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ON BETTER APPROXIMATION ORDER FOR THE NONLINEAR BLEIMANN–BUTZER–HAHN OPERATOR OF MAXIMUM PRODUCT KIND

SEZİN ÇİT[1] AND OGÜN DOĞRU[1*]

Dedicated to Professor Akif D. Gadjiev on the occasion of his 85 th birthday with high esteem.

Communicated by J.M. Aldaz

ABSTRACT. Using maximum instead of sum, a nonlinear Bleimann–Butzer–Hahn operator of maximum product kind was introduced. The present paper deals with the approximation processes for this operator. In a previous study, it was indicated that the order of approximation of this operator to a function f under the modulus is $\frac{(x+1)^{\frac{3}{2}}\sqrt{x}}{\sqrt{n}}$ and that it could not be improved except for some subclasses of functions. Contrary to this claim, under some special conditions, we show that a better order of approximation can be obtained with the help of classical and weighted modulus of continuities.

1. Introduction

For $f \in C[0,\infty)$, the classical Bleimann–Butzer–Hahn (BBH) operators defined as

$$L_n(f;x) = \frac{1}{(1+x)^n}\sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k}x^k,$$

were introduced in [13].

The construction logic of nonlinear maximum product type operators using the maximum instead of the sum is based on the studies [11, 12, 24] (for details, see also [9]).

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*Corresponding author.
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There are some other remarkable articles like [5–8, 10] that we will remind you in chronological order that various maximum product type nonlinear operators was introduced and that their approximation and rate of convergence properties were investigated. Also, some statistical approximation properties of maximum product type operators were given by Duman [18].

Especially, in [7], the approximation properties, rate of convergence, and shape preserving properties of BBH operator of maximum product kind are examined.

At this point, let us recall the following well-known concept of classical modulus of continuity:

$$\omega(f,\delta)=\max\{|f(x)-f(y)|:\ x,y\in I,\ \ |x-y|\leq\delta\}. \tag{1.1}$$

The order of approximation for the maximum product type BBH operator can be found in [7] by means of the modulus of continuity as $\omega\left(f;(1+x)^{\frac{3}{2}}\sqrt{x/n}\right)$. Also, Bede et al. indicated that the order of approximation under the modulus was $(1+x)^{\frac{3}{2}}\sqrt{x/n}$, and it could not be improved except for some subclasses of functions (see, for details, [7]).

Contrary to this claim, under some special conditions, we will show that a better order of approximation can be obtained with the help of classical and weighted modulus of continuities.

2. The concept of nonlinear maximum product operators

Before giving the main results, we will recall basic definitions and theorems about nonlinear operators given in [9–11].

Over the set of $\mathbb{R}_+$, we consider the operations $\vee$(maximum) and $''\cdot''$ product. Then $(\mathbb{R}_+,\vee,\cdot)$ has a semiring structure, and it is called a maximum product algebra.

Let $I\subset\mathbb{R}$ be bounded or unbounded interval, and

$$CB_+=\{f:I\to\mathbb{R}_+ : f \text{ continuous and bounded on } I\}.$$

Let us take the general form of $L_n: CB_+(I)\to CB_+(I)$, as

$$L_n(f)(x)=\bigvee_{i=0}^{n}K_n(x,x_i)\,f(x_i)\ ,$$

or

$$L_n(f)(x)=\bigvee_{i=0}^{\infty}K_n(x,x_i)\,f(x_i),$$

where $n\in\mathbb{N}$, $f\in CB_+(I)$, $K_n(\cdot,x_i)\in CB_+(I)$ and $x_i\in I$, for all i. These operators are nonlinear, positive operators, and moreover they satisfy the following pseudo-linearity condition of the form

$$L_n(\alpha f\vee\beta g)(x)=\alpha\,L_n(f)(x)\vee\beta\,L_n(g)(x)\,,\qquad \text{for all } \alpha,\beta\in\mathbb{R}_+,\ \ f,g:I\to\mathbb{R}_+.$$

In this section, we present some general results on these kinds of operators which will be used later.

Lemma 2.1 ([10]). *Let $I \subset \mathbb{R}$ be a bounded or an unbounded interval, let*

$$CB_{+} = \{ f : I \to \mathbb{R}_{+} : f \text{ continuous and bounded on } I \},$$

and let $L_n : CB_{+}(I) \to CB_{+}(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the following properties:

(i) *If $f, g \in CB_{+}(I)$ satisfy $f \leq g$, then $L_n(f) \leq L_n(g)$ for all $n \in \mathbb{N}$.*
(ii) *$L_n(f+g) \leq L_n(f) + L_n(g)$ for $f, g \in CB_{+}(I)$.*

Then for all $f, g \in CB_{+}(I)$, $n \in \mathbb{N}$, and $x \in I$, we have

$$|L_n(f)(x) - L_n(g)(x)| \leq L_n(|f-g|)(x).$$

Remark 2.2. (1) It is easy to see that the nonlinear BBH maximum product operator satisfies the conditions (i) and (ii) of Lemma 2.1. In fact, instead of (i), it also satisfies the following stronger condition:

$$L_n(f \vee g)(x) = L_n(f)(x) \vee L_n(g)(x), \qquad f, g \in CB_{+}(I).$$

Indeed, taking into consideration of the equality above, for $f \leq g$, $f, g \in CB_{+}(I)$, it easily follows $L_n(f)(x) \leq L_n(g)(x)$.

(2) In addition to this, it is positive homogeneous; that is, $L_n(\lambda f) = \lambda L_n(f)$ for all $\lambda \geq 0$.

After this point, let us denote the monomials $e_r(x) := x^r$, $r \in \mathbb{N}_0$. The first three monomials are also called the Korovkin test functions.

Corollary 2.3 ([10]). *Let $L_n : CB_{+}(I) \to CB_{+}(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the conditions* (i), (ii) *in Lemma 2.1 and in addition being positive homogeneous. Then for all $f \in CB_{+}(I)$, $n \in \mathbb{N}$, and $x \in I$, we have*

$$\begin{aligned} |L_n(f)(x) - f(x)| \ &\leq \ \left[\frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x) \right] \omega(f, \delta) \\ &\quad + f(x) \, |L_n(e_0)(x) - 1|, \end{aligned}$$

where $\omega(f, \delta)$ is the classical modulus of continuity defined by (1.1), $\delta > 0$, and $\varphi_x(t) = |t - x|$ for all $t \in I$, $x \in I$,. Moreover, if I is unbounded, then we suppose that there exists $L_n(\varphi_x)(x) \in \mathbb{R}_{+} \cup \{\infty\}$, for any $x \in I$, $n \in \mathbb{N}$.

A consequence of Corollary 2.3, we have the following result.

