Khayyam J. Math. 9 (2023), no. 2, 210-224
DOI: 10.22034/KJM.2023.350018.2587

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

GENERAL ENERGY DECAY FOR A VISCOELASTIC WAVE EQUATION WITH SPACE-TIME DAMPING COEFFICIENT IN $\mathbb{R}^n$

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Communicated by C. Zhang

Abstract. In this paper, we consider the following viscoelastic wave equation:

$$u_{tt} - \big(\Delta u - \int_0^t g(t-s)\Delta u(s)ds\big) + b(t,x)u_t = -|u|^{p-1}u,\ t > 0,\ x \in \mathbb{R}^n$$
$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \qquad x \in \mathbb{R}^n,$$

with space-time dependent potential and where the initial data $u_0(x)$, $u_1(x)$ have compact supports. Under suitable assumptions on the potential b and for a relaxation function g satisfying the condition $g'(t) \leq -\mu(t)g^r(t),\ t \geq 0$, $1 < r < \frac{3}{2}$, we obtain a general energy decay result that extends other results in the literature.

1. Introduction

In the case of energy decay in bounded domains, Messaoudi [14] considered the viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0 \quad \text{in} \quad \Omega \times (0,\infty)$$

and established a general decay result, which is not necessarily of exponential or polynomial type. Cavalcanti et al. [3] considered a damped wave equation of the form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + |u|^{\gamma}u = 0, \quad in \quad \Omega \times (0,\infty), \quad (1.1)$$

Date: Received: 3 July 2022; Revised: 28 February 2023; Accepted: 17 June 2023.
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2020 *Mathematics Subject Classification.* Primary 35L05; Secondary 35B40.
Key words and phrases. Energy decay, damping potential, relaxation function.

for $a : \Omega \to \mathbb{R}^+$ with $a(x) \geq a_0 > 0$. They established an exponential decay result when the relaxation function $g(t)$ decays exponentially. Song et al. [23] also considered (1.1) under certain suitable assumptions on a, g, and γ and proved energy decay results similar to that of [3, 14] using a new perturbed energy technique.

Under weaker conditions on the relaxation function, Messaoudi [13] considered the viscoelastic problem (1.1) with $a = 0$ and showed that the damping resulting from the integral term is sufficient to obtain polynomial as well as exponential decay result. This result was recently improved by Messaoudi and Al-Khulaifi [15] to the case where the relaxation function satisfies $g'(t) \leq -\mu(t)g^r(t), \quad t \geq 0,\ 1 \leq r < \frac{3}{2}$. A review on recent results on energy decay, global existence, and blow up of solutions to nonlinear wave equations in bounded domains is discussed in [16]. For other related results, the reader is referred to [1, 4, 11].

In the case of unbounded domains where the source term is absent, there is an extensive literature concerning total energy decay to the scalar valued wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + b(t,x)u_t = 0 \quad (t,x) \in [0,\infty) \times \mathbb{R}^n. \tag{1.2}$$

Ikehata et al. [10] considered the linear wave equation (1.2) with $g = 0$ and where (u_0, u_1) is compactly supported initial data in the energy space. They obtained polynomial energy decay under suitable assumptions on the potential $b(t,x)$. The result shows that for a potential of the form $V(x) \approx (1+|x|)^{-\alpha}$, $\alpha = 1$ is critical. The reader is referred to [5, 6, 18] for related results.

When $b = 0$, Said-Houari and Messaoudi [22] considered the viscoelastic problem (1.2) and obtained general decay estimates, using the energy technique in the Fourier space.

In the absence of the relaxation term but where the internal source term is present, Todorova and Yordanov [24] considered the problem

$$u_{tt} - \Delta u + b(t,x)u_t + |u|^{p-1}u = 0, \tag{1.3}$$

where $b(t,x) = b(x) \equiv b_0(1+|x|)^{-\alpha}$ with $\alpha \in [0,1)$ (the subcritical potential case) and obtained total energy decay rates, which are almost optimal. By modifying the technique due to Todorova and Yordanov [24], Ikehata and Inoue [8] considered the wave problem (1.3) and obtained total energy decay results in the case when $\alpha = 1$.

Mochizuki [17] considered the wave problem (1.3) and showed non-decay results for the energy function $E_u(t)$ in the case $b(t,x) \leq b_0(1+|x|)^{-1-\alpha}$ where $\alpha > 0$ (the supercritical potential case). For other related results, see [7, 9, 12, 25], and for time dependent potential $b(t,x) \equiv b_0(1+t)^{-1}$, see [21, 26, 27].

More recently, Ogbiyele and Arawomo [20] considered the viscoelastic problem

$$\begin{cases} u_{tt} - \big(\Delta u - \displaystyle\int_0^t g(t-s)\Delta u(s)ds\big) + b(t,x)u_t + |u|^{p-1}u = 0, & t > 0, \quad x \in \mathbb{R}^n \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

under suitable conditions on the damping potential $b(t,x)$ and where the relaxation function satisfies the condition $g'(t) \leq -\mu(t)g(t), \quad t \geq 0$. They obtained a general energy decay result from which the polynomial and exponential decay results are only special cases.

A structural link between energy decay and the decay of the memory kernel is observed in the literature. More precisely, for memory decay satisfying

$$g'(t) + \xi g(t) \leq 0, \quad \text{for a.e} \quad t \geq 0 \quad \xi > 0,$$

exponential decay of the solution is expected at ∞. When the memory kernel satisfies

$$g'(t) + \mu(t)g(t) \leq 0, \quad \text{for a.e} \quad t \geq 0$$

and μ is a strictly positive decreasing function, as in [20], where μ is a positive decreasing function, the energy decay rates in which exponential and polynomial decay are only special cases are expected. Moreover, when the kernel function satisfies

$$g'(t) + \xi g^{1+1/k}(t) \leq 0, \quad \text{for a.e} \quad t \geq 0 \quad \xi > 0,\ k \in (1,\infty),$$

a polynomial energy decay of the form $(1+t)^{-k}$ is guaranteed.

