space is understood from context, we use the notation $\mathrm{int}(A)$, $\mathrm{cl}(A)$, and ∂A to denote the interior, closure, and boundary of A, respectively. If we must specify the topological space X, we use $\mathrm{int}_X(A)$, $\mathrm{cl}_X(A)$, and $\partial_X A$.

Given a function $f : X \to Y$, we denote the graph of f by $\mathrm{gr}(f) = \{\langle x, f(x)\rangle : x \in X\}$. For a set X, we let $\wp(X)$ denote the set of subsets of X and $\wp^+(X) = \wp(X) \setminus \{\emptyset\}$. For sets X and Y, we let

$$\mathrm{Fn}(X,Y) = \bigcup_{A\in\wp^+(X)} Y^A;$$

that is, $\mathrm{Fn}(X,Y)$ is the collection of all Y-valued functions defined on nonempty subsets of X.

When a set X is implicitly serving as the parent space in context, given $A \subseteq X$, we will let $\mathbf{1}_A$ be the indicator function for A; that is, $\mathbf{1}_A : X \to \{0,1\}$ is defined by the rule

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A; \\ 0, & x \notin A. \end{cases}$$

For any set X, we let $X^{<\omega}$ denote the set of finite sequences of X, $[X]^{<\omega}$ denote the set of finite subsets of X, and, for any cardinal κ, $[X]^\kappa$ denote the set of κ-sized subsets of X.

For a space X, we let $K(X)$ denote the set of all nonempty compact subsets of X. We let $\mathbb{K}(X)$ denote the set $K(X)$ endowed with the Vietoris topology; that is, the topology with basis consisting of sets of the form

$$[U_1, U_2, \ldots, U_n] = \left\{ K \in \mathbb{K}(X) : K \subseteq \bigcup_{j=1}^{n} U_j \wedge K^n \cap \prod_{j=1}^{n} U_j \neq \emptyset \right\}.$$

For more about this topology, see [27].

Definition 1.1. For a set X, we say that a family $\mathcal{A} \subseteq \wp^+(X)$ is an *ideal of sets* if

- for $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$, and
- for every $x \in X$, $\{x\} \in \mathcal{A}$.

If X is a space, then we say that an ideal of sets $\mathcal{A}$ is an *ideal of closed sets* if $\mathcal{A}$ consists of closed sets.

Throughout, we will assume that any ideal of closed sets under consideration does not contain the entire space X. Two ideals of closed sets of primary interest are

- the collection of nonempty finite subsets of an infinite space X and
- the collection of nonempty compact subsets of a noncompact space X.

1.1. **Selection games.** General topological games have a long history, a lot of which can be gathered from Telgársky's survey [34]. In this paper, we will be dealing only with single-selection games of countable length.

Definition 1.2. Given sets $\mathcal{A}$ and $\mathcal{B}$, we define the *single-selection game* $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ as follows.

- For each $n \in \omega$, One chooses $A_n \in \mathcal{A}$ and Two responds with $x_n \in A_n$.
- Two is declared the winner if $\{x_n : n \in \omega\} \in \mathcal{B}$. Otherwise, One wins.

Definition 1.3. We define strategies of various strength below.

- We use two forms of *full-information* strategies.
  - A *strategy for player One* in $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ is a function $\sigma : (\bigcup \mathcal{A})^{<\omega} \to \mathcal{A}$. A strategy σ for One is called *winning* if whenever $x_n \in \sigma\langle x_k : k < n\rangle$ for all $n \in \omega$, $\{x_n : n \in \omega\} \notin \mathcal{B}$. If player One has a winning strategy, then we write $\mathrm{I} \uparrow \mathsf{G}_1(\mathcal{A}, \mathcal{B})$.
  - A *strategy for player Two* in $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ is a function $\tau : \mathcal{A}^{<\omega} \to \bigcup \mathcal{A}$. A strategy τ for Two is *winning* if whenever $A_n \in \mathcal{A}$ for all $n \in \omega$, $\{\tau(A_0, \ldots, A_n) : n \in \omega\} \in \mathcal{B}$. If player Two has a winning strategy, then we write $\mathrm{II} \uparrow \mathsf{G}_1(\mathcal{A}, \mathcal{B})$.
- We use two forms of *limited-information* strategies.
  - A *predetermined strategy* for One is a strategy that only considers the current turn number. We call this kind of strategy predetermined because One is not reacting to Two's moves. Formally it is a function $\sigma : \omega \to \mathcal{A}$. If One has a winning predetermined strategy, then we write $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathcal{A}, \mathcal{B})$.
  - A *Markov strategy* for Two is a strategy that only considers the most recent move of player One and the current turn number. Formally it is a function $\tau : \mathcal{A} \times \omega \to \bigcup \mathcal{A}$. If Two has a winning Markov strategy, then we write $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_1(\mathcal{A}, \mathcal{B})$.

The reader may be more familiar with selection principles than selection games. For more details on selection principles and relevant references, see [22, 31].

Definition 1.4. Let $\mathcal{A}$ and $\mathcal{B}$ be collections. The *single-selection principle* $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ for a space X is the following property. Given any $A \in \mathcal{A}^\omega$, there exists $\vec{x} \in \prod_{n\in\omega} A_n$ such that $\{\vec{x}_n : n \in \omega\} \in \mathcal{B}$.

As mentioned in [8, Prop. 15], $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ holds if and only if $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} \mathsf{G}_1(\mathcal{A}, \mathcal{B})$. Hence, we may establish equivalences between certain selection principles by addressing the corresponding selection games.

Definition 1.5. For a space X, an open cover $\mathscr{U}$ of X is said to be *nontrivial* if $\emptyset \notin \mathscr{U}$ and $X \notin \mathscr{U}$.

Definition 1.6. Let X be a space and let $\mathcal{A}$ be a set of closed subsets of X. We say that a nontrivial cover $\mathscr{U}$ of X is an $\mathcal{A}$-*cover* if, for every $A \in \mathcal{A}$, there exists $U \in \mathscr{U}$ such that $A \subseteq U$.

Definition 1.7. For a collection $\mathcal{A}$, we let $\neg\mathcal{A}$ denote the collection of sets that are not in $\mathcal{A}$. We also define the following classes for a space X and a collection $\mathcal{A}$ of closed subsets of X.

- $\mathscr{T}_X$ is the family of all proper nonempty open subsets of X.
- For $x \in X$, $\mathscr{N}_{X,x} = \{U \in \mathscr{T}_X : x \in U\}$.
- For $A \in \wp^+(X)$, $\mathscr{N}_X(A) = \{U \in \mathscr{T}_X : A \subseteq U\}$.
- $\mathscr{N}_X[\mathcal{A}] = \{\mathscr{N}_X(A) : A \in \mathcal{A}\}$,
- CD_X is the set of all closed discrete subsets of X.
- $\mathscr{D}_X$ is the set of all dense subsets of X.
- For $x \in X$, $\Omega_{X,x} = \{A \subseteq X : x \in \mathrm{cl}(A)\}$.
- For $x \in X$, $\Gamma_{X,x}$ is the set of all sequences of X converging to x.
- $\mathcal{O}_X$ is the set of all nontrivial open covers of X.
- $\mathcal{O}_X(\mathcal{A})$ is the set of all $\mathcal{A}$-covers.
- $\Lambda_X(\mathcal{A})$ is the set of all $\mathcal{A}$-covers $\mathscr{U}$ with the property that, for every $A \in \mathcal{A}$, $\{U \in \mathscr{U} : A \subseteq U\}$ is infinite.
- $\Gamma_X(\mathcal{A})$ is the set of all countable $\mathcal{A}$-covers $\mathscr{U}$ with the property that, for every $A \in \mathcal{A}$, $\{U \in \mathscr{U} : A \subseteq U\}$ is co-finite.

Note that, in our notation, $\mathcal{O}_X([X]^{<\omega})$ is the set of all ω-covers of X, which we will denote by Ω_X, and that $\mathcal{O}_X(K(X))$ is the set of all k-covers of X, which we will denote by $\mathcal{K}_X$. We also use $\Gamma_\omega(X)$ to denote $\Gamma_X([X]^{<\omega})$ and $\Gamma_k(X)$ to denote $\Gamma_X(K(X))$.

Also, note that $\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_X)$ is the Rothberger property and $\mathsf{G}_1(\mathcal{O}_X, \mathcal{O}_X)$ is the Rothberger game. If we let $\mathbb{P}_X = \{\mathscr{N}_{X,x} : x \in X\}$, then $\mathsf{G}_1(\mathbb{P}_X, \neg\mathcal{O})$ is isomorphic to the point-open game studied by Galvin [13] and Telgársky [33]. The games $\mathsf{G}_1(\mathscr{N}_{X,x}, \neg\Gamma_{X,x})$ and $\mathsf{G}_1(\mathscr{N}_{X,x}, \neg\Omega_{X,x})$ are two variants of Gruenhage's W-game (see [15]). We refer to $\mathsf{G}_1(\mathscr{N}_{X,x}, \neg\Gamma_{X,x})$ as Gruenhage's converging W-game and $\mathsf{G}_1(\mathscr{N}_{X,x}, \neg\Omega_{X,x})$ as Gruenhage's clustering W-game. The games $\mathsf{G}_1(\mathscr{T}_X, \neg\Omega_{X,x})$ and $\mathsf{G}_1(\mathscr{T}_X, \mathrm{CD}_X)$ were introduced by Tkachuk (see [36, 37]) and tied to Gruenhage's W-games in [9, 37]. The strong countable dense fan-tightness game at x is $\mathsf{G}_1(\mathscr{D}_X, \Omega_{X,x})$ and the strong countable fan-tightness game at x is $\mathsf{G}_1(\Omega_{X,x}, \Omega_{X,x})$ (see [1]).

Lemma 1.8 (see [3, Lemma 4])**.** *For a space X and an ideal of closed sets $\mathcal{A}$ of X, $\mathcal{O}_X(\mathcal{A}) = \Lambda_X(\mathcal{A})$.*

In what follows, we say that $\mathcal{G}$ is a *selection game* if there exist classes $\mathcal{A}$, $\mathcal{B}$ such that $\mathcal{G} = \mathsf{G}_1(\mathcal{A}, \mathcal{B})$.

Since we work with full- and limited-information strategies, we reflect this in our definitions of game equivalence and duality.

Definition 1.9. We say that two selection games $\mathcal{G}$ and $\mathcal{H}$ are *equivalent*, denoted by $\mathcal{G} \equiv \mathcal{H}$, if the following conditions hold:

- $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathcal{G} \iff \mathrm{II} \underset{\text{mark}}{\uparrow} \mathcal{H}$
- $\mathrm{II} \uparrow \mathcal{G} \iff \mathrm{II} \uparrow \mathcal{H}$
- $\mathrm{I} \not\uparrow \mathcal{G} \iff \mathrm{I} \not\uparrow \mathcal{H}$
- $\mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathcal{G} \iff \mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathcal{H}$

We also use a preorder on selection games.

Definition 1.10. Given selection games $\mathcal{G}$ and $\mathcal{H}$, we say that $\mathcal{G} \leq_{\mathrm{II}} \mathcal{H}$ if the following implications hold:

- $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathcal{G} \implies \mathrm{II} \underset{\text{mark}}{\uparrow} \mathcal{H}$
- $\mathrm{II} \uparrow \mathcal{G} \implies \mathrm{II} \uparrow \mathcal{H}$
- $\mathrm{I} \not\uparrow \mathcal{G} \implies \mathrm{I} \not\uparrow \mathcal{H}$
- $\mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathcal{G} \implies \mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathcal{H}$

Note that $\leq_{\mathrm{II}}$ is transitive and that if $\mathcal{G} \leq_{\mathrm{II}} \mathcal{H}$ and $\mathcal{H} \leq_{\mathrm{II}} \mathcal{G}$, then $\mathcal{G} \equiv \mathcal{H}$. We use the subscript of II since each implication in the definition of $\leq_{\mathrm{II}}$ is related to a transference of winning plays by Two.