Corollary 2.4 ([10]). *Suppose that in addition to the conditions in Corollary 2.3, the sequence $(L_n)_n$ satisfies $L_n(e_0) = e_0$, for all $n \in \mathbb{N}$. Then for all $f \in CB_{+}(I)$, $n \in \mathbb{N}$, and $x \in I$, we have*

$$|L_n(f)(x) - f(x)| \leq \left[1 + \frac{1}{\delta} L_n(\varphi_x)(x) \right] \omega(f, \delta),$$

where $\omega(f, \delta)$ is the classical modulus of continuity defined by (1.1) and $\delta > 0$.

3. Nonlinear BBH operator of maximum product kind

As we reminded in the first part, in the maximum product kind operators, the sum operator $\sum$ is replaced by the $\bigvee$ maximum operator.

So, the following maximum product type nonlinear BBH operator is introduced by Bede, Coroianu, and Gal [7]:

$$H_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} \binom{n}{k} x^k f\left(\frac{k}{n+1-k}\right)}{\bigvee_{k=0}^{n} \binom{n}{k} x^k}, \tag{3.1}$$

where $f \in C[0,\infty)$, $x \in [0,\infty)$, and $n \in \mathbb{N}$.

In [7], the approximation and shape preserving properties of the operator $H_n^{(M)}(f)(x)$ were also examined.

Lemma 3.1 ([7])**.** *For any arbitrary bounded function $f : [0,\infty) \to [0,\infty)$, the max-product operator $H_n^{(M)}(f)(x)$ is positive, bounded, continuous on $[0,\infty)$ and satisfies $H_n^{(M)}(f)(0) = f(0)$.*

Remark 3.2. It is clear that $H_n^{(M)}(f)(x)$ satisfies all conditions in Lemma 2.1, Corollaries 2.3 and 2.4 for $I = [0,\infty)$.

4. Auxiliary results

By Lemma 3.1, we know that $H_n^{(M)}(f)(0) - f(0) = 0$ for all n, so in this part, we will consider $x > 0$ in the notations, proofs, and statements of the all approximation result.

For all $k \in \{0,1,2,\ldots,n\}$, $j \in \{0,1,2,\ldots,n-1\}$, and $x \in \left[\frac{j}{n-j+1}, \frac{j+1}{n-j}\right]$ or $j = n$ and $x \in [0,\infty)$, $M_{k,n,j}(x)$ and $m_{k,n,j}(x)$ were defined in [7] by

$$M_{k,n,j}(x) := \frac{s_{n,k}(x)\left|\frac{k}{n+1-k} - x\right|}{s_{n,j}(x)},$$

$$m_{k,n,j}(x) := \frac{s_{n,k}(x)}{s_{n,j}(x)}.$$

It is clear that if $k \geq j+1$, then we get

$$M_{k,n,j}(x) = \frac{s_{n,k}(x)\left(\frac{k}{n+1-k} - x\right)}{s_{n,j}(x)},$$

and if $k \leq j$, then we have

$$M_{k,n,j}(x) = \frac{s_{n,k}(x)\left(x - \frac{k}{n+1-k}\right)}{s_{n,j}(x)},$$

where $s_{n,k}(x) = \binom{n}{k} x^k$.

The main result of this part is Lemma 4.2, which is proved by the induction method. Note that this proof is different from the proofs in [7].

Lemma 4.1 ([7])**.** *For all $k \in \{0,1,2,\dots,n\}$, $j \in \{0,1,2,\dots,n-1\}$, and $x \in \left[\frac{j}{n-j+1},\frac{j+1}{n-j}\right]$ or $j=n$ and $x \in [0,\infty)$, we have*

$$m_{k,n,j}(x) \le 1.$$

Lemma 4.2. (i) *Let $j \in \{0,1,2,\dots,n-1\}$, let $x \in \left[\frac{j}{n-j+1},\frac{j+1}{n-j}\right]$, and let $\alpha \in \{2,3,\dots\}$. If*

$$k \in \{j+1,j+2,\dots,n-1\}$$

is such that $k-(k+1)^{1/\alpha} \ge j$, then

$$M_{k,n,j}(x) \ge M_{k+1,n,j}(x).$$

(ii) *Let $j \in \{1,2,\dots,n-1\}$, let $x \in \left[\frac{j}{n-j+1},\frac{j+1}{n-j}\right]$, and let $\alpha \in \{2,3,\dots\}$. If $k \in \{1,2,\dots,j\}$ is such that $k+(k)^{1/\alpha} \le j$, then*

$$M_{k,n,j}(x) \ge M_{k-1,n,j}(x).$$

Proof. (i) After simple calculations, we obtain

$$\begin{aligned}
\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} &= \frac{s_{n,k}(x)\left(\frac{k}{n+1-k}-x\right)}{s_{n,j}(x)} \frac{s_{n,j}(x)}{s_{n,k+1}(x)\left(\frac{k+1}{n+1-k-1}-x\right)} \\
&= \frac{\binom{n}{k}x^k}{\binom{n}{k+1}x^{k+1}} \frac{\frac{k}{n+1-k}-x}{\frac{k+1}{n-k}-x} \\
&= \frac{n!}{k!\,(n-k)!} \frac{(k+1)!\,(n-k-1)!}{n!} \frac{1}{x} \frac{\frac{k}{n+1-k}-x}{\frac{k+1}{n-k}-x} \\
&= \frac{k+1}{n-k} \frac{1}{x} \frac{\frac{k}{n+1-k}-x}{\frac{k+1}{n-k}-x}.
\end{aligned}$$

If we denote

$$g_{n,k}(x) := \frac{1}{x} \frac{\frac{k}{n+1-k}-x}{\frac{k+1}{n-k}-x},$$

then the function $g_{n,k}(x)$ is nonincreasing. Really, since

$$x \le \frac{j+1}{n-j} \le \frac{k}{n-j},$$

we have

$$\begin{aligned}
g'_{n,k}(x) &= -\frac{1}{x^2} \frac{\frac{k}{n+1-k}-x}{\frac{k+1}{n-k}-x} + \frac{1}{x} \frac{-\left(\frac{k+1}{n-k}-x\right)+\left(\frac{k}{n+1-k}-x\right)}{\left(\frac{k+1}{n-k}-x\right)^2} \\
&= -\frac{1}{x^2} \frac{\frac{k}{n+1-k}-x}{\frac{k+1}{n-k}-x} + \frac{1}{x} \frac{\frac{-nk-n-k-1+k^2+k+nk-k^2}{(n-k)(n+1-k)}}{\left(\frac{k+1}{n-k}-x\right)^2} \\
&= -\frac{1}{x^2} \frac{\frac{k}{n+1-k}-x}{\frac{k+1}{n-k}-x} - \frac{1}{x} \frac{\frac{n+1}{(n-k)(n+1-k)}}{\left(\frac{k+1}{n-k}-x\right)^2} < 0.
\end{aligned}$$