In this paper, we consider the viscoelastic wave problem (1.4) with space-time dependent potential $b(t,x)$ and a power-type nonlinearity $|u|^{p-1}u$, where

$$1 < p < +\infty \quad (n=2) \quad \text{and} \quad 2 < p+1 < \frac{2n}{n-2} \quad (n \geq 3).$$

Under suitable assumptions on the damping potential b and when the relaxation function g satisfies, for some $1 \leq r < \frac{3}{2}$ the condition $g'(t) \leq -\mu(t)g^r(t),$ for all $t \geq 0$, we establish energy decay estimates where the initial data u_0 and u_1 are assumed to have compact support in a ball $B(L)$ of radius L about the origin, with L satisfying the condition $\operatorname{supp}\{u_0(x), u_1(x)\} \subset \{|x| \leq L\}$ and the solution satisfying the finite speed of propagation property

$$\operatorname{supp}\, u(t,x) \in B(L+t), \qquad t \in (0,\infty).$$

Our result improves on the results in the literature.

2. Preliminaries

In this section, we present some basic materials needed in the proof of our result. We use the standard Lebesgue space $L^q(\mathbb{R}^n)$, $1 \leq q \leq \infty$ and the Sobolev space $H^1(\mathbb{R}^n)$ as well as their usual norms and scalar products.

For the potential $b(t,x)$ and the relaxation function $g(t)$, we have the following assumptions:

(A_1) $\int_{B(L+t)} b(t,x)^{\frac{n}{2}} dx \in L^\infty_{loc}(J_T)$ where $J_T = (0,\infty)$.

(A_2) There exists a positive constant $c_{\alpha b}$ and a positive function $b_{_L}(t)$ such that
$b(t,x) \geq b_{_L}(t)$ for $x \in B(L+t)$ and $\alpha_{_L}(t) b_{_L}(t) \geq c_{\alpha b}$, where $\alpha_{_L}(t) = \left[\int_{B(L+t)} b(t,x)^{\frac{n}{2}} dx\right]^{2/n}$.

(A_3) g is a nonincreasing differentiable function satisfying

$$g(s) \geq 0, \qquad 1 - \int_0^\infty g(s)ds = \ell > 0, \qquad \text{for all} \quad s \geq 0.$$

(A_4) In addition, there exist positive constants λ and λ_μ and a positive nonincreasing differentiable function μ that satisfies for some $1 < r < \frac{3}{2}$,

$$g'(s) \leq -\mu(s)g^r(s), \quad \mu(s) \geq 0 \quad \text{and} \quad \mu'(s) \leq 0 \quad \text{for all } s \geq 0$$

such that

$$\int_0^\infty \frac{g^2}{\mu(s)g^r(s)}ds < \lambda.$$

Remark 2.1. From assumption (A_4), we observe that

$$g(t) \leq C\big(1 + \int_0^t \mu(s)ds\big)^{\frac{-1}{r-1}},$$

for a positive constant C and that

$$\mu(t)g^{\frac{1}{2}}(t) = \mu(t)g^{\frac{1}{2}-r}(t)g^r(t) \leq -g'(t)g^{\frac{1}{2}-r}(t).$$

Integrating this gives

$$\int_0^\infty \mu(s)g^{\frac{1}{2}}(s)ds \leq -\frac{g^{\frac{3}{2}-r}(t)}{\frac{3}{2}-r}\Big|_0^\infty < \infty \qquad \text{for} \quad r < \frac{3}{2}.$$

The condition $\int_0^\infty \mu(s)g^{\frac{1}{2}}(s)ds < \infty$ is same with that of [15].

Lemma 2.2 ([2])**.** *Suppose that $1 \leq q < n$. If $u \in W^{1,q}(\mathbb{R}^n)$, then $u \in L^{q^*}(\mathbb{R}^n)$ with*

$$\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}.$$

Moreover, there is a constant $k = k(n, q)$ such that

$$\|u\|_{q^*} \leq k\|\nabla u\|_q \qquad \text{for all } u \in W^{1,q}(\mathbb{R}^n).$$

For completeness, we state the existence result of [20].

Theorem 2.3. *Suppose that the assumptions (A_1)–(A_4) hold with initial data $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ having compact supports. Let $2 < p + 1 \leq \frac{2n}{n-2}$, if $n \geq 3$, then there exists a unique solution*

$$u \in C([0, T]; H^1(\mathbb{R}^n)) \qquad \text{and} \qquad u_t \in C([0, T]; L^2(\mathbb{R}^n)),$$

for some $T > 0$.

The proof follows that of [19, 20].

Lemma 2.4 ([20])**.** *Let $u(t, x)$ be the solution of (1.4) for $n \geq 3$. Then there exists a positive constant K such that*

$$\int_{\mathbb{R}^n} |u(t, x)|^2 dx \leq K^2(L + t)^2 \int_{\mathbb{R}^n} |\nabla u(t, x)|^2 dx. \tag{2.1}$$

We now define the modified energy functional $E(t)$ associated to problem (1.4) by

$$E(t) := \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\Big[1 - \int_0^t g(s)ds\Big]\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u) + \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \quad (2.2)$$

where for simplicity, we use the following notation:

$$(g \circ \nabla u) := \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2 ds.$$

Hence, for the functional $E(t)$, we state the following lemma.