Definition 1.11. We say that two selection games $\mathcal{G}$ and $\mathcal{H}$ are *dual* if the following conditions hold:

- $\mathrm{I} \uparrow \mathcal{G} \iff \mathrm{II} \uparrow \mathcal{H}$
- $\mathrm{II} \uparrow \mathcal{G} \iff \mathrm{I} \uparrow \mathcal{H}$
- $\mathrm{I} \underset{\text{pre}}{\uparrow} \mathcal{G} \iff \mathrm{II} \underset{\text{mark}}{\uparrow} \mathcal{H}$
- $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathcal{G} \iff \mathrm{I} \underset{\text{pre}}{\uparrow} \mathcal{H}$

We note one important way in which equivalence and duality interact.

Lemma 1.12. *Suppose $\mathcal{G}_1, \mathcal{G}_2, \mathcal{H}_1$, and $\mathcal{H}_2$ are selection games such that $\mathcal{G}_1$ is dual to $\mathcal{H}_1$ and $\mathcal{G}_2$ is dual to $\mathcal{H}_2$. Then, if $\mathcal{G}_1 \leq_{\mathrm{II}} \mathcal{G}_2$, then $\mathcal{H}_2 \leq_{\mathrm{II}} \mathcal{H}_1$. Consequently, if $\mathcal{G}_1 \equiv \mathcal{G}_2$, then $\mathcal{H}_1 \equiv \mathcal{H}_2$.*

We will use consequences of [8, Corollary 26] to see that a few classes of selection games are dual.

Lemma 1.13. *Let $\mathcal{A}$ be an ideal of closed sets of a space X and let $\mathcal{B}$ be a collection.*

(i) *By* [2, Corollary 3.4] *and* [8, Theorem 38], $\mathsf{G}_1(\mathcal{O}_X(\mathcal{A}), \mathcal{B})$ *and* $\mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathcal{B})$ *are dual (note that this is a general form of the duality of the Rothberger game and the point-open game).*

(ii) *By* [8, Corollary 33], $\mathsf{G}_1(\mathscr{D}_X, \mathcal{B})$ *and* $\mathsf{G}_1(\mathscr{T}_X, \neg\mathcal{B})$ *are dual.*

(iii) *By* [8, Corollary 35], *for $x \in X$,* $\mathsf{G}_1(\Omega_{X,x}, \mathcal{B})$ *and* $\mathsf{G}_1(\mathscr{N}_{X,x}, \neg\mathcal{B})$ *are dual.*

We now state the translation theorems, which we will be using to establish some game equivalences.

Theorem 1.14 ([3, Theorem 12])**.** *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be collections. Suppose that there are functions $\overleftarrow{T}_{\mathrm{I},n} : \mathcal{B} \to \mathcal{A}$ and $\overrightarrow{T}_{\mathrm{II},n} : (\bigcup \mathcal{A}) \times \mathcal{B} \to \bigcup \mathcal{B}$ for each $n \in \omega$, such that*

(i) *if* $x \in \overleftarrow{T}_{\mathrm{I},n}(B)$*, then* $\overrightarrow{T}_{\mathrm{II},n}(x, B) \in B$*, and*

(ii) *if* $\langle x_n : n \in \omega \rangle \in \prod_{n\in\omega} \overleftarrow{T}_{\mathrm{I},n}(B_n)$ *and* $\{x_n : n \in \omega\} \in \mathcal{C}$*, then*

$$\left\{ \overrightarrow{T}_{\mathrm{II},n}(x_n, B_n) : n \in \omega \right\} \in \mathcal{D}.$$

Then $\mathsf{G}_1(\mathcal{A}, \mathcal{C}) \leq_{\mathrm{II}} \mathsf{G}_1(\mathcal{B}, \mathcal{D})$.

Similar results to Theorem 1.14 for longer length games and finite-selection games, even with simplified hypotheses, can be found in [3–5].

We will also need a separation axiom for some results in what follows.

Definition 1.15. Let X be a space and let $\mathcal{A}$ be an ideal of closed subsets of X. We say that X is $\mathcal{A}$*-normal* if, given any $A \in \mathcal{A}$ and $U \subseteq X$ open with $A \subseteq U$, there exists an open set V such that $A \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$.

We also say that X is *functionally* $\mathcal{A}$*-normal* if, for $A \in \mathcal{A}$ and $U \subseteq X$ open with $A \subseteq U$, there exists a continuous function $f : X \to \mathbb{R}$ such that $f[A] = \{0\}$ and $f[X \setminus U] = \{1\}$.

Note that, if X is $\mathcal{A}$-normal, then X is regular. If $\mathcal{A} \subseteq K(X)$ and X is regular, then X is $\mathcal{A}$-normal. If X is Tychonoff and $\mathcal{A} \subseteq K(X)$, then X is functionally $\mathcal{A}$-normal.

1.2. **Uniform spaces.** To introduce the basics of uniform spaces, needed in this paper, we mostly follow [20, Chapter 6].

We recall the standard notation involved with uniformities. Let X be a set. The diagonal of X is $\Delta_X = \{\langle x, x\rangle : x \in X\}$. For $E \subseteq X^2$, $E^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in E\}$. If $E = E^{-1}$, then E is said to be *symmetric*. If $E, F \subseteq X^2$, then

$$E \circ F = \{\langle x, z\rangle : (\text{there exists } y \in X)\ \langle x, y\rangle \in F \wedge \langle y, z\rangle \in E\}.$$

For $E \subseteq X^2$, we let $E[x] = \{y \in X : \langle x, y\rangle \in E\}$ and $E[A] = \bigcup_{x\in A} E[x]$.

Definition 1.16. A *uniformity* on a set X is a set $\mathcal{E} \subseteq \wp^+(X^2)$ that satisfies the following properties:

- For every $E \in \mathcal{E}$, $\Delta_X \subseteq E$.
- For every $E \in \mathcal{E}$, $E^{-1} \in \mathcal{E}$.
- For every $E \in \mathcal{E}$, there exists $F \in \mathcal{E}$ such that $F \circ F \subseteq E$.
- For $E, F \in \mathcal{E}$, $E \cap F \in \mathcal{E}$.
- For $E \in \mathcal{E}$ and $F \subseteq X^2$, if $E \subseteq F$, $F \in \mathcal{E}$.

If, in addition, $\Delta_X = \bigcap \mathcal{E}$, we say that the uniformity $\mathcal{E}$ is *Hausdorff*. By an *entourage* of X, we mean $E \in \mathcal{E}$. The pair $(X, \mathcal{E})$ is called a *uniform space*.

Definition 1.17. For a set X, we say that $\mathcal{B} \subseteq \wp^+(X^2)$ is a *base for a uniformity* if

- for every $B \in \mathcal{B}$, $\Delta_X \subseteq B$;
- for every $B \in \mathcal{B}$, there is some $A \in \mathcal{B}$ such that $A \subseteq B^{-1}$;
- for every $B \in \mathcal{B}$, there is some $A \in \mathcal{B}$ such that $A \circ A \subseteq B$; and
- for $A, B \in \mathcal{B}$, there is some $C \in \mathcal{B}$ such that $C \subseteq A \cap B$.

If the uniformity generated by $\mathcal{B}$ is $\mathcal{E}$, then we say that $\mathcal{B}$ is a *base* for $\mathcal{E}$.

If $(X, \mathcal{E})$ is a uniform space, then the uniformity $\mathcal{E}$ generates a topology on X in the following way: $U \subseteq X$ is declared to be open provided that, for every $x \in U$, there is some $E \in \mathcal{E}$ such that $E[x] \subseteq U$. An important result about this topology reads as below.

Theorem 1.18 (see [20])**.** *A Hausdorff uniform space $(X, \mathcal{E})$ is metrizable if and only if $\mathcal{E}$ has a countable base.*

With the topology induced by the uniformity, we endow X^2 with the resulting product topology.

Lemma 1.19 (see [20])**.** *The family of open (or closed) symmetric entourages of a uniformity $\mathcal{E}$ is a base for $\mathcal{E}$.*

For a uniform space $(X, \mathcal{E})$, there is a natural way to define a uniformity on $K(X)$ that is directly analogous to the Pompeiu–Hausdorff distance defined in the context of metric spaces.

Definition 1.20. Let $(X, \mathcal{E})$ be a uniform space, and, for $E \in \mathcal{E}$, define

$$hE = \{\langle K, L\rangle \in K(X)^2 : K \subseteq E[L] \wedge L \subseteq E[K]\}.$$

Just as the Pompeiu–Hausdorff distance on compact subsets generates the Vietoris topology, the analogous uniformity also generates the Vietoris topology.

Theorem 1.21 (see [6, Chapter 2])**.** *For a uniform space $(X, \mathcal{E})$, $\mathcal{B} = \{hE : E \in \mathcal{E}\}$ is a base for a uniformity on $K(X)$; the topology generated by the uniform base $\mathcal{B}$ is the Vietoris topology.*

For the set of functions from a space X to a uniform space $(Y, \mathcal{E})$, we review the uniformity that generates the topology of uniform convergence on a family of subsets of X. For this review, we mostly follow [20, Chapter 7].

Definition 1.22. For the set Y^X of functions from a set X to a uniform space $(Y, \mathcal{E})$, we define, for $A \in \wp^+(X)$ and $E \in \mathcal{E}$,

$$\mathbf{U}(A, E) = \{\langle f, g\rangle \in (Y^X)^2 : (\text{for all } x \in A)\ \langle f(x), g(x)\rangle \in E\}.$$

For the set of functions $X \to \mathbb{K}(Y)$, we let $\mathbf{W}(A, E) = \mathbf{U}(A, hE)$.

If $\mathcal{B}$ is a base for a uniformity on Y and $\mathcal{A}$ is an ideal of subsets of X, then $\{\mathbf{U}(A, B) : A \in \mathcal{A}, B \in \mathcal{B}\}$ forms a base for a uniformity on Y^X. The corresponding topology generated by this base for a uniformity is the topology of uniform convergence on $\mathcal{A}$. Consequently, $\{\mathbf{W}(A, B) : A \in \mathcal{A}, B \in \mathcal{B}\}$ is a base for a uniformity on $\mathbb{K}(Y)^X$.

1.3. **Usco mappings.** In this section, we introduce the basic facts of usco mappings needed for this paper. Of primary use is Theorem 1.29, which offers a convenient characterization of minimal usco maps.

A *set-valued* function from X to Y is a function $\Phi : X \to \wp(Y)$. These are sometimes also referred to as *multi-functions*.

Definition 1.23. A set-valued function $\Phi : X \to \wp(Y)$ is said to be *upper semicontinuous* if, for every open $V \subseteq Y$,

$$\Phi^{\leftarrow}(V) := \{x \in X : \Phi(x) \subseteq V\}$$

is open in X. A *usco* map from a space X to Y is a set-valued map Φ from X to Y that is upper semicontinuous and whose range contained in $\mathbb{K}(Y)$. A usco map $\Phi : X \to \mathbb{K}(Y)$ is said to be *minimal* if its graph minimal with respect to the $\subseteq$ relation. Let $MU(X,Y)$ denote the collection of all minimal usco maps $X \to \mathbb{K}(Y)$.