Since $x \leq \frac{j+1}{n-j}$, we obtain

$$\begin{aligned} g_{n,k}(x) &\geq g_{n,k}\left(\frac{j+1}{n-j}\right) \\ &= \frac{n-j}{j+1}\frac{\frac{k}{n+1-k}-\frac{j+1}{n-j}}{\frac{k+1}{n-k}-\frac{j+1}{n-j}} \end{aligned}$$

for all $x \in \left[\frac{j}{n+1-j}, \frac{j+1}{n-j}\right]$. So we have

$$\begin{aligned} \frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} &\geq \frac{k+1}{n-k}\frac{n-j}{j+1}\frac{\frac{k}{n+1-k}-\frac{j+1}{n-j}}{\frac{k+1}{n-k}-\frac{j+1}{n-j}} \\ &= \frac{k+1}{j+1}\frac{\frac{n-j}{n+1-k}k-(j+1)}{k+1-\frac{n-k}{n-j}(j+1)}. \end{aligned}$$

Let

$$\begin{aligned} h_{k,j}(n) &= \frac{\frac{n-j}{n+1-k}k-(j+1)}{k+1-\frac{n-k}{n-j}(j+1)} \\ &= \frac{\frac{nk-kj-nj-n-j-1+kj+k}{n+1-k}}{\frac{nk+n-kj-j-nj-n+kj+k}{n-j}} \\ &= \frac{n(k-j-1)+k-j-1}{n+1-k}\frac{n-j}{k(n+1)-j(n+1)} \\ &= \frac{(k-j-1)(n+1)}{n+1-k}\frac{n-j}{(n+1)(k-j)} \\ &= \frac{k-j-1}{k-j}\frac{n-j}{n+1-k}. \end{aligned}$$

Then

$$\begin{aligned} h'_{k,j}(n) &= \frac{k-j-1}{k-j}\frac{(n+1-k)-(n-j)}{(n+1-k)^2} \\ &= \frac{-1}{k-j}\frac{(k-j-1)^2}{(n+1-k)^2} < 0, \end{aligned}$$

so $h_{k,j}$ is nonincreasing, and we have

$$\begin{aligned} \frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} &\geq \lim_{n\to\infty}\frac{k+1}{j+1}\frac{\frac{n-j}{n+1-k}k-(j+1)}{k+1-\frac{n-k+1}{n-j}(j+1)} \\ &= \frac{(k+1)(k-j-1)}{(j+1)(k-j)}. \end{aligned}$$

Now, by the induction method, let us show that

$$\frac{k+1}{j+1}\frac{k-j-1}{k-j} \geq 1 \tag{4.1}$$

for $k-(k+1)^{1/\alpha} \geq j$.

For $\alpha = 2$, since the condition $k-(k+1)^{1/2} \geq j$ holds, we have $(k-j)^2 \geq k+1$. So we get $(k+1)(k-j-1) \geq (j+1)(k-j)$ and $k^2 - kj - 1 \geq kj - j^2 + k$. Therefore we obtain the inequality (4.1) for $\alpha = 2$.

Now, we assume that the inequality (4.1) is provided for $\alpha - 1$. It follows $\frac{k+1}{j+1}\frac{k-j-1}{k-j} \geq 1$ when $k-(k+1)^{1/(\alpha-1)} \geq j$. This means that

$$(k-j)^{\alpha-1} \geq k+1 \Rightarrow (k-j)^{\alpha} \geq (k+1)(k-j).$$

Since $k \geq j+1$, $(k-j)^{\alpha} \geq (k+1)(k-j) \geq k+1$ is true for α, hence, for arbitrary $\alpha = 2, 3, \ldots$, the inequality (4.1) is provided when $k-(k+1)^{1/\alpha} \geq j$.

Since $k \geq j+1$, we have $\frac{n-j}{n-k} \geq 1$. So we obtain

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} \geq \frac{k+1}{n-k}\frac{n-j}{j+1}\frac{k-j-1}{k-j} \geq 1.$$

(ii) After simple calculations, we get

$$\begin{aligned}
\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} &= \frac{s_{n,k}(x)\left(x-\frac{k}{n+1-k}\right)}{s_{n,j}(x)}\frac{s_{n,j}(x)}{s_{n,k-1}(x)\left(x-\frac{k-1}{n+1-k}\right)} \\
&= \frac{\binom{n}{k}x^k}{\binom{n}{k-1}x^{k-1}}\frac{\frac{k}{n+1-k}-x}{\frac{k-1}{n+1-k+1}-x} \\
&= \frac{n!}{k!\,(n-k)!}\frac{(k-1)!\,(n-k+1)!}{n!}x\frac{x-\frac{k}{n+1-k}}{x-\frac{k-1}{n+2-k}} \\
&= \frac{n-k+1}{k}x\frac{x-\frac{k}{n+1-k}}{x-\frac{k-1}{n+2-k}}.
\end{aligned}$$

If we denote

$$g^*_{n,k}(x) := x\frac{x-\frac{k}{n+1-k}}{x-\frac{k-1}{n+2-k}},$$

then we see that the function $g^*_{n,k}(x)$ is nondecreasing. Really, since

$$\begin{aligned}
\frac{k}{n+1-k}-\frac{k-1}{n+2-k} &> \frac{k}{n+2-k}-\frac{k-1}{n+2-k} \\
&= \frac{1}{n+2-k} > 0
\end{aligned}$$

and

$$x \geq \frac{k}{n+1-k},$$

we have

$$\left(g^*_{n,k}\right)'(x) = \frac{x-\frac{k}{n+1-k}}{x-\frac{k-1}{n+2-k}} + x\frac{x-\frac{k-1}{n+2-k}-x+\frac{k}{n+1-k}}{\left(x-\frac{k-1}{n+2-k}\right)^2} > 0.$$

Using this property, since $x \geq \frac{j}{n-j+1}$, we obtain

$$g^*_{n,k}(x) \geq g^*_{n,k}\left(\frac{j}{n-j+1}\right)$$

$$= \frac{j}{n-j+1}\frac{\frac{j}{n-j+1}-\frac{k}{n+1-k}}{\frac{j}{n-j+1}-\frac{k-1}{n+2-k}}$$

for all $x \in \left[\frac{j}{n+1-j}, \frac{j+1}{n-j}\right]$. We have

$$\begin{aligned}
\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} &\geq \frac{n-k+1}{k}\frac{j}{n-j+1}\frac{\frac{j}{n-j+1}-\frac{k}{n+1-k}}{\frac{j}{n-j+1}-\frac{k-1}{n+2-k}} \\
&= \frac{j}{k}\frac{\frac{n-k+1}{n-j+1}j-k}{j-\frac{n-j+1}{n-k+2}(k-1)}.
\end{aligned}$$