Lemma 2.5. *Suppose that the assumptions (A_1)–(A_4) hold. Let u be a solution of problem (1.4). Then for any $t \geq 0$, the energy functional $E(t)$ satisfies*

$$E'(t) \leq -\int_{\mathbb{R}^n} b(t,x)|u_t|^2 dx - \frac{1}{2}g(t)\|\nabla u\|_2^2 + \frac{1}{2}\big(g' \circ \nabla u\big); \quad (2.3)$$

hence, we have

$$E(t) \leq E(0). \quad (2.4)$$

Proof. By multiplying (1.4) by u_t and integrating over $\mathbb{R}^n$, we obtain the estimate (2.3) for any regular solution. Thus, by using density arguments, the estimate remains valid for weak solutions. The boundedness result of (2.4) follows directly from (2.3), since under the assumptions given, $E'(t) \leq 0$. Integrating over $[0,t]$ gives $E(t) \leq E(0)$. □

Lemma 2.6 ([4]). *Assume that $g \in C([0,\infty])$, $w \in L^1_{loc}(0,\infty)$, and $0 \leq \sigma \leq 1$. Then*

$$\int_0^t |g(r)w(r)|dr \leq C\Big[\int_0^{+\infty} g^{1-\sigma}(s)|w(s)|ds\Big]^{\frac{r-1}{r-1+\sigma}}\Big(\int_0^t g^r(s)|w(s)|ds\Big)^{\frac{\sigma}{r-1+\sigma}}.$$

Remark 2.7. A consequence of Lemma 2.6 is that

$$\int_0^t |g(r)w(r)|dr \leq \frac{C}{\mu(t)}\Big[\int_0^{+\infty} \mu(s)g^{1-\sigma}(s)|w(s)|ds\Big]^{\frac{r-1}{r-1+\sigma}}\Big[\int_0^t \mu(s)g^r(s)|w(s)|ds\Big]^{\frac{\sigma}{r-1+\sigma}}.$$

3. General decay

In this section, we consider the decay of the solution energy of (1.4). To achieve this, we introduce the following functionals:

$$M(t) := \int_{\mathbb{R}^n} uu_t\, dx$$

and

$$L(t) := \beta(t)E(t) + \nu_1\rho(t)M(t), \quad (3.1)$$

where ν_1 is a positive constant to be determined later and β and ρ are positive functions depending on the support radius L. The weighted functions β and ρ are used to compensate for the lack of compactness in the case of unbounded region and as a control function in the case of variable coefficients. In addition, we state the following conditions on β and ρ:

(A_5) $0 < \beta(t), \quad \beta(t) \geq \rho(t)\alpha_{L}(t) \geq \rho(t)\alpha_{L}(0)$,
(A_6) There exist positive functions η_{L} and γ_{L} satisfying
  (i) $\eta_{L}(t)(L+t)^2 \leq \alpha_{L}(t)$ and $[\eta_{L}(t)(L+t)^2]^{-1} \leq c_\eta$
  (ii) $\gamma_{L}(t)(L+t)^2 \leq \alpha_{L}(t)b_{L}(t) \quad$ and $\quad \frac{1}{\gamma_L(t)}\left[\frac{\beta(t)}{\rho(t)}\left|\left(\frac{\rho(t)}{\beta(t)}\right)'\right|\right]^2 \leq c^2_{\rho\beta}$.

Lemma 3.1. *Suppose that the assumptions* $(A_5) - (A_6)$ *hold. Then there exist positive constants* k_1^* *and* k_2^* *such that the relation*

$$k_1^*\beta(t)E(t) \leq L(t) \leq k_2^*\beta(t)E(t) \tag{3.2}$$

is satisfied.

Proof. Using Hölder's, Sobolev's, and Young's inequalities and the assumptions (A_5) and (A_6), we obtain the following estimate:

$$\begin{aligned}
&|L(t) - \beta(t)E(t)|\\
\leq&\nu_1\rho(t)\int_{\mathbb{R}^n}|uu_t|dx\\
\leq&\nu_1\rho(t)\int_{\mathbb{R}^n}\left[\eta_{L}(t)\big(\omega_n(L+t)^n\big)^{\frac{2}{n}}|u_t|^2\right]^{\frac{1}{2}}\left[\eta_{L}^{-1}(t)\big(\omega_n(L+t)^n\big)^{\frac{-2}{n}}|u|^2\right]^{\frac{1}{2}}dx\\
\leq&\nu_1\rho(t)\left[\eta_{L}(t)\omega_n^{\frac{2}{n}}(L+t)^2\int_{\mathbb{R}^n}|u_t|^2dx\right]^{\frac{1}{2}}\left[\frac{1}{\eta_{L}(t)\omega_n^{\frac{2}{n}}(L+t)^2}\int_{\mathbb{R}^n}|u|^2dx\right]^{\frac{1}{2}}\\
\leq&\frac{\nu_2}{2}\rho(t)\eta_{L}(t)(L+t)^2\|u_t\|^2 + \frac{\nu_1k^2}{2\eta_{L}(t)}\rho(t)\|\nabla u\|^2\\
\leq&\rho(t)\eta_{L}(t)(L+t)^2\left[\frac{\nu_2}{2}\|u_t\|^2 + \frac{\nu_1k^2}{2(L+t)^2\eta_{L}^2(t)}\|\nabla u\|^2\right]\\
\leq&\rho(t)\eta_{L}(t)(L+t)^2\left[\frac{\nu_2}{2}\|u_t\|^2 + \frac{\nu_1k^2c_\eta}{2}\|\nabla u\|^2\right]\\
\leq&\frac{k_0^*\beta(t)}{2}\left[\|u_t\|^2 + \ell\|\nabla u\|^2\right] \leq k_0^*\beta(t)E(t),
\end{aligned}$$

where $k_0^* = \max\{\nu_2,\ \frac{\nu_1}{\ell}k^2c_\eta\}$ and $\nu_2 = \nu_2(\omega_n, \nu_1)$. Therefore, we obtain (3.2). □

The following lemma will be useful in the proof of the main result.