It is clear that any map $\Phi : X \to K(Y)$ is usco if and only if Φ is continuous relative to the *upper Vietoris topology* on $K(Y)$, which is the topology generated by the sets $\{K \in K(Y) : K \subseteq U\}$ for open $U \subseteq Y$. However, inspired by Theorem 1.21, we maintain that the full Vietoris topology is the desirable topology. At minimum, for most spaces of interest, the full Vietoris topology is Hausdorff. Moreover, the full Vietoris topology on the set of compact subsets has other desirable properties that are intimately related with X, like being metrizable when, and only when, X is metrizable.

As above, it is also clear that any continuous $\Phi : X \to \mathbb{K}(Y)$ is usco and that there are continuous $\Phi : X \to \mathbb{K}(Y)$ that are not minimal. As Example 1.31 will demonstrate, there are minimal usco maps that are not continuous.

Definition 1.24. Suppose $\Phi : X \to \wp^+(Y)$. We say that a function $f : X \to Y$ is a *selection* of Φ if $f(x) \in \Phi(x)$ for every $x \in X$. We let $\mathrm{sel}(\Phi)$ be the set of all selections of Φ.

If $D \subseteq X$ is dense and $f : D \to Y$ is with $f(x) \in \Phi(x)$ for each $x \in D$, then we say that f is a *densely defined selection* of Φ.

Recall that a point x is said to be an *accumulation point* of a net $\langle x_\lambda : \lambda \in \Lambda\rangle$ if, for every open neighborhood U of x and every $\lambda \in \Lambda$, there exists $\mu \geq \lambda$ such that $x_\mu \in U$.

The notion of subcontinuity was introduced by Fuller [12], which can be extended to so-called densely defined functions in the following way. See also [23].

Definition 1.25. Suppose that $D \subseteq X$ is dense. We say that a function $f : D \to Y$ is *subcontinuous* if, for every $x \in X$ and every net $\langle x_\lambda : \lambda \in \Lambda\rangle$ in D with $x_\lambda \to x$, $\langle f(x_\lambda) : \lambda \in \Lambda\rangle$ has an accumulation point.

The notion of semi-open sets was introduced by Levine [24].

Definition 1.26. For a space X, a set $A \subseteq X$ is said to be *semi-open* if $A \subseteq \mathrm{cl}\,\mathrm{int}(A)$.

The notion of quasicontinuity was introduced by Kempisty [21] and surveyed by Neubrunn [28].

Definition 1.27. A function $f : X \to Y$ is said to be *quasicontinuous* if, for each open $V \subseteq Y$, $f^{-1}(V)$ is semi-open in X.

If $D \subseteq X$ is dense and $f : D \to Y$, then we will say that f is quasicontinuous if it is quasicontinuous on D with the subspace topology.

Definition 1.28. For $f \in \mathrm{Fn}(X,Y)$, define $\overline{f} : X \to \wp(Y)$ by the rule

$$\overline{f}(x) = \{y \in Y : \langle x, y\rangle \in \mathrm{cl}\ \mathrm{gr}(f)\}.$$

Theorem 1.29 (see [16,17]). *Suppose that Y is regular and that $\Phi : X \to \wp^+(Y)$. Then the following conditions are equivalent:*

(i) *Φ is minimal usco.*
(ii) *Every selection f of Φ is subcontinuous, quasicontinuous, and $\Phi = \overline{f}$.*
(iii) *There exists a selection f of Φ that is subcontinuous, quasicontinuous, and $\Phi = \overline{f}$.*
(iv) *There exists a densely defined selection f of Φ that is subcontinuous, quasicontinuous, and $\Phi = \overline{f}$.*

A consequence of Theorem 1.29 we will use is that, for $\Phi \in MU(X,\mathbb{R})$, the function $f : X \to \mathbb{R}$ defined by $f(x) = \max \Phi(x)$ is a selection of Φ, and hence subcontinuous and quasicontinuous.

We will be using the following to construct certain functions.

Lemma 1.30. *Let $f, g : X \to Y$ and $U \in \mathscr{T}_X$ and define $h : X \to Y$ by the rule*

$$h(x) = \begin{cases} f(x), & x \in \mathrm{cl}(U); \\ g(x), & x \notin \mathrm{cl}(U). \end{cases}$$

(i) *If f and g are subcontinuous, then h is subcontinuous.*
(ii) *If f is constant and g is quasicontinuous, then h is quasicontinuous.*

Consequently, if f is constant and g is both subcontinuous and quasicontinuous, then h is both subcontinuous and quasicontinuous, which implies that $\overline{h}$ is minimal usco.

Proof. (i) Suppose that $\langle x_\lambda : \lambda \in \Lambda\rangle$ is such that $x_\lambda \to x$. If there is a cofinal subnet of $\langle x_\lambda : \lambda \in \Lambda\rangle$ that is contained in $X \setminus \mathrm{cl}(U)$, then we can appeal to the subcontinuity of g to see that $\langle h(x_\lambda) : \lambda \in \Lambda\rangle$ has an accumulation point. Otherwise, we can let $\lambda_0 \in \Lambda$ be such that, for all $\lambda \geq \lambda_0$, $x_\lambda \in \mathrm{cl}(U)$. Then $\langle x_\lambda : \lambda \geq \lambda_0\rangle$ is a net contained in $\mathrm{cl}(U)$, so we can use the subcontinuity of f to establish that $\langle h(x_\lambda) : \lambda \in \Lambda\rangle$ has an accumulation point.

(ii) Suppose that $V \subseteq Y$ is open such that $h^{-1}(V) \neq \emptyset$. Let $x \in h^{-1}(V)$, and suppose that W is open with $x \in W$. We proceed by cases.

If $x \in \mathrm{cl}(U)$, then $W \cap U \neq \emptyset$. Note also that $U \subseteq h^{-1}(V)$ in this case since f is constant. So $W \cap \mathrm{int}(h^{-1}(V)) \neq \emptyset$.

Now, suppose $x \notin \mathrm{cl}(U)$, and observe that $x \in W \setminus \mathrm{cl}(U)$. Since $h(x) \in V$ and $x \notin \mathrm{cl}(U)$, $h(x) = g(x) \in V$. By the quasicontinuity of g, it must be the case that

$$(W \setminus \mathrm{cl}(U)) \cap \mathrm{int}(g^{-1}(V)) \neq \emptyset.$$

Observe that $\mathrm{int}(g^{-1}(V)) \setminus \mathrm{cl}(U) \subseteq h^{-1}(V)$, so $W \cap \mathrm{int}(h^{-1}(V)) \neq \emptyset$.

It follows that, in either case, $h^{-1}(V)$ is semi-open, establishing the quasicontinuity of h.

For the remainder of the proof, combine (i) and (ii) with Theorem 1.29. □

Note that, if $F \subseteq X$ is closed and nowhere dense, then $\mathbf{1}_F$ is not quasicontinuous since F is not semi-open. Thus, the requirement that we use the closure of an open set in Lemma 1.30(ii) is, in general, necessary. As a similar example, $\mathbf{1}_{(0,1)\cup(1,2)}$ is not quasicontinuous at 1.

Moreover, consider $U \in \mathscr{T}_X$ with $\partial U \neq \emptyset$ and $\mathrm{cl}(U) \neq X$. Then $f : X \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \mathbf{1}_{X\setminus U}(x), & x \in \mathrm{cl}(U); \\ \mathbf{1}_U(x), & x \notin \mathrm{cl}(U). \end{cases}$$

is $\mathbf{1}_{\partial U}$. So the requirement that f be constant (or something stronger than quasicontinuity, at least) in Lemma 1.30(ii) is, in general, necessary.

Example 1.31. Consider $MU(\mathbb{R},\mathbb{R})$. By Lemma 1.30, $\Phi := \overline{\mathbf{1}}_{[0,1]}$ is minimal usco. However, Φ is not continuous since

$$\{0, 1\} = \{x \in \mathbb{R} : \Phi(x) \in [(-0.5, 0.5), (0.5, 1.5)]\}.$$

Hence, when Y is metrizable, studying the space $MU(X, Y)$ is, in general, different than studying the space of continuous functions into a metrizable space.

We will also be using the following corollary often.

Corollary 1.32. *Suppose that $\Phi, \Psi \in MU(X, Y)$ and $U \in \mathscr{T}_X$ are such that there exist $f \in \mathrm{sel}(\Phi)$ and $g \in \mathrm{sel}(\Psi)$ with the property that $f \restriction_U = g \restriction_U$. Then $\Phi \restriction_U = \Psi \restriction_U$.*

Proof. We will show that $\Phi(x) \subseteq \Psi(x)$ for each $x \in U$. By symmetry, this will establish that $\Phi(x) = \Psi(x)$ for each $x \in U$.

So let $x \in U$ and $y \in \Phi(x)$, which by Theorem 1.29 means that $y \in \overline{f}(x)$. Now, consider any neighborhood $V \times W$ of $\langle x, y\rangle$. Without loss of generality, we may assume that $V \subseteq U$. Then we can find $\langle z, f(z)\rangle \in V \times W$. Since $z \in U$, $f(z) = g(z)$, and, as $V \times W$ was arbitrary, we see that $\langle x, y\rangle \in \mathrm{cl}\ \mathrm{gr}(g)$. Hence, $y \in \overline{g}(x) = \Psi(x)$. □

Corollary 1.33. *If $A \subseteq X$ is nonempty, $U, V \in \mathscr{T}_X$ are such that $A \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$, $\Phi \in MU(X, Y)$, and $f \in \mathrm{sel}(\Phi)$, then, for $y_0 \in Y$, $g : X \to Y$ defined by*

$$g(x) = \begin{cases} y_0, & x \in \mathrm{cl}(X \setminus \mathrm{cl}(V)); \\ f(x), & \text{otherwise}, \end{cases}$$

has the property that $\Psi := \overline{g} \in MU(X, Y)$, $\langle \Phi, \Psi\rangle \in \mathbf{W}(A, E)$ for any entourage E of Y, and $g[X \setminus U] = \{y_0\}$.

Proof. Note that Theorem 1.29 and Lemma 1.30 imply that g is subcontinuous and quasicontinuous. Then $\Psi := \overline{g} \in MU(X, Y)$.

Since V is open, $\mathrm{cl}(X \setminus \mathrm{cl}(V)) \subseteq X \setminus V$. Then $g(x) = f(x)$ for all $x \in V$. By Corollary 1.32, we see that $\Phi \restriction_A = \Psi \restriction_A$ as $A \subseteq V$. Also, since $X \setminus U \subseteq X \setminus \mathrm{cl}(V) \subseteq \mathrm{cl}(X \setminus \mathrm{cl}(V))$, we see that $g[X \setminus U] = \{y_0\}$. □

As a final note in this section, we offer the following generalization of [18, Lemma 3.1] to general uniform spaces. Recall that Lemma 1.19 allows us to restrict our attention to closed entourages.

Corollary 1.34. *Let X be a space and let Y be a uniform space. If $\Phi, \Psi \in MU(X,Y)$, a closed entourage E of Y, and a dense $D \subseteq X$ are such that $\langle \Phi(x), \Psi(x) \rangle \in hE$ for all $x \in D$, then $\langle \Phi(x), \Psi(x) \rangle \in hE$ for all $x \in X$.*

Proof. Define $F : X \to \wp(Y)$ by $F(x) = E[\Phi(x)]$.