Let

$$\begin{aligned}
h^*_{k,j}(n) &= \frac{\frac{n-k+1}{n-j+1}j-k}{j-\frac{n-j+1}{n-k+2}(k-1)} \\
&= \frac{\frac{nj-kj+j-nk+kj-k}{n-j+1}}{\frac{nj-kj+2j-nk+n+kj-j-k+1}{n-k+2}} \\
&= \frac{j(n+1)-k(n+1)}{n-j+1}\frac{n-k+2}{n(j-k)+j-k+n+1} \\
&= \frac{(n+1)(j-k)}{n-j+1}\frac{n-k+2}{(n+1)(j-k+1)} \\
&= \frac{j-k}{j-k+1}\frac{n-k+2}{n-j+1}.
\end{aligned}$$

Then

$$\begin{aligned}
\left(h^*_{k,j}\right)'(n) &= \frac{j-k}{j-k+1}\frac{n-j+1-n+k-2}{(n-j+1)^2} \\
&= \frac{j-k}{j-k+1}\frac{k-j-1}{(n-j+1)^2} \\
&= -\frac{j-k}{(n-j+1)^2} < 0,
\end{aligned}$$

and we obtain

$$\begin{aligned}
\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} &\geq \lim_{n\to\infty}\frac{j}{k}\frac{\frac{n-k+1}{n-j+1}j-k}{j-\frac{n-j+1}{n-k+2}(k-1)} \\
&= \frac{j}{k}\frac{j-k}{j-k+1}.
\end{aligned}$$

Now, as the same as proof of (i), using the induction method, let us show that

$$\frac{j}{k}\frac{j-k}{j-k+1} \geq 1 \tag{4.2}$$

holds for $k+(k)^{1/\alpha} \leq j$.

For $\alpha = 2$, because of the condition $k + (k)^{1/2} \leq j$, we get $k \leq (j-k)^2$ and $j(j-k) \geq k(j-k+1)$. Thus we have $j^2 - kj \geq kj - k^2 + k$, and so we see that (4.2) is satisfied.

Now, we assume that (4.2) is correct for $\alpha - 1$. Since $\frac{j}{k}\frac{j-k}{j-k+1} \geq 1$ for $k + (k)^{1/(\alpha-1)} \leq j$, we obtain $k \leq (j-k)^{\alpha-1} \Rightarrow k(j-k) \leq (j-k)^{\alpha}$.

Since $k \leq j-1$, $k \leq k(j-k) \leq (j-k)^{\alpha}$ is true for $\alpha > 1$, the desired inequality is provided for $k + (k)^{1/\alpha} \leq j$. So we obtain

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} \geq \frac{n-k-1}{k}\frac{j}{n+1-j}\frac{j-k}{j-k+1} \geq 1,$$

which gives the desired result. □

At this point, we also recall the following lemma.

Lemma 4.3 ([7])**.** *Denoting $s_{n,k}(x) = \binom{n}{k}x^k$, we have*

$$\bigvee_{k=0}^{n} s_{n,k}(x) = s_{n,j}(x), \quad \text{for all } x \in \left[\frac{j}{n-j+1}, \frac{j+1}{n-j}\right],\ j = 0, 1, \ldots, n-1$$

and

$$\bigvee_{k=0}^{n} s_{n,k}(x) = s_{n,n}(x) \quad \text{if } x \in [n, \infty).$$

5. Pointwise rate of convergence

Let us take x_0 as fixed point on the interval $[0, \infty)$. The main aim of this section is to obtain a better order of pointwise approximation for the operators $H_n^{(M)}(f)(x_0)$ to the function $f(x_0)$ by means of the classical modulus of continuity. According to the following theorem, we can say that the order of pointwise approximation can be improved when the α is big enough. Moreover if we choose $\alpha = 2$, these approximation results turn out to be the results in [7].

Theorem 5.1. *Let $f : [0, \infty) \to \mathbb{R}_+$ be continuous. Then for any fixed point x_0 on the interval $[0, \infty)$ satisfying*

$$(n+1)^{\alpha-1} \geq \max\left\{[2(1+x_0)]^{\alpha-1}, 2^{\alpha+2}x_0(1+x_0)^{\alpha-1}\right\},$$

we have the following order of approximation for the operators (3.1) to the function f by means of the modulus of continuity:

$$\left|H_n^{(M)}(f)(x_0) - f(x_0)\right| \leq (1 + 8(1+x_0)^{2-\frac{1}{\alpha}} x_0^{\frac{1}{\alpha}})\ \omega\left(f; \frac{1}{(n+1)^{1-\frac{1}{\alpha}}}\right),$$

for all $n \in \mathbb{N}$, where $\omega(f; \delta)$ is the classical modulus of continuity defined by (1.1) and $\alpha = 2, 3, \ldots$.

Proof. Since nonlinear maximum product BBH operators satisfy the conditions in Corollary 2.4, for any $x_0 \in [0, \infty)$, using the properties of $\omega(f; \delta)$, we get

$$\left|H_n^{(M)}(f)(x_0) - f(x_0)\right| \leq \left[1 + \frac{1}{\delta_n} H_n^{(M)}(\varphi_{x_0})(x_0)\right] \omega(f, \delta), \tag{5.1}$$

where $\varphi_{x_0}(t) = |t - x_0|$. At this point, let us denote

$$E_n(x_0) := H_n^{(M)}(\varphi_{x_0})(x_0) = \frac{\bigvee_{k=0}^{n} s_{n,k}(x_0)\left|\frac{k}{n+1-k} - x_0\right|}{\bigvee_{k=0}^{n} s_{n,k}(x_0)}, \qquad x_0 \in [0, \infty).$$

Let $x_0 \in \left[\frac{j}{n-j+1}, \frac{j+1}{n-j}\right]$, where $j \in \{0, 1, \ldots, n-1\}$ is fixed, arbitrary. By Lemma 4.3 we easily obtain

$$E_n(x_0) = \max_{k=0,1,\ldots,n} \{M_{k,n,j}(x_0)\}, \qquad x_0 \in \left[\frac{j}{n-j+1}, \frac{j+1}{n-j}\right].$$

Firstly let us examine for $j = 0$, where $x_0 \in \left[0, \frac{1}{n}\right]$ and $\alpha = 2, 3, \ldots$.