Lemma 3.2. *Let* u *be a solution of problem* (1.4). *Suppose that the assumptions* (A_2) *and* (A_3) *hold. Then the functional*

$$M(t) := \int_{\mathbb{R}^n} uu_t\, dx$$

satisfies the following estimate:

$$\begin{aligned}[\rho(t)M(t)]' \leq& \rho(t)\|u_t\|^2 - \rho(t)\Big(\ell - \frac{[1+k^2]}{2\delta}\Big)\|\nabla u\|^2 \\ &+ \frac{\delta}{2}\rho(t)\alpha_{_L}(t)\int_{\mathbb{R}^n} b(t,x)|u_t|^2dx + \frac{\lambda\delta}{2}\rho(t)(\mu g^r\circ\nabla u) \\ &- \rho(t)\|u\|_{p+1}^{p+1} + \rho'(t)M(t).\end{aligned} \tag{3.3}$$

Proof. Differentiating $M(t)$ and using (1.4), we obtain

$$\begin{aligned}M'(t) =& \|u_t\|^2 - \|\nabla u\|^2 + \int_0^t g(t-s)\int_{\mathbb{R}^n}\nabla u(s)\nabla u(t)dxds \\ &- \int_{\mathbb{R}^n} b(t,x)u_t u dx - \|u\|_{p+1}^{p+1}.\end{aligned} \tag{3.4}$$

Multiplying this later inequality by $\rho(t)$ gives

$$\begin{aligned}[\rho(t)M(t)]' =& \rho(t)\|u_t\|^2 - \rho(t)\|\nabla u\|^2 + \rho(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}\nabla u(s)\nabla u(t)dxds \\ &- \rho(t)\int_{\mathbb{R}^n} b(t,x)u_t u dx - \rho(t)\|u\|_{p+1}^{p+1} + \rho'(t)M(t).\end{aligned} \tag{3.5}$$

For the third term on the right-hand side of (3.5), using Young's inequality, we get

$$\begin{aligned}&\int_0^t g(t-s)\int_{\mathbb{R}^n}\nabla u(s)\nabla u(t)dxds \\ &\leq \int_0^t g(t-s)\int_{\mathbb{R}^n}|\nabla u(s)-\nabla u(t)||\nabla u(t)|dxds + \int_0^t g(s)ds\|\nabla u\|^2 \\ &\leq (\frac{1}{2\delta} + \int_0^t g(s)ds)\|\nabla u\|^2 + \frac{\delta}{2}\int_0^t \frac{1}{\mu(s)}g^{2-r}(s)ds\big(\mu g^r\circ\nabla u).\end{aligned} \tag{3.6}$$

For the fourth term on the right hand side of (3.5), using (2.1) and Hölder's, Young's, and Sobolev's inequalities, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} b(t,x)u_t u dx \\ &\leq \Big[\int_{\mathbb{R}^n} b(t,x)|u_t|^2dx\Big]^{\frac{1}{2}}\Big[\Big[\int_{\mathbb{R}^n}|u|^{\frac{2n}{n-2}}dx\Big]^{\frac{n-2}{n}}\Big[\int_{B(L+t)} b(t,x)^{\frac{n}{2}}dx\Big]^{\frac{2}{n}}\Big]^{\frac{1}{2}} \\ &\leq \frac{\delta}{2}\alpha_{_L}(t)\int_{\mathbb{R}^n} b(t,x)|u_t|^2dx + \frac{k^2}{2\delta}\|\nabla u\|^2.\end{aligned} \tag{3.7}$$

Substituting the estimates (3.6)–(3.7) into (3.5) gives

$$\begin{aligned}[\rho(t)M(t)]' \leq& \rho(t)\|u_t\|^2 - \rho(t)\Big(\ell - \frac{[1+k^2]}{2\delta}\Big)\|\nabla u\|^2 + \frac{\delta}{2}\rho(t)\alpha_{_L}(t)\int_{\mathbb{R}^n} b(t,x)|u_t|^2dx \\ &+ \frac{\lambda\delta}{2}\rho(t)(\mu g^r\circ\nabla u) - \rho(t)\|u\|_{p+1}^{p+1} + \rho'(t)M(t),\end{aligned}$$

where $\int_0^t g(s)ds \leq \int_0^\infty g(s)ds = 1-\ell$ and $\int_0^t \frac{g^{2-r}(s)}{\mu(s)}ds \leq \int_0^\infty \frac{g^{2-r}(s)}{\mu(s)}ds < \lambda$. □

We now present the main result on decay of the energy.

Theorem 3.3. *Suppose that the assumptions* (A_1)–(A_6) *hold, and let* $u_0 \in H^1(\mathbb{R}^n)$ *and* $u_1 \in L^2(\mathbb{R}^n)$ *with compact supports. Then for some* $c_\mu > 0$ *and* $1 < r < \frac{3}{2}$, *the solution energy* (2.2) *satisfies for all* $t \geq 0$,

$$E(t) \leq C\Big[1+\int_0^t \frac{\rho(s)}{\beta(s)}ds\Big]^{\frac{-1}{2(r-1)}}, \qquad \text{if} \quad \alpha_{_L}(t)\mu^{2r-1}(t) \geq c_\mu, \tag{3.8}$$

$$E(t) \leq C\Big[1+\int_0^t \frac{\mu^{2r-1}(s)\rho(s)}{\beta(s)}ds\Big]^{\frac{-1}{2(r-1)}}, \qquad \text{if} \quad \alpha_{_L}(t)\mu^{2r-1}(t) < c_\mu. \tag{3.9}$$