We first show that the graph of F is closed. Suppose $\langle x, y \rangle \in \mathrm{cl}\ \mathrm{gr}(F)$, and let $\langle \langle x_\lambda, y_\lambda \rangle : \lambda \in \Lambda \rangle$ be a net in $\mathrm{gr}(F)$ such that $\langle x_\lambda, y_\lambda \rangle \to \langle x, y \rangle$. Since $y_\lambda \in E[\Phi(x_\lambda)]$, we can let $w_\lambda \in \Phi(x_\lambda)$ be such that $y_\lambda \in E[w_\lambda]$. Observe that, since $x_\lambda \to x$ and $w_\lambda \in \Phi(x_\lambda)$ for each $\lambda \in \Lambda$, by Theorem 1.29, $\langle w_\lambda : \lambda \in \Lambda \rangle$ has an accumulation point $w \in \Phi(x)$. Since $y_\lambda \to y$ and w is an accumulation point of $\langle w_\lambda : \lambda \in \Lambda \rangle$, $\langle w, y \rangle$ is an accumulation point of $\langle \langle w_\lambda, y_\lambda \rangle : \lambda \in \Lambda \rangle$. Moreover, as $\langle w_\lambda, y_\lambda \rangle \in E$ for all $\lambda \in \Lambda$ and E is closed, we see that $\langle w, y \rangle \in E$. Hence, $y \in E[w] \subseteq E[\Phi(x)] = F(x)$. That is, $\langle x, y \rangle \in \mathrm{gr}(F)$, which establishes that $\mathrm{gr}(F)$ is closed.

Now, by Theorem 1.29, we can let $g : D \to Y$ be subcontinuous and quasicontinuous such that $g(x) \in \Psi(x)$ for each $x \in D$ and $\Psi = \overline{g}$. Since $\mathrm{gr}(F)$ is closed and $\mathrm{gr}(g) \subseteq \mathrm{gr}(F)$, we see that $\mathrm{cl}\ \mathrm{gr}(g) \subseteq \mathrm{gr}(F)$. That is, $\Psi(x) \subseteq F(x) = E[\Phi(x)]$ for all $x \in X$.

A symmetric argument shows that $\Phi(x) \subseteq E[\Psi(x)]$ for all $x \in X$, finishing the proof. □

2. Results

We first note that the clustering version of Gruenhage's W-game is equivalent to an entourage selection game in the realm of topological groups. Such a result holds, for example, for $C_{\mathcal{A}}(X)$ where we define $C_{\mathcal{A}}(X)$ to be the space of continuous real-valued functions on X endowed with the topology of uniform convergence on $\mathcal{A}$, an ideal of closed subsets of X. This topology is generated by the uniformity outlined in Definition 1.22.

Recall that a topological group is a (multiplicative) group G with a topology for which the operations $\langle g, h \rangle \mapsto gh$, $G^2 \to G$, and $g \mapsto g^{-1}$, $G \to G$, are continuous. Let $\mathbf{i}$ be the identity element of G. Also, recall that

$$\left\{ \{ \langle g, h \rangle \in G^2 : gh^{-1} \in U \} : U \in \mathscr{N}_{G,\mathbf{i}} \right\}$$

is a basis for a uniformity on G. Let $\mathscr{E}_G$ be the set of all entourages of G with the generated uniformity. Also, let

$$\mathbf{\Omega}_\Delta = \left\{ A \subseteq G^2 : \{ gh^{-1} : \langle g, h \rangle \in A \} \in \Omega_{G,\mathbf{i}} \right\}.$$

Theorem 2.1. *If G is a (multiplicative) topological group and $\mathbf{i}$ is the identity, then, for any $g \in G$,*

$$\mathsf{G}_1(\mathscr{N}_{G,g}, \neg\Omega_{G,g}) \equiv \mathsf{G}_1(\mathscr{N}_{G,\mathbf{i}}, \neg\Omega_{G,\mathbf{i}}) \equiv \mathsf{G}_1(\mathscr{E}_G, \neg\mathbf{\Omega}_\Delta).$$

Proof. The equivalence

$$\mathsf{G}_1(\mathscr{N}_{G,g}, \neg\Omega_{G,g}) \equiv \mathsf{G}_1(\mathscr{N}_{G,\mathbf{i}}, \neg\Omega_{G,\mathbf{i}})$$

can be seen by using the homeomorphism $x \mapsto gx$, $G \to G$.

We first show that

$$\mathsf{G}_1(\mathscr{N}_{G,\mathbf{i}}, \neg\Omega_{G,\mathbf{i}}) \leq_{\mathrm{II}} \mathsf{G}_1(\mathscr{E}_G, \neg\mathbf{\Omega}_\Delta).$$

Define $\overleftarrow{T}_{\mathrm{I},n} : \mathscr{E}_G \to \mathscr{N}_{G,\mathbf{i}}$ such that

$$\left\{\langle g,h\rangle \in G^2 : gh^{-1} \in \overleftarrow{T}_{\mathrm{I},n}(E)\right\} \subseteq E.$$

We now define $\overrightarrow{T}_{\mathrm{II},n} : G \times \mathscr{E}_G \to G^2$ in the following way. For $g \in \overleftarrow{T}_{\mathrm{I},n}(E)$, let $\overrightarrow{T}_{\mathrm{II},n}(g,E) = \langle g, \mathbf{i}\rangle$; otherwise, let $\overrightarrow{T}_{\mathrm{II},n}(g,E) = \langle \mathbf{i}, \mathbf{i}\rangle$. By the above definition, we see that $\overrightarrow{T}_{\mathrm{II},n}(g,E) \in E$ when $g \in \overleftarrow{T}_{\mathrm{I},n}(E)$.

Suppose that $\langle g_n : n \in \omega\rangle \in \prod_{n\in\omega} \overleftarrow{T}_{\mathrm{I},n}(E_n)$ for a sequence $\langle E_n : n \in \omega\rangle$ of $\mathscr{E}_G$ is such that $\{g_n : n \in \omega\} \notin \Omega_{G,\mathbf{i}}$. Then let $U \in \mathscr{N}_{G,\mathbf{i}}$ be such that $U \cap \{g_n : n \in \omega\} = \emptyset$. Note that $\overrightarrow{T}_{\mathrm{II},n}(g_n, E_n) \notin \mathbf{\Omega}_\Delta$. Thus, Theorem 1.14 applies.

We now show that

$$\mathsf{G}_1(\mathscr{E}_G, \neg\mathbf{\Omega}_\Delta) \leq_{\mathrm{II}} \mathsf{G}_1(\mathscr{N}_{G,\mathbf{i}}, \neg\Omega_{G,\mathbf{i}}).$$

Define $\overleftarrow{T}_{\mathrm{II},n} : \mathscr{N}_{G,\mathbf{i}} \to \mathscr{E}_G$ by the rule

$$\overleftarrow{T}_{\mathrm{II},n}(U) = \{\langle g,h\rangle \in G^2 : gh^{-1} \in U\}$$

and $\overrightarrow{T}_{\mathrm{II},n} : G^2 \times \mathscr{N}_{G,\mathbf{i}} \to G$ by $\overrightarrow{T}_{\mathrm{II},n}(\langle g,h\rangle, U) = gh^{-1}$. Note that, if $\langle g,h\rangle \in \overleftarrow{T}_{\mathrm{II},n}(U)$, then $\overrightarrow{T}_{\mathrm{II},n}(\langle g,h\rangle, U) \in U$. So suppose that

$$\langle\langle g_n, h_n\rangle : n \in \omega\rangle \in \prod_{n\in\omega} \overleftarrow{T}_{\mathrm{II},n}(U_n)$$

for a sequence $\langle U_n : n \in \omega\rangle$ of $\mathscr{N}_{G,\mathbf{i}}$ such that $\{\langle g_n, h_n\rangle : n \in \omega\} \notin \mathbf{\Omega}_\Delta$. Evidently, $\overrightarrow{T}_{\mathrm{II},n}(\langle g_n, h_n\rangle, U_n) \notin \Omega_{G,\mathbf{i}}$. Again, Theorem 1.14 applies. □

For the remainder of the paper, we will be interested only in real set-valued functions; so we will let $MU(X) = MU(X, \mathbb{R})$. We also use, for $\varepsilon > 0$,

$$\Delta_\varepsilon = \{\langle x,y\rangle \in \mathbb{R}^2 : |x-y| < \varepsilon\}.$$

For $A \subseteq X$, we will use $\mathbf{U}(A,\varepsilon) = \mathbf{U}(A, \Delta_\varepsilon)$ and $\mathbf{W}(A,\varepsilon) = \mathbf{W}(A,\Delta_\varepsilon)$. For $Y \subseteq \mathbb{R}$, let $\mathbb{B}(Y,\varepsilon) = \bigcup_{y\in Y} B(y,\varepsilon)$ and note that

$$\mathbf{W}(A,\varepsilon) = \left\{\langle \Phi,\Psi\rangle \in \mathbb{K}(\mathbb{R})^X : (\text{for all } x \in A)[\Phi(x) \subseteq \mathbb{B}(\Psi(x),\varepsilon) \wedge \Psi(x) \subseteq \mathbb{B}(\Phi(x),\varepsilon)]\right\}.$$

Then, if $\mathcal{A}$ is an ideal of closed subsets of X, then we will use $MU_{\mathcal{A}}(X)$ to denote the set $MU(X)$ with the topology generated by the base for a uniformity $\{\mathbf{W}(A,\varepsilon) : A \in \mathcal{A}, \varepsilon > 0\}$. When $\mathcal{A} = [X]^{<\omega}$, we use $MU_p(X)$, and when $\mathcal{A} = K(X)$, we use $MU_k(X)$. For $\Phi \in MU(X)$, $A \subseteq X$, and $\varepsilon > 0$, we let $[\Phi; A, \varepsilon] = \mathbf{W}(A,\varepsilon)[\Phi]$. We will use $\mathbf{0}$ to denote the function that is constantly 0 when dealing with real-valued functions and the function that is constantly $\{0\}$ when dealing with usco maps.

Theorem 2.2. *Let X be regular and let $\mathcal{A}$ and $\mathcal{B}$ be ideals of closed subsets of X. Then,*

(i) $\mathsf{G}_1(\mathcal{O}_X(\mathcal{A}), \Lambda_X(\mathcal{B})) \leq_{\mathrm{II}} \mathsf{G}_1(\Omega_{MU_{\mathcal{A}}(X),\mathbf{0}}, \Omega_{MU_{\mathcal{B}}(X),\mathbf{0}})$,

(ii) $\mathsf{G}_1(\Omega_{MU_{\mathcal{A}}(X),\mathbf{0}}, \Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}) \leq_{\mathrm{II}} \mathsf{G}_1(\mathscr{D}_{MU_{\mathcal{A}}(X)}, \Omega_{MU_{\mathcal{B}}(X),\mathbf{0}})$, *and*

(iii) *if X is $\mathcal{A}$-normal,* $\mathsf{G}_1(\mathscr{D}_{MU_{\mathcal{A}}(X)}, \Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}) \leq_{\mathrm{II}} \mathsf{G}_1(\mathcal{O}_X(\mathcal{A}), \Lambda_X(\mathcal{B}))$.
Thus, if X is $\mathcal{A}$-normal, then the three games are equivalent.