For $k = 0$, we get

$$M_{0,n,0}(x_0) = x_0 = x_0^{\frac{1}{\alpha}} x_0^{1-\frac{1}{\alpha}} \leq \frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}.$$

Also, for $k \geq 1$, we get

$$\begin{aligned} M_{k,n,0}(x_0) &= \binom{n}{k} x_0^k \left(\frac{k}{n+1-k} - x_0\right) \\ &\leq \binom{n}{k} x_0^k \frac{k}{n+1-k} \\ &= \frac{n!}{(n-k)!k!} x_0^k \frac{k}{n+1-k} \\ &= \frac{n!}{(n+1-k)!\,(k-1)!} x_0^k \\ &= \binom{n}{k-1} x_0^{k-1} x_0. \end{aligned}$$

Since $(1+x_0)^n = 1 + \sum_{k=1}^{\infty} \binom{n}{k} x_0^k$, then $\binom{n}{k-1} x_0^{k-1} \leq (1+x_0)^n$. Using this result in the last inequality and since $x_0 \leq \frac{1}{n}$, we obtain

$$\begin{aligned} M_{k,n,0}(x_0) &\leq (1+x_0)^n x_0 \\ &\leq \left(1 + \frac{1}{n}\right)^n x_0 \\ &\leq e\, x_0^{\frac{1}{\alpha}} x_0^{1-\frac{1}{\alpha}} \\ &\leq e\, \frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}. \end{aligned}$$

So, we find an upper estimate for any $k = 0, 1, \ldots, n$, $E_n(x_0) \leq e \frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}$ when $j = 0$.

Now, it remains to find an upper estimate for each $M_{k,n,j}(x_0)$ when $j = 1,2,\ldots,n-1$, is fixed, $x_0 \in \left[\frac{j}{n-j+1}, \frac{j+1}{n-j}\right]$, $k \in \{0,1,\ldots,n\}$ and $\alpha = 2,3,\ldots$.

In fact, we will prove that

$$M_{k,n,j}(x_0) \le 8\frac{(1+x_0)^{2-\frac{1}{\alpha}} x_0^{\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}} \tag{5.2}$$

for all $x_0 \in \left[\frac{j}{n-j+1}, \frac{j+1}{n-j}\right]$, $k = 0,1,2,\ldots,n$, which directly will imply that

$$E_n(x_0) \le 8\frac{(1+x_0)^{2-\frac{1}{\alpha}} x_0^{\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}}, \qquad \text{for all } n \in \mathbb{N}.$$

Taking $\delta_n = \frac{1}{(n+1)^{1-\frac{1}{\alpha}}}$ in (5.1), we obtain the estimate in the statement immediately.

So, in order to completing the proof of (5.2), we consider the following cases:

(i*) $k \in \{j+1, j+2, \ldots, n\}$ with $j \in \{1,2,\ldots,n-1\}$,

(ii*) $k \in \{0,1,\ldots,j\}$ with $j \in \{1,2,\ldots,n-1\}$.

Case (i*). Subcase (a). Assume first that $k-(k+1)^{\frac{1}{\alpha}} \le j$. Then we get

$$\begin{aligned} M_{k,n,j}(x_0) &= m_{k,n,j}(x_0)\left(\frac{k}{n+1-k} - x_0\right) \\ &\le \frac{k}{n+1-k} - x_0 \\ &\le \frac{k}{n+1-k} - \frac{j}{n-j+1} \\ &= \frac{nk-kj+k-nj+kj-j}{(n-j+1)(n+1-k)} \\ &= \frac{(k-j)(n+1)}{(n-j+1)(n+1-k)}. \end{aligned}$$

Now, using $k-(k+1)^{\frac{1}{\alpha}} \le j \Longrightarrow k-j \le (k+1)^{\frac{1}{\alpha}}$ and $-k \ge -(k+1)^{\frac{1}{\alpha}} - j$, we have

$$\begin{aligned} M_{k,n,j}(x_0) &\le \frac{(k+1)^{\frac{1}{\alpha}}(n+1)}{(n-j+1)(n+1-k)} \\ &\le \frac{(k+1)^{\frac{1}{\alpha}}(n+1)}{(n-j+1)\left(n+1-j-(k+1)^{\frac{1}{\alpha}}\right)}. \end{aligned}$$

Taking into account that $\frac{j}{n-j+1} \le x_0$, we have $\frac{n+1}{n-j+1} \le 1+x_0$. Then we obtain

$$M_{k,n,j}(x_0) \le (1+x_0)\frac{(k+1)^{\frac{1}{\alpha}}}{\left(n+1-j-(k+1)^{\frac{1}{\alpha}}\right)}.$$

We observe that $k-(k+1)^{\frac{1}{\alpha}} \leq j$ gives $k+1 \leq 4j$. In fact, if we assume that $k+1>4j$, then $4j-1-(4j)^{\frac{1}{\alpha}} < k-(k+1)^{\frac{1}{\alpha}} \leq j$, which implies $3j-1<(4j)^{\frac{1}{\alpha}}$, and this is false if $j \geq 1$. Also, since $\frac{j}{n-j+1} \leq x_0$, we have $j \leq \frac{(n+1)x_0}{1+x_0}$, and for $n+1>j+(4j)^{\frac{1}{\alpha}}$, we get

$$\begin{aligned}
M_{k,n,j}(x_0) &\leq (1+x_0)\frac{(4j)^{\frac{1}{\alpha}}}{\left(n+1-j-(4j)^{\frac{1}{\alpha}}\right)} \\
&\leq 4^{\frac{1}{\alpha}}(1+x_0)\frac{\left[\frac{(n+1)x_0}{1+x}\right]^{\frac{1}{\alpha}}}{n+1-\frac{(n+1)x_0}{1+x_0}-4^{\frac{1}{\alpha}}\left[\frac{(n+1)x_0}{1+x_0}\right]^{\frac{1}{\alpha}}} \\
&= 4^{\frac{1}{\alpha}}(1+x_0)^{1-\frac{1}{\alpha}}\frac{x_0^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}}{\frac{n+1}{1+x_0}-4^{\frac{1}{\alpha}}\left[\frac{(n+1)x_0}{1+x_0}\right]^{\frac{1}{\alpha}}} \\
&= 4^{\frac{1}{\alpha}}(1+x_0)^{1-\frac{1}{\alpha}}\frac{x_0^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}(1+x_0)}{n+1-4^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}(1+x_0)^{1-\frac{1}{\alpha}}} \\
&= 4^{\frac{1}{\alpha}}(1+x_0)^{2-\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}\frac{(n+1)^{\frac{1}{\alpha}}}{n+1-4^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}(1+x_0)^{1-\frac{1}{\alpha}}}.
\end{aligned}$$

If $(n+1)^{\alpha-1} \geq 2^{2+\alpha}x_0(1+x_0)^{\alpha-1}$, then we observe that

$$\frac{(n+1)^{\frac{1}{\alpha}}}{n+1-4^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}(1+x_0)^{1-\frac{1}{\alpha}}} \leq \frac{2}{(n+1)^{1-\frac{1}{\alpha}}}.$$