If, in addition,

$$\int_0^\infty \mu(s)E(t-s)ds < \infty,$$

then, the energy of solution to (1.4) *satisfies*

$$E(t) \leq C\Big[1+\int_0^t \frac{\rho(s)}{\beta(s)}ds\Big]^{\frac{-1}{r-1}}, \qquad \text{if} \quad \alpha_{_L}(t)\mu^{r}(t) \geq c_\mu, \tag{3.10}$$

$$E(t) \leq C\Big[1+\int_0^t \frac{\mu^{r}(s)\rho(s)}{\beta(s)}ds\Big]^{\frac{-1}{r-1}}, \qquad \text{if} \quad \alpha_{_L}(t)\mu^{r}(t) < c_\mu. \tag{3.11}$$

Proof. Using (2.3), we obtain

$$[\beta(t)E(t)]' \leq -\beta(t)\int_0^t b(t,x)|u_t|^2dx + \frac{\beta(t)}{2}\big(g'\circ\nabla u\big) + \beta'(t)E(t). \tag{3.12}$$

By combining (3.3) and (3.12), we get

$$\begin{aligned} L'(t) \leq &-\Big[\beta(t) - \frac{\nu_1\delta}{2}\rho(t)\alpha_{_L}(t)\Big]\int_{\mathbb{R}^n} b(t,x)|u_t|^2dx + \nu_1\rho(t)\|u_t\|^2 \\ &- \nu_1\rho(t)\Big[\ell - \frac{[1+k^2]}{2\delta}\Big]\|\nabla u\|^2 - \nu_1\rho(t)\|u\|_{p+1}^{p+1} + \frac{\beta(t)}{2}\big(g'\circ\nabla u\big) \\ &+ \frac{\nu_1\lambda\delta}{2}\rho(t)(\mu g^r\circ\nabla u) + \beta'(t)E(t) + \nu_1\rho'(t)M(t). \end{aligned} \tag{3.13}$$

The last two terms in the right-hand side of (3.13) can be estimated in terms of $L(t)$ and $M(t)$, using (3.1), as follows:

$$\begin{aligned} \beta'(t)E(t) + \nu_1\rho'(t)M(t) &= \frac{\beta'(t)}{\beta(t)}L(t) - \nu_1\frac{\rho(t)\beta'(t)}{\beta(t)}M(t) + \nu_1\rho'(t)M(t) \\ &= \frac{\beta'(t)}{\beta(t)}L(t) + \nu_1\beta(t)\Big[\frac{\rho(t)}{\beta(t)}\Big]'M(t). \end{aligned} \tag{3.14}$$

Next, for the term $\nu_1\beta(t)\big[\frac{\rho(t)}{\beta(t)}\big]'M(t)$, using Hölder's, Young's, and Sobolev's inequalities and assumption (A_6)(ii), we have

$$\begin{aligned}
&\nu_1\beta(t)\Big|\Big(\frac{\rho(t)}{\beta(t)}\Big)'\Big|\int_{\mathbb{R}^n}|u_t u|dx \\
&\leq \frac{\nu_2\delta}{2}\rho(t)\gamma_{_L}(t)(L+t)^2\|u_t\|^2+\frac{\nu_1 k^2}{2\delta\gamma_{_L}(t)}\Big[\frac{\beta(t)}{\rho(t)}\Big|\Big(\frac{\rho(t)}{\beta(t)}\Big)'\Big|\Big]^2\rho(t)\|\nabla u\|^2 \qquad (3.15)\\
&\leq \frac{\nu_2\delta}{2}\rho(t)\alpha_{_L}(t)b_{_L}(t)\|u_t\|^2+\frac{\nu_1 k^2 c^2_{_{\rho\beta}}}{2\delta}\rho(t)\|\nabla u\|^2.
\end{aligned}$$

Now, combining (3.13)–(3.15) and employing (A_4) and (A_5), we obtain

$$\begin{aligned}
L'(t)\leq &-\rho(t)\Big[\big[1-\frac{\delta(\nu_1+\nu_2)}{2}\big]\alpha_{_L}(t)b_{_L}(t)-\nu_1\Big]\|u_t\|^2\\
&-\nu_1\rho(t)\Big[\ell-\frac{[1+(1+c^2_{_{\rho\beta}})k^2]}{2\delta}\Big]\|\nabla u\|^2-\nu_1\rho(t)\|u\|^{p+1}_{p+1} \qquad (3.16)\\
&+\frac{\nu_1\lambda\delta}{2}\rho(t)(\mu g^r\circ\nabla u)+\frac{\rho(t)\alpha_{_L}(t)}{2}\big(g'\circ\nabla u\big)+\frac{\beta'(t)}{\beta(t)}L(t).
\end{aligned}$$

At this point, we choose δ large enough such that

$$\frac{[1+(1+c^2_{_{\rho\beta}})k^2]}{2\delta}<\ell.$$

Therefore, using assumption (A_2), the estimate (3.16) reduces to

$$\begin{aligned}
L'(t)\leq &-\rho(t)\Big[\big[1-\frac{\delta(\nu_1+\nu_2)}{2}\big]c_{\alpha b}-\nu_1\Big]\|u_t\|^2\\
&-\nu_1\rho(t)\Big[\ell-\frac{[1+(1+c^2_{_{\rho\beta}})k^2]}{2\delta}\Big]\|\nabla u\|^2-\nu_1\rho(t)\|u\|^{p+1}_{p+1} \qquad (3.17)\\
&+\frac{\nu_1\lambda\delta}{2}\rho(t)(\mu g^r\circ\nabla u)+\frac{\rho(t)\alpha_{_L}(t)}{2}\big(g'\circ\nabla u\big)+\frac{\beta'(t)}{\beta(t)}L(t).
\end{aligned}$$