Proof. We first address (i). Fix some $\mathscr{U}_0 \in \mathcal{O}_X(\mathcal{A})$ and let $W_{\Phi,n} = \Phi^{\leftarrow}[(-2^{-n}, 2^{-n})]$ for $\Phi \in MU(X)$ and $n \in \omega$. Define $\overleftarrow{T}_{\mathrm{I},n} : \Omega_{MU_{\mathcal{A}}(X),\mathbf{0}} \to \mathcal{O}_X(\mathcal{A})$ by the rule

$$\overleftarrow{T}_{\mathrm{I},n}(\mathscr{F}) = \begin{cases} \{W_{\Phi,n} : \Phi \in \mathscr{F}\}, & (\text{for all } \Phi \in \mathscr{F})\ W_{\Phi,n} \neq X; \\ \mathscr{U}_0, & \text{otherwise.} \end{cases}$$

To see that $\overleftarrow{T}_{\mathrm{I},n}$ is defined, let $\mathscr{F} \in \Omega_{MU_{\mathcal{A}}(X),\mathbf{0}}$ be such that $W_{\Phi,n} \neq X$ for every $\Phi \in \mathscr{F}$. Let $A \in \mathcal{A}$ be arbitrary, and choose $\Phi \in [\mathbf{0}; A, 2^{-n}] \cap \mathscr{F}$. It follows that $A \subseteq W_{\Phi,n}$. Hence, $\overleftarrow{T}_{\mathrm{I},n}(\mathscr{F}) \in \mathcal{O}_X(\mathcal{A})$.

We now define

$$\overrightarrow{T}_{\mathrm{II},n} : \mathscr{T}_X \times \Omega_{MU_{\mathcal{A}}(X),\mathbf{0}} \to MU(X)$$

in the following way. Let

$$\mathfrak{T}_n = \{\mathscr{F} \in \Omega_{MU_{\mathcal{A}}(X),\mathbf{0}} : (\text{there exists } \Phi \in \mathscr{F})\ W_{\Phi,n} = X\}$$

and $\mathfrak{T}_n^\star = \Omega_{MU_{\mathcal{A}}(X),\mathbf{0}} \setminus \mathfrak{T}_n$. For each $\langle U, \mathscr{F}\rangle \in \mathscr{T}_X \times \mathfrak{T}_n$, let $\overrightarrow{T}_{\mathrm{II},n}(U, \mathscr{F}) \in \mathscr{F}$ be such that $W_{\overrightarrow{T}_{\mathrm{II},n}(U,\mathscr{F}),n} = X$. For $\langle U, \mathscr{F}\rangle \in \mathscr{T}_X \times \mathfrak{T}_n^\star$ with $U \in \overleftarrow{T}_{\mathrm{I},n}(\mathscr{F})$, let $\overrightarrow{T}_{\mathrm{II},n}(U, \mathscr{F}) \in \mathscr{F}$ be such that $U = W_{\overrightarrow{T}_{\mathrm{II},n}(U,\mathscr{F}),n}$. For $\langle U, \mathscr{F}\rangle \in \mathscr{T}_X \times \mathfrak{T}_n^\star$ with $U \notin \overleftarrow{T}_{\mathrm{I},n}(\mathscr{F})$, let $\overrightarrow{T}_{\mathrm{II},n}(U, \mathscr{F}) = \mathbf{0}$. By the construction, if $U \in \overleftarrow{T}_{\mathrm{I},n}(\mathscr{F})$, then $\overrightarrow{T}_{\mathrm{II},n}(U, \mathscr{F}) \in \mathscr{F}$.

To finish this application of Theorem 1.14, assume that we have

$$\langle U_n : n \in \omega\rangle \in \prod_{n\in\omega} \overleftarrow{T}_{\mathrm{I},n}(\mathscr{F}_n)$$

for some sequence $\langle \mathscr{F}_n : n \in \omega\rangle$ of $\Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}$ such that $\{U_n : n \in \omega\} \in \Lambda_X(\mathcal{B})$. For each $n \in \omega$, let $\Phi_n = \overrightarrow{T}_{\mathrm{II},n}(U_n, \mathscr{F}_n)$. Now, let $B \in \mathcal{B}$ and $\varepsilon > 0$ be arbitrary. Choose $n \in \omega$ such that $2^{-n} < \varepsilon$ and $B \subseteq U_n$. If $\mathscr{F}_n \in \mathfrak{T}_n$, then Φ_n has the property that $X = \Phi_n^{\leftarrow}[(-2^{-n}, 2^{-n})]$; hence, $\Phi_n \in [\mathbf{0}; B, \varepsilon]$. Otherwise, $B \subseteq U_n = \Phi_n^{\leftarrow}[(-2^{-n}, 2^{-n})]$, which also implies that $\Phi_n \in [\mathbf{0}; B, \varepsilon]$. Thus, $\{\Phi_n : n \in \omega\} \in \Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}$.

(ii) holds since $\mathscr{D}_{MU_{\mathcal{A}}(X)} \subseteq \Omega_{MU_{\mathcal{A}}(X),\mathbf{0}}$.

Lastly, we address (iii). We define $\overleftarrow{T}_{\mathrm{I},n} : \mathcal{O}_X(\mathcal{A}) \to \mathscr{D}_{MU_{\mathcal{A}}(X)}$ by the rule

$$\overleftarrow{T}_{\mathrm{I},n}(\mathscr{U}) = \{\Phi \in MU(X) : (\text{there exist } U \in \mathscr{U} \text{ and } f \in \mathrm{sel}(\Phi))\ f[X \setminus U] = \{1\}\}.$$

To see that $\overleftarrow{T}_{\mathrm{I},n}$ is defined, let $\mathscr{U} \in \mathcal{O}_X(\mathcal{A})$ and consider a basic open set $[\Phi; A, \varepsilon]$. Then let $U \in \mathscr{U}$ be such that $A \subseteq U$ and, by $\mathcal{A}$-normality, let V be open such that $A \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$. Define $f : X \to \mathbb{R}$ by the rule

$$f(x) = \begin{cases} 1, & x \in \mathrm{cl}(X \setminus \mathrm{cl}(V)); \\ \max \Phi(x), & \text{otherwise.} \end{cases}$$

By Corollary 1.33, $\overline{f} \in [\Phi; A, \varepsilon] \cap \overleftarrow{T}_{\mathrm{I},n}(\mathscr{U})$. Hence, $\overleftarrow{T}_{\mathrm{I},n}(\mathscr{U}) \in \mathscr{D}_{MU_{\mathcal{A}}(X)}$.

We define $\overrightarrow{T}_{\mathrm{II},n} : MU(X) \times \mathcal{O}_X(\mathcal{A}) \to \mathscr{T}_X$ in the following way. Fix some $U_0 \in \mathscr{T}_X$. For $\langle \Phi, \mathscr{U} \rangle \in MU(X) \times \mathcal{O}_X(\mathcal{A})$, if

$$\{U \in \mathscr{U} : (\text{there exists } f \in \operatorname{sel}(\Phi))\ f[X \setminus U] = \{1\}\} \neq \emptyset,$$

let $\overrightarrow{T}_{\mathrm{II},n}(\Phi, \mathscr{U}) \in \mathscr{U}$ be such that there exists $f \in \operatorname{sel}(\Phi)$ with the property that $f[X \setminus \overrightarrow{T}_{\mathrm{II},n}(\Phi, \mathscr{U})] = \{1\}$; otherwise, let $\overrightarrow{T}_{\mathrm{II},n}(\Phi, \mathscr{U}) = U_0$. By the construction, if $\Phi \in \overleftarrow{T}_{\mathrm{I},n}(\mathscr{U})$, then $\overrightarrow{T}_{\mathrm{II},n}(\Phi, \mathscr{U}) \in \mathscr{U}$.

Suppose we have

$$\langle \Phi_n : n \in \omega \rangle \in \prod_{n \in \omega} \overleftarrow{T}_{\mathrm{I},n}(\mathscr{U}_n)$$

for a sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ of $\mathcal{O}_X(\mathcal{A})$ with the property that $\{\Phi_n : n \in \omega\} \in \Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}$. For each $n \in \omega$, let $U_n = \overrightarrow{T}_{\mathrm{II},n}(\Phi_n, \mathscr{U}_n)$. Since $\mathcal{B}$ is an ideal of sets, we need only to show that $\langle U_n : n \in \omega \rangle$ is a $\mathcal{B}$-cover. So let $B \in \mathcal{B}$ be arbitrary and let $n \in \omega$ be such that $\Phi_n \in [\mathbf{0}; B, 1]$. Then we can let $f \in \operatorname{sel}(\Phi_n)$ be such that $f[X \setminus U_n] = \{1\}$. Since $\Phi_n \in [\mathbf{0}; B, 1]$, we see that, for each $x \in B$, $f(x) \in \Phi_n(x) \subseteq (-1, 1)$. Hence, $B \cap (X \setminus U_n) = \emptyset$, which is to say that $B \subseteq U_n$. So Theorem 1.14 applies. □

We now establish some relationships between these games and games on the space of continuous real-valued functions.

Corollary 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be ideals of closed subsets of X, and suppose that X is $\mathcal{A}$-normal. Then*

$$\begin{aligned}\mathcal{G} := \mathsf{G}_1(\mathcal{O}_X(\mathcal{A}), \mathcal{O}_X(\mathcal{B})) &\equiv \mathsf{G}_1(\Omega_{MU_{\mathcal{A}}(X),\mathbf{0}}, \Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}) \\ &\equiv \mathsf{G}_1(\mathscr{D}_{MU_{\mathcal{A}}(X)}, \Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}),\end{aligned}$$

$$\begin{aligned}\mathcal{H} := \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathcal{O}_X(\mathcal{B})) &\equiv \mathsf{G}_1(\mathscr{N}_{MU_{\mathcal{A}}(X),\mathbf{0}}, \neg\Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}) \\ &\equiv \mathsf{G}_1(\mathscr{T}_{MU_{\mathcal{A}}(X)}, \neg\Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}),\end{aligned}$$

and $\mathcal{G}$ is dual to $\mathcal{H}$. If X is functionally $\mathcal{A}$-normal, then

$$\begin{aligned}\mathsf{G}_1(\mathcal{O}_X(\mathcal{A}), \mathcal{O}_X(\mathcal{B})) &\equiv \mathsf{G}_1(\Omega_{MU_{\mathcal{A}}(X),\mathbf{0}}, \Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}) \\ &\equiv \mathsf{G}_1(\Omega_{C_{\mathcal{A}}(X),\mathbf{0}}, \Omega_{C_{\mathcal{B}}(X),\mathbf{0}}) \\ &\equiv \mathsf{G}_1(\mathscr{D}_{MU_{\mathcal{A}}(X)}, \Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}) \\ &\equiv \mathsf{G}_1(\mathscr{D}_{C_{\mathcal{A}}(X)}, \Omega_{C_{\mathcal{B}}(X),\mathbf{0}})\end{aligned}$$

and

$$\begin{aligned}\mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathcal{O}_X(\mathcal{B})) &\equiv \mathsf{G}_1(\mathscr{N}_{MU_{\mathcal{A}}(X),\mathbf{0}}, \neg\Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}) \\ &\equiv \mathsf{G}_1(\mathscr{N}_{C_{\mathcal{A}}(X),\mathbf{0}}, \neg\Omega_{C_{\mathcal{B}}(X),\mathbf{0}}) \\ &\equiv \mathsf{G}_1(\mathscr{T}_{MU_{\mathcal{A}}(X)}, \neg\Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}) \\ &\equiv \mathsf{G}_1(\mathscr{T}_{C_{\mathcal{A}}(X)}, \neg\Omega_{C_{\mathcal{B}}(X),\mathbf{0}}).\end{aligned}$$

Proof. Apply Theorem 2.2, Lemmas 1.8, 1.12, and 1.13, and [3, Corollary 14]. □

In [4, Theorem 31], inspired by Li [25], the game $\mathsf{G}_1(\mathcal{O}_X(\mathcal{A}), \mathcal{O}_X(\mathcal{B}))$ is shown to be equivalent to the selective separability game on certain hyperspaces of X, which we only note in passing here for the interested reader.

Recall that a subset A of a topological space is *sequentially compact* if every sequence in A has a subsequence that converges to a point of A.