Indeed,

$$n+1 \leq 2n+2-4^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}2(1+x_0)^{1-\frac{1}{\alpha}},$$

$$\begin{aligned}
4^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}2(1+x_0)^{1-\frac{1}{\alpha}} &\leq n+1, \\
4^{\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}2(1+x_0)^{1-\frac{1}{\alpha}} &\leq (n+1)^{1-\frac{1}{\alpha}}, \\
2^{\alpha}4\ x_0(1+x_0)^{\alpha-1} &\leq (n+1)^{\alpha-1}.
\end{aligned}$$

Also, the same condition ensures $n+1>j+(4j)^{\frac{1}{\alpha}}$. Finally we obtain

$$\begin{aligned}
M_{k,n,j}(x_0) &\leq 4^{\frac{1}{\alpha}}(1+x_0)^{2-\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}\frac{2}{(n+1)^{1-\frac{1}{\alpha}}} \\
&= 2^{1+\frac{2}{\alpha}}\frac{(1+x_0)^{2-\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}},
\end{aligned}$$

for any $(n+1)^{\alpha-1} \geq 2^{2+\alpha}x_0(1+x_0)^{\alpha-1}$.

Subcase (b). Assume now that $k-(k+1)^{\frac{1}{\alpha}} > j$. For the function $g(x_0) := x_0-(x_0+1)^{\frac{1}{\alpha}}$, we have

$$\left.\frac{dg(x)}{dx}\right|_{x=x_0} = g'(x_0) = 1-\left(1/\alpha\,(x_0+1)^{1-\frac{1}{\alpha}}\right) > 0.$$

Thus we can say that the function g is nondecreasing on the interval $[0,\infty)$. It follows that there exists a maximum value $\bar{k}\in\{0,1,2,\ldots,n\}$ satisfying the inequality $\bar{k}-\left(\bar{k}+1\right)^{\frac{1}{\alpha}} \le j$. Then, for $k_1=\bar{k}+1$, we have $k_1-(k_1+1)^{\frac{1}{\alpha}} > j$. Also, we have $k_1 \ge j+1$. Indeed, this is a consequence of the fact that g is nondecreasing, and it is easy to see that $f(j) = j-(x_0+1)^{\frac{1}{\alpha}} < j$. Then we get

$$\begin{aligned}
M_{\bar{k}+1,n,j}(x_0) &= m_{\bar{k}+1,n,j}(x_0)\left(\frac{\bar{k}+1}{n-\bar{k}+1-1}-x_0\right)\\
&\le \frac{\bar{k}+1}{n-\bar{k}}-\frac{j}{n-j+1}\\
&= \frac{n\bar{k}+n-j\bar{k}-j+\bar{k}+1-nj+j\bar{k}}{\left(n-\bar{k}\right)(n-j+1)}\\
&= \frac{(n+1)\left(\bar{k}-j+1\right)}{\left(n-\bar{k}\right)(n-j+1)}.
\end{aligned}$$

Since $\frac{n+1}{n-j+1} \le x_0+1$ and $\bar{k}-j \le \left(\bar{k}+1\right)^{\frac{1}{\alpha}}$, we have

$$\begin{aligned}
M_{\bar{k}+1,n,j}(x_0) &\le \frac{(x_0+1)\left(\bar{k}-j+1\right)}{\left(n-\bar{k}\right)}\\
&\le \frac{(x_0+1)\left(\left(\bar{k}+1\right)^{\frac{1}{\alpha}}+1\right)}{\left(n-\bar{k}\right)}\\
&\le \frac{2(x_0+1)\left(\bar{k}+1\right)^{\frac{1}{\alpha}}}{\left(n-\bar{k}\right)}.
\end{aligned}$$

Moreover, $k_1 \le 4j$ and similar to subcase $a)$ and $j \le \frac{x_0(n+1)}{1+x_0}$, we obtain

$$\begin{aligned}
M_{\bar{k}+1,n,j}(x_0) &\le \frac{2(x_0+1)(4j)^{\frac{1}{\alpha}}}{\left(n-j-(4j)^{\frac{1}{\alpha}}\right)}\\
&\le \frac{4^{\frac{1}{\alpha}}2(x_0+1)\left(\frac{x_0(n+1)}{1+x_0}\right)^{\frac{1}{\alpha}}}{n-\frac{x_0(n+1)}{1+x_0}-4^{\frac{1}{\alpha}}\left(\frac{x_0(n+1)}{1+x_0}\right)^{\frac{1}{\alpha}}}\\
&= \frac{2^{1+\frac{2}{\alpha}}(x_0+1)^{2-\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}}{n-x_0-4^{\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}(x_0+1)^{1-\frac{1}{\alpha}}}.
\end{aligned}$$

We observe that if $(n+1)^{\alpha-1} \geq 2^{\alpha+2} x_0 (1+x_0)^{\alpha-1}$, then

$$\frac{(n+1)^{\frac{1}{\alpha}}}{n - x_0 - 4^{\frac{1}{\alpha}} x_0^{\frac{1}{\alpha}} (n+1)^{\frac{1}{\alpha}} (x_0+1)^{1-\frac{1}{\alpha}}} \leq \frac{2}{(n+1)^{1-\frac{1}{\alpha}}}.$$

Indeed,

$$n+1 \leq 2n - 2x_0 - 4^{\frac{1}{\alpha}} x_0^{\frac{1}{\alpha}} 2 (n+1)^{\frac{1}{\alpha}} (x_0+1)^{1-\frac{1}{\alpha}},$$

$$\begin{aligned} 4^{\frac{1}{\alpha}} x_0^{\frac{1}{\alpha}} 2 (n+1)^{\frac{1}{\alpha}} (x_0+1)^{1-\frac{1}{\alpha}} &\leq n - 1 - 2x_0 \leq n+1, \\ 4^{\frac{1}{\alpha}} x^{\frac{1}{\alpha}} 2 (x_0+1)^{1-\frac{1}{\alpha}} &\leq (n+1)^{1-\frac{1}{\alpha}}, \\ 2^{\alpha} 4 x_0 (x_0+1)^{\alpha-1} &\leq (n+1)^{\alpha-1}. \end{aligned}$$

Using the last inequality, eventually we get

$$\begin{aligned} M_{\bar{k}+1,n,j}(x) &\leq \frac{2^{1+\frac{2}{\alpha}} 2 (x+1)^{2-\frac{1}{\alpha}} x^{\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}} \\ &= \frac{2^{2+\frac{2}{\alpha}} (x+1)^{2-\frac{1}{\alpha}} x^{\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}}. \end{aligned}$$

By Lemma 4.2, (i), it follows that

$$M_{\bar{k}+1,n,j}(x_0) \geq M_{\bar{k}+2,n,j}(x_0) \geq \cdots \geq M_{n,n,j}(x_0).$$

Thus we have $M_{k,n,j}(x_0) \leq \frac{2^{2+\frac{2}{\alpha}} (x_0+1)^{2-\frac{1}{\alpha}} x_0^{\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}}$ for any $k \in \left\{\bar{k}+1, \bar{k}+2, \ldots, n\right\}$.