We then choose ν_1 and ν_2 so small that $\big[1-\frac{\delta(\nu_1+\nu_2)}{2}\big]c_{\alpha b}-\nu_1>0$. Hence, there exists a positive constant k^* satisfying

$$\begin{aligned}
&\big[1-\frac{\delta(\nu_1+\nu_2)}{2}\big]c_{\alpha b}-\nu_1\geq\frac{k^*}{2},\quad \nu_1\Big[\ell-\frac{[1+(1+c^2_{_{\rho\beta}})k^2]}{2\delta}\Big]\geq\frac{k^*}{2},\\
&\text{and}\quad \nu_1\geq\frac{k^*}{p+1}.
\end{aligned}$$

Hence, (3.17) yields

$$\begin{aligned}
L'(t)\leq &-k^*\rho(t)\Big[\frac{1}{2}\big[\|u_t\|^2+\|\nabla u\|_2^2\big]+\frac{1}{p+1}\|u\|^{p+1}_{p+1}\Big]+\frac{\nu_1\lambda\delta}{2}\rho(t)(\mu g^r\circ\nabla u)\\
&+\frac{\rho(t)\alpha_{_L}(t)}{2}\big(g'\circ\nabla u\big)+\frac{\beta'(t)}{\beta(t)}L(t).
\end{aligned}$$

By the use of the integrating factor $\frac{1}{\beta(t)}$ and assumption (A_4), we get

$$\begin{aligned}\Big[\frac{1}{\beta(t)}L(t)\Big]' \leq &- k^*\frac{\rho(t)}{\beta(t)}\Big[\frac{1}{2}\big[\|u_t\|^2+\|\nabla u\|_2^2\big]+\frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big]\\ &+\frac{\rho(t)}{\beta(t)}\Big[\frac{\nu_1\lambda\delta}{2}\Big](\mu g^r\circ\nabla u)-\frac{1}{2}\frac{\rho(t)}{\beta(t)}\alpha_L(t)\big(\mu g^r\circ\nabla u\big).\end{aligned} \tag{3.18}$$

From Remark 2.7 and (2.2), we see that, for $\varrho>1$,

$$\begin{aligned}E^{\varrho}(t) \leq & k_1E^{\varrho-1}(0)\Big[\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big]+k_2\big[(g\circ\nabla u)\big]^{\varrho}\\ \leq & k_1E^{\varrho-1}(0)\Big[\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big]\\ &+\frac{k_3}{\mu^{\varrho}(t)}\Big[\int_0^t\mu(s)g^{1-\sigma}(s)\|\nabla u(t)-\nabla u(t-s)\|_2^2ds\Big]^{\frac{\varrho(r-1)}{[r-1+\sigma]}}\\ &\times\Big[\big(\mu g^r\circ\nabla u\big)(t)\Big]^{\sigma\varrho/[r-1+\sigma]},\end{aligned} \tag{3.19}$$

where k_1,k_2,k_3 are positive constants. For the last term on the right side of (3.19), we have

$$\begin{aligned}&\int_0^t\mu(s)g^{1-\sigma}(s)\|\nabla u(t)-\nabla u(t-s)\|_2^2ds\\ &\leq C_1\int_0^t\mu(s)g^{1-\sigma}(s)\big[\|\nabla u(t)\|_2^2+\|\nabla u(t-s)\|_2^2\big]ds\\ &\leq C_2\int_0^t\mu(s)g^{1-\sigma}(s)\big[E(t)+E(t-s)\big]ds\\ &\leq 2C_2\int_0^t\mu(s)g^{1-\sigma}(s)E(t-s)ds\leq C_3E(0)\int_0^t\mu(s)g^{1-\sigma}(s)ds,\end{aligned} \tag{3.20}$$

consequently, (3.19) takes the form

$$\begin{aligned}E^{\varrho}(t) \leq & k_1E^{\varrho-1}(0)\Big[\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big]\\ &+\frac{k_4}{\mu^{\varrho}(t)}E^{\frac{\varrho(r-1)}{[r-1+\sigma]}}(0)\Big[\int_0^{\infty}\mu(s)g^{1-\sigma}(s)ds\Big]^{\frac{\varrho(r-1)}{[r-1+\sigma]}}\Big[\big(\mu g^r\circ\nabla u\big)(t)\Big]^{\frac{\sigma\varrho}{[r-1+\sigma]}}.\end{aligned} \tag{3.21}$$

Choosing $\sigma=\frac{1}{2}$ and $\varrho=2r-1$ in (3.21), then we have, for some $k_5,k_6>0$,

$$\Big[\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big]\geq k_5E^{2r-1}(t)-\frac{k_6}{\mu^{2r-1}(t)}\big(\mu g^r\circ\nabla u\big)(t). \tag{3.22}$$

Substituting (3.22) in (3.18), we obtain

$$\begin{aligned} \Big[\frac{1}{\beta(t)}L(t)\Big]' \leq & -k_7\frac{\rho(t)}{\beta(t)}E^{2r-1}(t) \\ & + \frac{\rho(t)}{\mu^{2r-1}(t)\beta(t)}\Big[\frac{\nu_1\lambda\delta}{2}\mu^{2r-1}(0)+k_8-\frac{1}{2}\alpha_{_L}(t)\mu^{2r-1}(t)\Big]\big(\mu g^r\circ\nabla u\big). \end{aligned} \tag{3.23}$$