Lemma 2.4. *Suppose that $\Phi \in MU(X)$ and that $A \subseteq X$ is sequentially compact. Then $\Phi[A]$ is bounded.*

Proof. Suppose that $\Phi : X \to \mathbb{K}(\mathbb{R})$ is unbounded on $A \subseteq X$, which is sequentially compact. For each $n \in \omega$, let $x_n \in A$ be such that there is some $y \in \Phi(x_n)$ with $|y| \geq n$. Then let $y_n \in \Phi(x_n)$ be such that $|y_n| \geq n$, and define $f : X \to \mathbb{R}$ to be a selection of Φ such that $f(x_n) = y_n$ for $n \in \omega$. Since A is sequentially compact, we can find $x \in A$ and a subsequence $\langle x_{n_k} : k \in \omega \rangle$ such that $x_{n_k} \to x$. Note that $\langle f(x_{n_k}) : k \in \omega \rangle$ does not have an accumulation point. Therefore f is not subcontinuous, and by Theorem 1.29, Φ is not a minimal usco map. ☐

Theorem 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be ideals of closed subsets of X. If X is $\mathcal{A}$-normal and $\mathcal{B}$ consists of sequentially compact sets, then*

$$\mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\Lambda_X(\mathcal{B})) \leq_{\mathrm{II}} \mathsf{G}_1(\mathscr{T}_{MU_{\mathcal{A}}(X)}, \mathrm{CD}_{MU_{\mathcal{B}}(X)}).$$

Consequently,

$$\begin{aligned}\mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathcal{O}_X(\mathcal{B})) &\equiv \mathsf{G}_1(\mathscr{N}_{MU_{\mathcal{A}}(X),\mathbf{0}}, \neg\Omega_{MU_{\mathcal{B}}(X),\mathbf{0}})\\ &\equiv \mathsf{G}_1(\mathscr{T}_{MU_{\mathcal{A}}(X)}, \neg\Omega_{MU_{\mathcal{B}}(X),\mathbf{0}})\\ &\equiv \mathsf{G}_1(\mathscr{T}_{MU_{\mathcal{A}}(X)}, \mathrm{CD}_{MU_{\mathcal{B}}(X)}).\end{aligned}$$

Proof. Let $\pi_1 : MU(X) \times \mathcal{A} \times \mathbb{R} \to MU(X)$, $\pi_2 : MU(X) \times \mathcal{A} \times \mathbb{R} \to \mathcal{A}$, and $\pi_3 : MU(X) \times \mathcal{A} \times \mathbb{R} \to \mathbb{R}$ be the standard coordinate projection maps. Define a choice function $\gamma : \mathscr{T}_{MU_{\mathcal{A}}(X)} \to MU(X) \times \mathcal{A} \times \mathbb{R}$ such that

$$[\pi_1(\gamma(W)); \pi_2(\gamma(W)), \pi_3(\gamma(W))] \subseteq W.$$

Let $\Psi_W = \pi_1(\gamma(W))$ and $A_W = \pi_2(\gamma(W))$. Now we define $\overleftarrow{T}_{\mathrm{I},n} : \mathscr{T}_{MU_{\mathcal{A}}(X)} \to \mathscr{N}_X[\mathcal{A}]$ by $\overleftarrow{T}_{\mathrm{I},n}(W) = \mathscr{N}_X(A_W)$.

We now define $\overrightarrow{T}_{\mathrm{II},n} : \mathscr{T}_X \times \mathscr{T}_{MU_{\mathcal{A}}(X)} \to MU(X)$ in the following way. For $A \in \mathcal{A}$ and $U \in \mathscr{N}_X(A)$, let $V_{A,U}$ be open such that

$$A \subseteq V_{A,U} \subseteq \mathrm{cl}(V_{A,U}) \subseteq U.$$

For $W \in \mathscr{T}_{MU_{\mathcal{A}}(X)}$ and $U \in \overleftarrow{T}_{\mathrm{I},n}(W)$, define $f_{W,U,n} : X \to \mathbb{R}$ by the rule

$$f_{W,U,n}(x) = \begin{cases} n, & x \in \mathrm{cl}(X \setminus \mathrm{cl}(V_{A_W,U})); \\ \max \Psi_W(x), & \text{otherwise.} \end{cases}$$

Then we set

$$\overrightarrow{T}_{\mathrm{II},n}(U, W) = \begin{cases} \overline{f}_{W,U,n}, & U \in \overleftarrow{T}_{\mathrm{I},n}(W); \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

By Corollary 1.33, $\overrightarrow{T}_{\mathrm{II},n}(U,W) \in MU(X)$, and if $U \in \overleftarrow{T}_{\mathrm{I},n}(W)$, then

$$\overrightarrow{T}_{\mathrm{II},n}(U,W) \in [\Psi_W; A_W, \pi_3(\gamma(W))] \subseteq W.$$

Suppose we have a sequence

$$\langle U_n : n \in \omega \rangle \in \prod_{n\in\omega} \overleftarrow{T}_{\mathrm{I},n}(W_n)$$

for a sequence $\langle W_n : n \in \omega \rangle$ of $\mathscr{T}_{MU_{\mathcal{A}}(X)}$ such that $\{U_n : n \in \omega\} \notin \Lambda_X(\mathcal{B})$. Let $\Phi_n = \overrightarrow{T}_{\mathrm{II},n}(U_n, W_n)$ for each $n \in \omega$. We can find $N \in \omega$ and $B \in \mathcal{B}$ such that, for every $n \geq N$, $B \not\subseteq U_n$. Now, suppose that $\Phi \in MU(X)\setminus\{\Phi_n : n \in \omega\}$ is arbitrary. By Lemma 2.4, $\Phi[B]$ is bounded, so let $M > \sup|\Phi[B]|$ and $n \geq \max\{N, M+1\}$. Now, for $x \in B \setminus U_n$, note that $n \in \Phi_n(x)$ and that, for $y \in \Phi(x)$,

$$y \leq \sup|\Phi[B]| < M \leq n-1 \implies y - n < -1 \implies |y-n| > 1.$$

In particular, $\Phi_n(x) \not\subseteq \mathbb{B}(\Phi(x), 1)$, which establishes that $\Phi_n \notin [\Phi; B, 1]$. Hence, $\{\Phi_n : n \in \omega\}$ is closed and discrete, and Theorem 1.14 applies.

For what remains, observe that

$$\mathsf{G}_1(\mathscr{T}_{MU_{\mathcal{A}}(X)}, \mathrm{CD}_{MU_{\mathcal{B}}(X)}) \leq_{\mathrm{II}} \mathsf{G}_1(\mathscr{T}_{MU_{\mathcal{A}}(X)}, \neg\Omega_{MU_{\mathcal{B}}(X),\mathbf{0}})$$

since, if Two can produce a closed discrete set, then Two can avoid clustering around $\mathbf{0}$. Hence, by Corollary 2.3 we obtain that

$$\begin{aligned}\mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathcal{O}_X(\mathcal{B})) &= \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\Lambda_X(\mathcal{B}))\\ &\leq_{\mathrm{II}} \mathsf{G}_1(\mathscr{T}_{MU_{\mathcal{A}}(X)}, \mathrm{CD}_{MU_{\mathcal{B}}(X)})\\ &\leq_{\mathrm{II}} \mathsf{G}_1(\mathscr{T}_{MU_{\mathcal{A}}(X)}, \neg\Omega_{MU_{\mathcal{B}}(X),\mathbf{0}})\\ &\equiv \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathcal{O}_X(\mathcal{B})).\end{aligned}$$

□

Corollary 2.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be ideals of closed subsets of X. If X is functionally $\mathcal{A}$-normal and $\mathcal{B}$ consists of sequentially compact sets, then*

$$\begin{aligned}\mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathcal{O}_X(\mathcal{B})) &\equiv \mathsf{G}_1(\mathscr{N}_{MU_{\mathcal{A}}(X),\mathbf{0}}, \neg\Omega_{MU_{\mathcal{B}}(X),\mathbf{0}})\\ &\equiv \mathsf{G}_1(\mathscr{N}_{C_{\mathcal{A}}(X),\mathbf{0}}, \neg\Omega_{C_{\mathcal{B}}(X),\mathbf{0}})\\ &\equiv \mathsf{G}_1(\mathscr{T}_{MU_{\mathcal{A}}(X)}, \neg\Omega_{MU_{\mathcal{B}}(X),\mathbf{0}})\\ &\equiv \mathsf{G}_1(\mathscr{T}_{C_{\mathcal{A}}(X)}, \neg\Omega_{C_{\mathcal{B}}(X),\mathbf{0}})\\ &\equiv \mathsf{G}_1(\mathscr{T}_{MU_{\mathcal{A}}(X)}, \mathrm{CD}_{MU_{\mathcal{B}}(X)})\\ &\equiv \mathsf{G}_1(\mathscr{T}_{C_{\mathcal{A}}(X)}, \mathrm{CD}_{C_{\mathcal{B}}(X)}).\end{aligned}$$

Proof. Apply Theorem 2.5 and [3, Corollary 15]. □

We now offer some relationships related to Gruenhage's W-games.

Proposition 2.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be ideals of closed subsets of X. Then*

(i) $\mathsf{G}_1(\mathscr{N}_{MU_{\mathcal{A}}(X),\mathbf{0}}, \neg\Omega_{MU_{\mathcal{B}}(X),\mathbf{0}}) \leq_{\mathrm{II}} \mathsf{G}_1(\mathscr{N}_{MU_{\mathcal{A}}(X),\mathbf{0}}, \neg\Gamma_{MU_{\mathcal{B}}(X),\mathbf{0}})$ *and*
(ii) $\mathsf{G}_1(\mathscr{N}_{MU_{\mathcal{A}}(X),\mathbf{0}}, \neg\Gamma_{MU_{\mathcal{B}}(X),\mathbf{0}}) \leq_{\mathrm{II}} \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\Gamma_X(\mathcal{B}))$.

Proof. (i) is evident since, if Two can avoid clustering at $\mathbf{0}$, they can surely avoid converging to $\mathbf{0}$.

(ii) Fix $U_0 \in \mathscr{T}_X$, and define $\overleftarrow{T}_{\mathrm{I},n} : \mathscr{N}_X[\mathcal{A}] \to \mathscr{N}_{MU_{\mathcal{A}}(X),\mathbf{0}}$ by $\overleftarrow{T}_{\mathrm{I},n}(\mathscr{N}_X(A)) = [\mathbf{0}; A, 2^{-n}]$. Then define $\overrightarrow{T}_{\mathrm{II},n} : MU(X) \times \mathscr{N}_X[\mathcal{A}] \to \mathscr{T}_X$ by $\overrightarrow{T}_{\mathrm{II},n}(\Phi, \mathscr{N}_X(A)) = \Phi^{\leftarrow}[(-2^{-n}, 2^{-n})]$. Note that, if $\Phi \in [\mathbf{0}; A, 2^{-n}] = \overleftarrow{T}_{\mathrm{I},n}(\mathscr{N}_X(A))$, then $A \subseteq \Phi^{\leftarrow}[(-2^{-n}, 2^{-n})]$, which establishes that $\overrightarrow{T}_{\mathrm{II},n}(\Phi, \mathscr{N}_X(A)) \in \mathscr{N}_X(A)$.