Case (ii*). Subcase (a). Assume first that $k + k^{\frac{1}{\alpha}} > j$. Then we obtain

$$\begin{aligned} M_{k,n,j}(x_0) &= m_{k,n,j}(x_0) \left(x_0 - \frac{k}{n+1-k} \right) \\ &\leq x_0 - \frac{k}{n+1-k} \\ &\leq \frac{j+1}{n-j} - \frac{k}{n+1-k} \\ &= \frac{nj + n + j + 1 - kj - k - nk + kj}{(n-j)(n+1-k)} \\ &= \frac{-k(n+1) + j(n+1) + n + 1}{(n-j)(n+1-k)} \\ &= \frac{(n+1)(j-k+1)}{(n-j)(n+1-k)}. \end{aligned}$$

Since $j - k < k^{\frac{1}{\alpha}}$, $j \geq k$, and $\frac{j}{n-j+1} \leq x_0$, we have $j \leq \frac{x_0(n+1)}{x_0+1}$. When we use these inequalities, respectively, we could have

$$M_{k,n,j}(x_0) \leq \frac{(n+1)\left(k^{\frac{1}{\alpha}}+1\right)}{(n-j)\left(n+1-k^{\frac{1}{\alpha}}-j\right)}$$

$$
\begin{aligned}
&\leq \frac{(n+1)\left(j^{\frac{1}{\alpha}}+1\right)}{(n-j)\left(n+1+j^{\frac{1}{\alpha}}-j\right)} \\
&\leq \frac{2(n+1)\ j^{\frac{1}{\alpha}}}{(n-j)\left(n+1+j^{\frac{1}{\alpha}}-j\right)} \\
&\leq \frac{2(n+1)\left(\frac{x_0(n+1)}{x_0+1}\right)^{\frac{1}{\alpha}}}{\left(n-\frac{x_0(n+1)}{x_0+1}\right)\left(n+1+\left(\frac{x_0(n+1)}{x_0+1}\right)^{\frac{1}{\alpha}}-\frac{x_0(n+1)}{x_0+1}\right)} \\
&= \frac{2(n+1)^{1+\frac{1}{\alpha}}\, x_0^{\frac{1}{\alpha}}\,(1+x_0)^{-\frac{1}{\alpha}}}{\left(\frac{nx_0+n-nx_0-x_0}{x_0+1}\right)\left(\frac{nx_0+x_0+n+1-nx_0-x_0}{x_0+1}+\frac{x_0^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}(x_0+1)^{1-\frac{1}{\alpha}}}{x_0+1}\right)} \\
&= \frac{2(n+1)^{1+\frac{1}{\alpha}}\, x_0^{\frac{1}{\alpha}}\,(1+x_0)^{2-\frac{1}{\alpha}}}{(n-x_0)\left(n+1+x_0^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}(x_0+1)^{1-\frac{1}{\alpha}}\right)}.
\end{aligned}
$$

Now, we observe that

$$\frac{(n+1)^{1+\frac{1}{\alpha}}}{\left(n+1+x_0^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}(x_0+1)^{1-\frac{1}{\alpha}}\right)} \leq (n+1)^{\frac{1}{\alpha}}.$$

Indeed,

$$
\begin{aligned}
n+1 &\leq n+1+x_0^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}(x_0+1)^{1-\frac{1}{\alpha}}, \\
0 &\leq x_0^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}(x_0+1)^{1-\frac{1}{\alpha}}.
\end{aligned}
$$

So, we have

$$M_{k,n,j}(x_0) \leq 2x_0^{\frac{1}{\alpha}}(1+x_0)^{2-\frac{1}{\alpha}}\frac{(n+1)^{\frac{1}{\alpha}}}{(n-x_0)}.$$

Also, if $n \geq 1+2x_0$, that is $(n+1)^{\alpha-1} \geq [2(1+x_0)]^{\alpha-1}$, then we have

$$\frac{(n+1)^{\frac{1}{\alpha}}}{(n-x_0)} \leq \frac{2}{(n+1)^{1-\frac{1}{\alpha}}}.$$

So we get

$$1+2x_0 \leq n.$$

Finally we obtain

$$
\begin{aligned}
M_{k,n,j}(x_0) &\leq 2x_0^{\frac{1}{\alpha}}(1+x_0)^{2-\frac{1}{\alpha}}\frac{2}{(n+1)^{1-\frac{1}{\alpha}}} \\
&= 4\frac{x_0^{\frac{1}{\alpha}}(1+x_0)^{2-\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}}.
\end{aligned}
$$

Subcase (b). Assume now that $k+k^{\frac{1}{\alpha}} \leq j$. Let $\tilde{k} \in \{0,1,2,\ldots,n\}$ be the minimum value such that $\tilde{k}+\left(\tilde{k}\right)^{\frac{1}{\alpha}} > j$. Then $k_2=\tilde{k}-1$ satisfies $k_2+(k_2)^{\frac{1}{\alpha}} < j$ and

$$\begin{aligned}
M_{\tilde{k}-1,n,j}\left(x_0\right) &= m_{\tilde{k}-1,n,j}\left(x_0\right)\left(x_0-\frac{\tilde{k}-1}{n-\tilde{k}+2}\right) \\
&\leq x_0-\frac{\tilde{k}-1}{n-\tilde{k}+2} \\
&\leq \frac{j+1}{n-j}-\frac{\tilde{k}-1}{n-\tilde{k}+2} \\
&= \frac{nj+n-\tilde{k}j-\tilde{k}+2j+2-n\tilde{k}+n+\tilde{k}j-j}{(n-j)\left(n-\tilde{k}+2\right)} \\
&= \frac{(n+1)\left(j-\tilde{k}+2\right)}{(n-j)\left(n-\tilde{k}+2\right)} \\
&\leq \frac{(n+1)\left(\left(\tilde{k}\right)^{\frac{1}{\alpha}}+2\right)}{(n-j)\left(n+\left(\tilde{k}\right)^{\frac{1}{\alpha}}-j+2\right)} \\
&\leq \frac{3(n+1)\left(\tilde{k}\right)^{\frac{1}{\alpha}}}{(n-j)\left(n+\left(\tilde{k}\right)^{\frac{1}{\alpha}}-j+2\right)} \\
&\leq \frac{3(n+1)(j)^{\frac{1}{\alpha}}}{(n-j)\left(n+(j)^{\frac{1}{\alpha}}-j\right)}.
\end{aligned}$$