Case 1: If $\alpha_{_L}(t)\mu^{2r-1}(t)\geq c_\mu$ and $\frac{\nu_1\lambda\delta}{2}\mu^{2r-1}(0)+k_8\leq\frac{c_\mu}{2}$, then the estimate (3.23) reduces to

$$\Big[\frac{1}{\beta(t)}L(t)\Big]'\leq -k_7\frac{\rho(t)}{\beta(t)}E^{2r-1}(t). \tag{3.24}$$

So (3.2) and (3.24) give

$$\Big[\frac{1}{\beta(t)}L(t)\Big]'\leq -k_9\frac{\rho(t)}{\beta(t)}\Big[\frac{1}{\beta(t)}L(t)\Big]^{2r-1}. \tag{3.25}$$

Define $G(t)$ by

$$G(t):=\frac{1}{\beta(t)}L(t).$$

Then (3.25) reduces to

$$G'(t)\leq -\frac{k_9\rho(t)}{\beta(t)}G^{2r-1}(t). \tag{3.26}$$

Integrating (3.26) over $[0,t]$ and using the fact that $G(t)$ is equivalent to $E(t)$, we obtain

$$E(t)\leq C\Big[1+\int_0^t\frac{\rho(s)}{\beta(s)}ds\Big]^{\frac{-1}{2(r-1)}} \qquad \text{for all } t\geq 0.$$

Case 2: If $\alpha_{_L}(t)\mu^{2r-1}(t)<c_\mu$, then multiplying the estimate (3.23) by $\mu^{2r-1}(t)$ and using (2.3) together with assumption (A_4) lead to

$$\begin{aligned} \Big[\frac{\mu^{2r-1}(t)}{\beta(t)}L(t)\Big]' &\leq -k_7\frac{\mu^{2r-1}(t)\rho(t)}{\beta(t)}E^{2r-1}(t)+k_{10}\frac{\rho(t)}{\beta(t)}\big(\mu g^r\circ\nabla u\big) \\ &\leq -k_7\frac{\mu^{2r-1}(t)\rho(t)}{\beta(t)}E^{2r-1}(t)-\frac{k_{11}\rho(t)}{\beta(t)}E'(t). \end{aligned} \tag{3.27}$$

Since $E'\leq 0$ and we have from assumption (A_5) that $\frac{\rho(t)}{\beta(t)}\leq c_{\alpha_0}$ where $c_{\alpha_0}=c(\alpha_L(0))$, then rearranging (3.27), we get

$$\Big[\frac{\mu^{2r-1}(t)}{\beta(t)}L(t)+k_{12}E(t)\Big]'\leq -\frac{k_7\rho(t)\mu^{2r-1}(t)}{\beta(t)}E^{2r-1}(t). \tag{3.28}$$

Let

$$F(t):=\frac{\mu^{2r-1}(t)}{\beta(t)}L(t)+k_{12}E(t).$$

Recalling (3.2), it is easy to show that $F(t)\sim E(t)$. Hence, the estimate (3.28) reduces to

$$F'(t)\leq -\frac{k_{13}\rho(t)\mu^{2r-1}(t)}{\beta(t)}F^{2r-1}(t). \tag{3.29}$$

Integrating (3.29) over $(0, t)$, and using the fact that $F(t) \sim E(t)$, we obtain the following estimate:

$$E(t) \leq C\Big[1 + \int_0^t \frac{\mu^{2r-1}(s)\rho(s)}{\beta(s)} ds\Big]^{\frac{-1}{2(r-1)}} \qquad \text{for all } t \geq 0.$$

Case 3: if $\int_0^\infty \mu(s)E(t-s)ds < \infty$, then using (3.20) in (3.19), with $\sigma = 1$, we obtain

$$\begin{aligned} E^\varrho(t) \leq & k_1 E^{\varrho-1}(0)\Big[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big] \\ & + \frac{k_{14}}{\mu^\varrho(t)}\Big\{\int_0^t \mu(s)E(t-s)ds\Big\}^{\frac{\varrho(r-1)}{r}} \Big[\big(\mu g^r \circ \nabla u\big)(t)\Big]^{\varrho/r}. \end{aligned}$$

Setting $\varrho = r$, we obtain

$$\Big[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big] \geq k_{15}E^r(t) - \frac{k_{16}}{\mu^r(t)}\big(\mu g^r \circ \nabla u\big)(t). \tag{3.30}$$

Substituting (3.30) in (3.18), we get

$$\begin{aligned} \Big[\frac{1}{\beta(t)}L(t)\Big]' \leq & -k_{17}\frac{\rho(t)}{\beta(t)}E^r(t) \\ & + \frac{\rho(t)}{\mu^r(t)\beta(t)}\Big[\frac{\nu_1\lambda\delta}{2\mu^{-r}(0)} + k_{18} - \frac{1}{2}\alpha_{_L}(t)\mu^r(t)\Big]\big(\mu g^r \circ \nabla u\big). \end{aligned} \tag{3.31}$$

If $\alpha_{_L}(t)\mu^r(t) \geq c_\mu$ and $\frac{\nu_1\lambda\delta\mu^r(0)}{2} + k_{18} \leq \frac{c_\mu}{2}$, then the estimate (3.31) reduces to

$$\Big[\frac{1}{\beta(t)}L(t)\Big]' \leq -k_{17}\frac{\rho(t)}{\beta(t)}E^r(t).$$

Following the same argument as in Case 1, we have the estimate

$$E(t) \leq C\Big[1 + \int_0^t \frac{\rho(s)}{\beta(s)} ds\Big]^{\frac{-1}{r-1}}, \qquad t \geq 0.$$

Likewise, in the case where $\alpha_{_L}(t)\mu^r(t) < c_\mu$, multiplying the estimate (3.31) by $\mu^r(t)$ and following the same argument as in Case 2, we obtain

$$E(t) \leq C\Big[1 + \int_0^t \frac{\mu^r(s)\rho(s)}{\beta(s)} ds\Big]^{\frac{-1}{(r-1)}}, \qquad t \geq 0.$$

□

Remark 3.4. Our result extends the result of [20] to the case of non-necessarily exponentially decaying relaxation functions.