Suppose we have

$$\langle \Phi_n : n \in \omega \rangle \in \prod_{n \in \omega} \overleftarrow{T}_{\mathrm{I},n}(\mathscr{N}_X(A_n))$$

for a sequence $\langle A_n : n \in \omega \rangle$ of $\mathcal{A}$ such that $\langle \Phi_n : n \in \omega \rangle \notin \Gamma_{MU_{\mathcal{B}}(X),\mathbf{0}}$. Then we can find $B \in \mathcal{B}$, $\varepsilon > 0$, and $N \in \omega$ such that $2^{-N} < \varepsilon$ and, for all $n \geq N$, $\Phi_n \notin [\mathbf{0}; B, \varepsilon]$.

To finish this application of Theorem 1.14, we need to show that

$$B \not\subseteq \overrightarrow{T}_{\mathrm{II},n}(\Phi_n, \mathscr{N}_X(A_n))$$

for all $n \geq N$. So let $n \geq N$ and note that, since $\Phi_n \notin [\mathbf{0}; B, \varepsilon]$, there are some $x \in B$ and $y \in \Phi_n(x)$ such that $|y| \geq \varepsilon > 2^{-N} \geq 2^{-n}$. That is, $\Phi_n(x) \not\subseteq (-2^{-n}, 2^{-n})$ and so $x \notin \overrightarrow{T}_{\mathrm{II},n}(\Phi_n, \mathscr{N}_X(A_n))$. □

Though particular applications of Corollaries 2.3 and 2.6 abound, we record a few that capture the general spirit using ideals of usual interest after recalling some other facts and some names for particular selection principles. We suppress the relationships with $C_{\mathcal{A}}(X)$ in the following applications in the interest of space.

Definition 2.8. We identify some particular selection principles by name.

- $\mathsf{S}_1(\Omega_{X,x}, \Omega_{X,x})$ is known as the *strong countable fan-tightness* property for X at x.
- $\mathsf{S}_1(\mathscr{D}_X, \Omega_{X,x})$ is known as the *strong countable dense fan-tightness* property for X at x.
- $\mathsf{S}_1(\mathscr{T}_X, \mathrm{CD}_X)$ is known as the *discretely selective* property for X.
- We refer to $\mathsf{S}_1(\Omega_X, \Omega_X)$ as the *ω-Rothberger* property and $\mathsf{S}_1(\mathcal{K}_X, \mathcal{K}_X)$ as the *k-Rothberger* property.

Definition 2.9. For a partially ordered set $(\mathbb{P}, \leq)$ and collections $\mathcal{A}, \mathcal{B} \subseteq \mathbb{P}$ such that, for every $B \in \mathcal{B}$, there exists some $A \in \mathcal{A}$ with $B \subseteq A$, we define the *cofinality of $\mathcal{A}$ relative to $\mathcal{B}$* by

$$\mathrm{cof}(\mathcal{A}; \mathcal{B}, \leq) = \min\{\kappa \in \mathrm{CARD} : (\text{there exists } \mathscr{F} \in [\mathcal{A}]^{\kappa}) \\ (\text{for all } B \in \mathcal{B})(\text{there exists } A \in \mathscr{F})\ B \subseteq A\}$$

where CARD is the class of cardinals.

Lemma 2.10. *Let $\mathcal{A}, \mathcal{B} \subseteq \wp^+(X)$ for a space X.*

As long as X is T_1,

$$\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathcal{O}_X(\mathcal{B})) \iff \mathrm{cof}(\mathcal{A}; \mathcal{B}, \subseteq) \leq \omega$$

(*see* [14, 35], *and* [3, Lemma 23]).

If $\mathcal{A}$ consists of G_δ sets, then

$$\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathcal{O}_X(\mathcal{B})) \iff \mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathcal{O}_X(\mathcal{B})) \iff \mathrm{cof}(\mathcal{A}; \mathcal{B}, \subseteq) \leq \omega$$

(*see* [13, 33] *and* [3, Lemma 24]).

Observe that Lemma 2.10 informs us that, for a T_1 space X,

- $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathbb{P}_X, \neg\mathcal{O}_X)$ if and only if X is countable,
- $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathcal{O}_X)$ if and only if X is σ-compact, and
- $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathcal{K}_X)$ if and only if X is hemicompact.

Corollary 2.11. *For an ideal $\mathcal{A}$ of closed subsets of a T_1 space X,*

$$\mathrm{cof}(\mathcal{A}; \mathcal{A}, \subseteq) \leq \omega$$

if and only if $MU_{\mathcal{A}}(X)$ is metrizable.

Proof. If $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ is such that, for every $A \in \mathcal{A}$, there is $n \in \omega$ with $A \subseteq A_n$, then the family $\{\mathbf{W}(A_n, 2^{-m}) : n, m \in \omega\}$ is a countable base for the uniformity on $MU_{\mathcal{A}}(X)$; so Theorem 1.18 demonstrates that $MU_{\mathcal{A}}(X)$ is metrizable.

Now, suppose that $MU_{\mathcal{A}}(X)$ is metrizable, which implies that $MU_{\mathcal{A}}(X)$ is first-countable. Using a descending countable basis at $\mathbf{0}$, we see that

$$\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_{MU_{\mathcal{A}}(X),\mathbf{0}}, \neg\Gamma_{MU_{\mathcal{A}}(X),\mathbf{0}}),$$

and, in particular,

$$\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_{MU_{\mathcal{A}}(X),\mathbf{0}}, \neg\Omega_{MU_{\mathcal{A}}(X),\mathbf{0}}).$$

By Corollary 2.6, we see that

$$\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathcal{O}_X(\mathcal{A})).$$

So, by Lemma 2.10, $\mathrm{cof}(\mathcal{A}; \mathcal{A}, \subseteq) \leq \omega$. □

As a particular consequence of this, we see the following result.

Corollary 2.12. *For any regular space X, the following conditions are equivalent:*

(i) *X is countable.*
(ii) *$MU_p(X)$ is metrizable.*
(iii) *$MU_p(X)$ is not discretely selective.*
(iv) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_1(\Omega_X, \Omega_X)$.
(v) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_1(\Omega_{MU_p(X),\mathbf{0}}, \Omega_{MU_p(X),\mathbf{0}})$.
(vi) $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} \mathsf{G}_1(\mathscr{D}_{MU_p(X)}, \Omega_{MU_p(X),\mathbf{0}})$.

Also, the following properties are equivalent.

(i) *X is hemicompact.*
(ii) *$\mathbb{K}(X)$ is hemicompact* (*see* [2, Theorem 3.22]).

(iii) $MU_k(X)$ *is metrizable* (*see* [19, Corollary 4.5]).
(iv) $MU_k(X)$ *is not discretely selective.*
(v) $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$.
(vi) $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\Omega_{MU_k(X),\mathbf{0}}, \Omega_{MU_k(X),\mathbf{0}})$.
(vii) $\mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\mathscr{D}_{MU_k(X)}, \Omega_{MU_k(X),\mathbf{0}})$.

Before the next corollary, we recall Tkachuk's strategy strengthening for the generalized point-open games.

Theorem 2.13 (see [3, Corollary 11] and [35])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be ideals of closed subsets of X. Then*

$$\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathcal{O}_X(\mathcal{B})) \iff \mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\Gamma_X(\mathcal{B}))$$

and

$$\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathcal{O}_X(\mathcal{B})) \iff \mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\Gamma_X(\mathcal{B}))$$

Corollary 2.14. *For any regular space X, the following conditions are equivalent:*

(i) $\mathrm{II} \uparrow \mathsf{G}_1(\Omega_X, \Omega_X)$.
(ii) $\mathrm{II} \uparrow \mathsf{G}_1(\Omega_{MU_p(X),\mathbf{0}}, \Omega_{MU_p(X),\mathbf{0}})$.
(iii) $\mathrm{II} \uparrow \mathsf{G}_1(\mathscr{D}_{MU_p(X)}, \Omega_{MU_p(X),\mathbf{0}})$.
(iv) $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{T}_{MU_p(X)}, \mathrm{CD}_{MU_p(X)})$.
(v) $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[[X]^{<\omega}], \neg\Omega_X)$.
(vi) $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[[X]^{<\omega}], \neg\Gamma_\omega(X))$.

Also, the following are conditions equivalent.

(i) $\mathrm{II} \uparrow \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$.
(ii) $\mathrm{II} \uparrow \mathsf{G}_1(\Omega_{MU_k(X),\mathbf{0}}, \Omega_{MU_k(X),\mathbf{0}})$.
(iii) $\mathrm{II} \uparrow \mathsf{G}_1(\mathscr{D}_{MU_k(X)}, \Omega_{MU_k(X),\mathbf{0}})$.
(iv) $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{T}_{MU_k(X)}, \mathrm{CD}_{MU_k(X)})$.
(v) $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathcal{K}_X)$.
(vi) $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\Gamma_k(X)$.

In general, Corollaries 2.12 and 2.14 are strictly separate, as the following example demonstrates.

Example 2.15. Let X be the one-point Lindelöfication of ω_1 with the discrete topology. In [2, Example 3.24], it is shown that X has the property that $\mathrm{II} \uparrow \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$, but $\mathrm{II} \underset{\text{mark}}{\not\uparrow} \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$.

However, according to Theorem 2.16, if Two can win against predetermined strategies in some Rothberger-like games, Two can actually win against full-information strategies in those games.

Theorem 2.16. *Let X be any space.*

(i) *By Pawlikowski* [29],

$$\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathcal{O}_X, \mathcal{O}_X) \iff \mathrm{I} \uparrow \mathsf{G}_1(\mathcal{O}_X, \mathcal{O}_X).$$

(ii) *By Scheepers* [32] (*see also* [5, Corollary 4.12]),

$$\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\Omega_X, \Omega_X) \iff \mathrm{I} \uparrow \mathsf{G}_1(\Omega_X, \Omega_X).$$

(iii) *By* [5, Theorem 4.21],

$$\mathrm{I} \underset{\mathrm{pre}}{\uparrow} \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X) \iff \mathrm{I} \uparrow \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X).$$

Corollary 2.17. *For any regular space X, the following conditions are equivalen:*

(i) *X is ω-Rothberger.*
(ii) *$X^{<\omega}$ is Rothberger, where $X^{<\omega}$ is the disjoint union of X^n for all $n \geq 1$* (*see* [30] *and* [5, Corollary 3.11]).
(iii) *$\mathcal{P}_{\mathrm{fin}}(X)$ is Rothberger, where $\mathcal{P}_{\mathrm{fin}}(X)$ is the set $[X]^{<\omega}$ with the subspace topology inherited from $\mathbb{K}(X)$* (*see* [5, Corollary 4.11]).
(iv) $\mathrm{I} \not\uparrow \mathsf{G}_1(\Omega_X, \Omega_X)$.
(v) *$MU_p(X)$ has strong countable fan-tightness at* $\mathbf{0}$.
(vi) *$MU_p(X)$ has strong countable dense fan-tightness at* $\mathbf{0}$.
(vii) $\mathrm{II} \underset{\mathrm{mark}}{\not\uparrow} \mathsf{G}_1(\mathscr{N}_X[[X]^{<\omega}], \neg\Omega_X)$.
(viii) $\mathrm{II} \not\uparrow \mathsf{G}_1(\mathscr{N}_X[[X]^{<\omega}], \neg\Omega_X)$.
(ix) $\mathrm{II} \underset{\mathrm{mark}}{\not\uparrow} \mathsf{G}_1(\mathscr{T}_{MU_p(X)}, \mathrm{CD}_{MU_p(X)})$.
(x) $\mathrm{II} \not\uparrow \mathsf{G}_1(\mathscr{T}_{MU_p(X)}, \mathrm{CD}_{MU_p(X)})$.
(xi) $\mathrm{II} \underset{\mathrm{mark}}{\not\uparrow} \mathsf{G}_1(\mathscr{N}_{MU_p(X),\mathbf{0}}, \neg\Omega_{MU_p(X),\mathbf{0}})$.
(xii) $\mathrm{II} \not\uparrow \mathsf{G}_1(\mathscr{N}_{MU_p(X),\mathbf{0}}, \neg\Omega_{MU_p(X),\mathbf{0}})$.