Examples

Assume that $b(t, x) \approx C(1+t)^{-m}$ for $m = 1$. Then

$$\frac{\rho(t)}{\beta(t)} \approx C(L+t)^{-1} \quad \text{and} \quad \alpha_L(t) \approx C(L+t).$$

(i) Let $g(t) = \nu(1+t)^{-\omega}$, where $0 < \nu < 1$ and $\omega > 2$ are constants chosen so that $\int_0^\infty g(t) < 1$. Then

$$g'(t) = -\omega\nu^{-1/\omega}\big(\nu(1+t)^{-\omega}\big)^{1+\frac{1}{\omega}} = -\omega\nu^{-1/\omega}g^{1+\frac{1}{\omega}},$$

with $r = 1+\frac{1}{\omega}$, $\mu(t) = \omega\nu^{-1/\omega}$ and there exists a positive constant C such that $\mu^{1+\frac{2}{\omega}}(t)\alpha_L(t) \geq C$. Hence, from (3.8), the energy of the solution satisfies

$$\begin{aligned} E(t) \leq& C[1 + \int_0^t (1+s)^{-1}ds]^{\frac{-\omega}{2}} \\ =& C[1+\ln(1+t)]^{\frac{-\omega}{2}}. \end{aligned}$$

Since $\int_0^\infty \varrho\nu^{-1/\omega}E(t-s)ds < \infty$, we have

$$E(t) \leq C[1+\ln(1+t)]^{-\omega}$$

(ii) Let $g(t) = \nu(e+t)^{-\sigma}\big(\ln(e+t)\big)^{-\varrho}$, where $\varrho > \sigma > 2$ and $\nu > 0$ are constants chosen so that $\int_0^\infty g(t) < 1$. Then

$$\begin{aligned} g'(t) =& -\nu(e+t)^{-(\sigma+1)}\big(\varrho(\ln(e+t))^{-(\varrho+1)} + \sigma(\ln(e+t))^{-\varrho}\big) \\ \leq& -\nu C(e+t)^{-(\sigma+1)}(\ln(e+t))^{-(\varrho+1)} \\ =& -\nu^{\frac{-1}{\varrho}}C(e+t)^{-1+\frac{\sigma}{\varrho}}\Big(\nu(e+t)^{-\sigma}\big(\ln(e+t)\big)^{-\varrho}\Big)^{1+\frac{1}{\varrho}} \\ =& -\nu^{\frac{-1}{\varrho}}C(e+t)^{-1+\frac{\sigma}{\varrho}}g^{1+\frac{1}{\varrho}}(t), \end{aligned}$$

where $C = \min\{\sigma,\varrho\}$, $r = 1+\frac{1}{\varrho}$, and $\mu(t) = \nu^{\frac{-1}{\varrho}}C(e+t)^{-1+\frac{\sigma}{\varrho}}$ since $\alpha_L(t)\mu^{2r-1}(t) \approx C(L+t)^{\frac{\varrho(\sigma-2)+2\sigma}{\varrho^2}} \geq CL^{\frac{\varrho(\sigma-2)+2\sigma}{\varrho^2}}$ and $\varrho(\sigma-2)+2\sigma > 0$. Then, from (3.8), we have

$$\begin{aligned} E(t) \leq& C[1 + \int_0^t (1+s)^{-1}ds]^{\frac{-\varrho}{2}} \\ =& C[1+\ln(1+t)]^{\frac{-\varrho}{2}}. \end{aligned}$$

Remark 3.5. Observe that for $b(t,x) \approx C(1+t)^{-m}$, where $0 < m < 1$, $\frac{\rho(t)}{\beta(t)} \approx (1+s)^{-2+m}$, the energy behavior for $|x| \leq L+t$ satisfies

$$\begin{aligned} E(t) \leq& C[1 + \int_0^t (1+s)^{-2+m}ds]^{\frac{-1}{2(r-1)}} \\ \leq& C\Big[1-(1+t)^{-1+m}\Big]^{\frac{-1}{2(r-1)}}, \end{aligned}$$

whenever $\mu^{2r-1}(t)\alpha_L(t) \geq C$. Furthermore, within the ball ($|x| \leq L$), $\beta(t) \approx$ *constant* and $\rho(t) \approx C(1+t)^{-m}$, and in the first case; that is,

- when $g(t) = \nu(1+t)^{-\omega}$, $\mu(t) = \omega\nu^{-1/\omega}$ and $\frac{\mu^r(t)}{\rho(t)} \approx C(1+t)^m > c_\mu$. Then from (3.8), we have $\int_0^\infty \mu(s)E(t-s)ds < \infty$, and the energy decay satisfies

$$E(t) \leq C[1+(1+t)^{1-m}]^{-\omega}.$$

In the second case,

- $g(t) = \nu(e+t)^{-\sigma}\big(\ln(e+t)\big)^{-\varrho}$, where $\varrho > \sigma > 2$ and $\nu > 0$, $\mu(t) = \nu^{\frac{-1}{\varrho}}C(e+t)^{-1+\frac{\sigma}{\varrho}}$, and there exists a constant C such that $\frac{\mu^r(t)}{\rho(t)} < C$ if $(\sigma-1)\varrho+\sigma < (1-m)\varrho^2$. Then, (3.11) gives the estimate
$$E(t) \le C[1+(e+t)^{-m+\frac{\varrho(\sigma-1)+\sigma}{\varrho^2}}]^{-\varrho}$$
since $\int_0^\infty \mu(s)E(t-s)ds < \infty$.

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