Also, the following conditions are equivalent:

(i) *X is k-Rothberger.*
(ii) $\mathrm{I} \not\uparrow \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$.
(iii) *$MU_k(X)$ has strong countable fan-tightness at* $\mathbf{0}$.
(iv) *$MU_k(X)$ has strong countable dense fan-tightness at* $\mathbf{0}$.
(v) $\mathrm{II} \underset{\mathrm{mark}}{\not\uparrow} \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathcal{K}_X)$.
(vi) $\mathrm{II} \not\uparrow \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathcal{K}_X)$.
(vii) $\mathrm{II} \underset{\mathrm{mark}}{\not\uparrow} \mathsf{G}_1(\mathscr{T}_{MU_k(X)}, \mathrm{CD}_{MU_k(X)})$.
(viii) $\mathrm{II} \not\uparrow \mathsf{G}_1(\mathscr{T}_{MU_k(X)}, \mathrm{CD}_{MU_k(X)})$.
(ix) $\mathrm{II} \underset{\mathrm{mark}}{\not\uparrow} \mathsf{G}_1(\mathscr{N}_{MU_k(X),\mathbf{0}}, \neg\Omega_{MU_k(X),\mathbf{0}})$.
(x) $\mathrm{II} \not\uparrow \mathsf{G}_1(\mathscr{N}_{MU_k(X),\mathbf{0}}, \neg\Omega_{MU_k(X),\mathbf{0}})$.

Using the techniques of Gerlits and Nagy [14], Galvin [13], and Telgársky [33], we offer an analog of Lemma 2.10 related to the so-called *weak k-covering number* in [19].

Definition 2.18. For a collection $\mathcal{A}$ of closed subsets of X, define the *weak covering number* of $\mathcal{A}$ to be

$$wkc(\mathcal{A}) = \min\left\{\kappa \in \mathrm{CARD} : (\text{there exists } \mathscr{F} \in [\mathcal{A}]^{\kappa})\ X = \mathrm{cl}\left(\bigcup \mathscr{F}\right)\right\}.$$

Definition 2.19. For a space X, let DO_X be the set of all $\mathscr{U} \subseteq \mathscr{T}_X$ such that $X = \mathrm{cl}\left(\bigcup \mathscr{U}\right)$.

Lemma 2.20. *Let $\mathcal{A} \subseteq \wp^+(X)$, where X is a regular space. Then*

$$\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathrm{DO}_X) \iff wkc(\mathcal{A}) \leq \omega.$$

If X is metrizable and $\mathcal{A} \subseteq K(X)$, then

$$\begin{aligned} \mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathrm{DO}_X) &\iff \mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathrm{DO}_X) \\ &\iff wkc(\mathcal{A}) \leq \omega. \end{aligned}$$

Proof. The implications

$$wkc(\mathcal{A}) \leq \omega \implies \mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathrm{DO}_X)$$

and

$$\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathrm{DO}_X) \implies \mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathrm{DO}_X)$$

are evident and hold with no assumptions on X.

Suppose that X is regular and that $wkc(\mathcal{A}) > \omega$. Any predetermined strategy for One in $\mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathrm{DO}_X)$ corresponds to a sequence $\langle A_n : n \in \omega\rangle \in \mathcal{A}^\omega$. We show that no such strategy can be winning for One. Since $\bigcup\{A_n : n \in \omega\}$ is not dense in X, we can find $U \in \mathscr{T}_X$ such that $U \cap \bigcup\{A_n : n \in \omega\} = \emptyset$. As X assumed to be regular, without loss of generality, we can assume that $\mathrm{cl}(U) \cap \bigcup\{A_n : n \in \omega\} = \emptyset$. It follows that $A_n \subseteq X \setminus \mathrm{cl}(U) =: V$ for all $n \in \omega$. Hence, $\langle \mathscr{N}_X(A_n) : n \in \omega\rangle$ is not a winning strategy for One. That is, $\mathrm{I} \underset{\text{pre}}{\not\uparrow} \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathrm{DO}_X)$.

Now suppose that X is metrizable, $\mathcal{A} \subseteq K(X)$, and $\mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathrm{DO}_X)$. Let σ be a winning strategy for One that is coded by elements of $\mathcal{A}$. Let $\mathbb{T}_0 = \langle\rangle$, and for $n \in \omega$, let

$$\mathbb{T}_{n+1} = \left\{ w^\frown \left\langle \sigma(w), \mathbb{B}\left(\sigma(w), 2^{-\ell}\right)\right\rangle : w \in \mathbb{T}_n \wedge \ell \in \omega \right\}.$$

Note that $\mathscr{F} = \bigcup_{n\in\omega}\{\sigma(w) : w \in \mathbb{T}_n\}$ is a countable subset of $\mathcal{A}$.

We show that $X = \mathrm{cl}\left(\bigcup \mathscr{F}\right)$ by way of contradiction. Suppose $X \setminus \mathrm{cl}\left(\bigcup \mathscr{F}\right) \neq \emptyset$, and let $V \in \mathscr{T}_X$ be open such that $\mathrm{cl}(V) \cap \mathrm{cl}\left(\bigcup\mathscr{F}\right) = \emptyset$. For $A_0 := \sigma(\emptyset)$, let $\ell_0 \in \omega$ be such that $\mathrm{cl}(V) \cap \mathbb{B}\left(A_0, 2^{-\ell_0}\right) = \emptyset$ and set $w_0 = \langle A_0, \mathbb{B}\left(A_0, 2^{-\ell_0}\right)\rangle \in \mathbb{T}_1$.

For $n \in \omega$, suppose that $w_n \in \mathbb{T}_{n+1}$ is defined. Now, for $A_{n+1} := \sigma(w_n)$, let $\ell_{n+1} \in \omega$ be such that $\mathrm{cl}(V) \cap \mathbb{B}\left(A_{n+1}, 2^{-\ell_{n+1}}\right) = \emptyset$. Then define $w_{n+1} = w_n^\frown \left\langle A_{n+1}, \mathbb{B}\left(A_{n+1}, 2^{-\ell_{n+1}}\right)\right\rangle \in \mathbb{T}_{n+2}$.

Now, we have a run of the game according to σ with the property that

$$\mathrm{cl}(V) \cap \bigcup_{n\in\omega} \mathbb{B}\left(A_n, 2^{-\ell_n}\right) = \emptyset.$$

It follows that

$$V \cap \mathrm{cl}\left(\bigcup_{n\in\omega} \mathbb{B}\left(A_n, 2^{-\ell_n}\right)\right) = \emptyset,$$

which contradicts the assumption that σ is a winning strategy. $\square$

Consequently, for a regular space X,

- $\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[[X]^{<\omega}], \neg\mathrm{DO}_X)$ if and only if X is separable and
- $\mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathrm{DO}_X)$ if and only if X admits a countable collection of compact subsets whose union is dense.

Also, by Lemma 1.13, $\mathsf{G}_1(\mathscr{N}_X[\mathcal{A}], \neg\mathrm{DO}_X)$ is dual to $\mathsf{G}_1(\mathscr{O}_X(\mathcal{A}), \mathrm{DO}_X)$. Hence, we obtain the following result.

Corollary 2.21. *If X is a regular space, then*

$$\begin{aligned} X \text{ is separable} &\iff \mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[[X]^{<\omega}], \neg\mathrm{DO}_X) \\ &\iff \mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\Omega_X, \mathrm{DO}_X) \end{aligned}$$

and

$$\begin{aligned} wkc(K(X)) \le \omega &\iff \mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathrm{DO}_X) \\ &\iff \mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\mathcal{K}_X, \mathrm{DO}_X). \end{aligned}$$

If X is, in addition, metrizable, then

$$\begin{aligned} X \text{ is separable} &\iff \mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[[X]^{<\omega}], \neg\mathrm{DO}_X) \\ &\iff \mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[[X]^{<\omega}], \neg\mathrm{DO}_X) \\ &\iff \mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\Omega_X, \mathrm{DO}_X) \\ &\iff \mathrm{II} \uparrow \mathsf{G}_1(\Omega_X, \mathrm{DO}_X) \end{aligned}$$

and

$$\begin{aligned} wkc(K(X)) \le \omega &\iff \mathrm{I} \underset{\text{pre}}{\uparrow} \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathrm{DO}_X) \\ &\iff \mathrm{I} \uparrow \mathsf{G}_1(\mathscr{N}_X[K(X)], \neg\mathrm{DO}_X) \\ &\iff \mathrm{II} \underset{\text{mark}}{\uparrow} \mathsf{G}_1(\mathcal{K}_X, \mathrm{DO}_X) \\ &\iff \mathrm{II} \uparrow \mathsf{G}_1(\mathcal{K}_X, \mathrm{DO}_X). \end{aligned}$$

3. Questions

It is tempting to conjecture that an analogous result to Theorem 2.1 holds for the space of minimal usco maps, except that the natural candidate of pointwise set difference between two minimal usco maps may not generally produce a minimal usco map. For $A, B \subseteq \mathbb{R}$, let $A - B = \{x - y : x \in A, y \in B\}$, and consider $\Phi := \overline{\mathbf{1}}_{[0,1]}$. Note that the pointwise difference $\Phi - \Phi : X \to \mathbb{R}$ defined by $\Phi(x) - \Phi(x)$ has the property that

$$\Phi(x) - \Phi(x) = \begin{cases} \{0\}, & x \notin \{0,1\}; \\ \{-1, 0, 1\}, & x \in \{0,1\}. \end{cases}$$

Since $\mathrm{gr}(\mathbf{0}) \subsetneq \mathrm{gr}(\Phi - \Phi)$, we see that $\Phi - \Phi$ is not minimal.

We also show that the pointwise Pompeiu–Hausdorff distance between two minimal usco maps need not be quasicontinuous. For compact $A, B \subseteq \mathbb{R}$, let

$$H_d(A, B) = \inf\{\varepsilon > 0 : A \subseteq \mathbb{B}(B, \varepsilon) \wedge B \subseteq \mathbb{B}(A, \varepsilon)\}.$$

We can define $f : X \to \mathbb{R}$ by $f(x) = H_d\left(\overline{\mathbf{1}}_{[0,1]}(x), \overline{\mathbf{1}}_{\mathbb{R}\setminus[0,1]}(x)\right)$, and observe that

$$f(x) = \begin{cases} 1, & x \notin \{0, 1\}; \\ 0, & x \in \{0, 1\}. \end{cases}$$

So f is not quasicontinuous. However, this does not mean that the pointwise Pompeiu–Hausdorff distance cannot be used to establish an analogous result to Theorem 2.1 for $MU_{\mathcal{A}}(X)$.

We end with a few questions.

Problem 3.1. Is there a result similar to Theorem 2.1 for general uniform spaces? Along these lines, are there selection games that characterize the diagonal degree and the uniformity degree of a uniform space?

Problem 3.2. Can results similar to Theorems 2.2 and 2.5 be established relative to $\Omega_{MU_{\mathcal{A}}(X),\Phi}$ for any $\Phi \in MU(X)$?

Problem 3.3. How many of the equivalences and dualities of this paper can be established for games of longer length and for finite-selection games?

Problem 3.4. How much of this theory can be recovered when we study $MU(X, Y)$ for $Y \neq \mathbb{R}$, for example, when Y is $[0, 1]$, any metrizable space, any topological group, or any uniform space?

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