Since $j \leq \frac{x_0(n+1)}{x_0+1}$, we have

$$\begin{aligned}
M_{\tilde{k}-1,n,j}\left(x_0\right) &\leq \frac{3(n+1)\left(\frac{x_0(n+1)}{x_0+1}\right)^{\frac{1}{\alpha}}}{\left(n-\frac{x_0(n+1)}{x_0+1}\right)\left(n+\left(\frac{x_0(n+1)}{x_0+1}\right)^{\frac{1}{\alpha}}-\frac{x_0(n+1)}{x_0+1}\right)} \\
&= \frac{3(n+1)^{1+\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}\left(x_0+1\right)^{-\frac{1}{\alpha}}}{\left(\frac{nx_0+n-nx_0-x_0}{x_0+1}\right)\left(\frac{nx_0+n-nx_0-x_0}{x_0+1}+\frac{x_0^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}(x_0+1)^{1-\frac{1}{\alpha}}}{x_0+1}\right)} \\
&= 3x_0^{\frac{1}{\alpha}}\left(x_0+1\right)^{2-\frac{1}{\alpha}}\frac{(n+1)^{1+\frac{1}{\alpha}}}{\left(n-x_0\right)\left(n-x_0+x_0^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}\left(x_0+1\right)^{1-\frac{1}{\alpha}}\right)}.
\end{aligned}$$

After simple computations, we observe that

$$\frac{(n+1)^{1+\frac{1}{\alpha}}}{n-x_0+x_0^{\frac{1}{\alpha}}(n+1)^{\frac{1}{\alpha}}(x_0+1)^{1-\frac{1}{\alpha}}} \leq (n+1)^{\frac{1}{\alpha}}.$$

So we obtain

$$M_{\tilde{k}-1,n,j}(x_0) \leq 3x_0^{\frac{1}{\alpha}}(x_0+1)^{2-\frac{1}{\alpha}}\frac{(n+1)^{\frac{1}{\alpha}}}{n-x_0}.$$

Also, using $\frac{(n+1)^{\frac{1}{\alpha}}}{n-x_0} \leq \frac{2}{(n+1)^{1-\frac{1}{\alpha}}}$ in the last inequality, we get

$$\begin{aligned} M_{\tilde{k}-1,n,j}(x_0) &\leq 3x_0^{\frac{1}{\alpha}}(x_0+1)^{2-\frac{1}{\alpha}}\frac{2}{(n+1)^{1-\frac{1}{\alpha}}} \\ &= 6\frac{x_0^{\frac{1}{\alpha}}(x_0+1)^{2-\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}}. \end{aligned}$$

In the light of Lemma 4.2(ii), it follows that

$$M_{\tilde{k}-1,n,j}(x_0) \geq M_{\tilde{k}-2,n,j}(x_0) \geq \cdots \geq M_{0,n,j}(x_0).$$

Thus we obtain

$$M_{k,n,j}(x_0) \leq 6\frac{x_0^{\frac{1}{\alpha}}(x_0+1)^{2-\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}}$$

for any $k \leq j$ and $x_0 \in \left[\frac{j}{n-j+1}, \frac{j+1}{n-j}\right]$.

So, taking into consideration the fact that

$$\max\left\{\begin{array}{c} \frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}, e\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}, 2^{1+\frac{2}{\alpha}}\frac{(1+x_0)^{2-\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}}, 2^{2+\frac{2}{\alpha}}\frac{(x_0+1)^{2-\frac{1}{\alpha}}x_0^{\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}}, \\ 4\frac{x_0^{\frac{1}{\alpha}}(1+x_0)^{2-\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}}, 6\frac{x_0^{\frac{1}{\alpha}}(x_0+1)^{2-\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}} \end{array}\right\} \leq 8\frac{x_0^{\frac{1}{\alpha}}(x_0+1)^{2-\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}},$$

we have the desired result. □

6. Weighted rate of convergence

In the previous section, pointwise convergence properties of the operators were given at a fixed point x_0. Nevertheless, if we want to obtain a uniform approximation order on infinite intervals, then we should use the weighted modulus of continuities.

Before giving useful properties about these type of modulus of continuities, let us recall the following spaces and norm (see, for instance, [20, 21]):

$$\begin{aligned} B_\rho(\mathbb{R}) &= \{f : \mathbb{R} \to \mathbb{R} | \text{ a constant } M_f \text{ depending on } f \text{ exists} \\ &\quad \text{such that } |f| \leq M_f\rho\}, \end{aligned}$$

$$C_\rho(\mathbb{R}) = \{f \in B_\rho(\mathbb{R}) | \ f \text{ continuous on } \mathbb{R}\},$$

endowed with the norm

$$\|f\|_\rho = \sup_{x\geq 0 \geq} \frac{|f(x)|}{\rho(x)}.$$

In order to obtain the rate of weighted approximation of the positive linear operators defined on infinite intervals, various weighted modulus of continuities are introduced. Some of them include the term h in the denominator of the supremum expression. In the chronological order, let us refer to some related papers as [1, 4, 16, 19, 22, 23, 25, 27].

In order to obtain weighted approximation properties of Szász–Mirakjan operators on $\mathbb{R}_+$, a weighted modulus was defined in [1].

In [23], the second author, jointly with Gadjieva, introduced the following modulus of continuity:

$$\Omega(f;\delta) = \sup_{0\le x, |h|\le\delta} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}. \tag{6.1}$$

There are some papers including rates of weighted approximation with the help of $\Omega(f;\delta)$. (see, for instance, [3, 14, 17, 26]).

Then the second author [16] defined the following modulus of continuity:

$$\omega_\rho(f;\delta) = \sup_{0\le x, |h|\le\delta} \frac{|f(x+h)-f(x)|}{\rho(x+h)}, \tag{6.2}$$

where $\rho(x) \ge \max(1,x)$.

In [16], a generalization of the Gadjiev–Ibragimov operators was introduced, which includes many well-known operators, and its rate of weighted convergence with the help of $\omega_\rho(f;\delta)$ defined in (6.2) was obtained.

Moreno [27] introduced another type of modulus of continuity in (6.2) as follows:

$$\overline{\Omega}_\alpha(f;\delta) = \sup_{0\le x, |h|\le\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^\alpha}.$$

Gadjiev and Aral [22] defined the following modulus of continuity:

$$\widetilde{\Omega}_\rho(f;\delta) = \sup_{x,t\in\mathbb{R}_+, |\rho(t)-\rho(x)|\le\delta} \frac{|f(t)-f(x)|}{\left(|\rho(t)-\rho(x)|+1\right)\rho(x)},$$

where $\rho(0)=1$ and $\inf_{x\ge 0}\rho(x)\ge 1$.

By choosing $\alpha = 2$, in the definition of $\overline{\Omega}_\alpha(f;\delta)$, then we obtain $\overline{\Omega}_2(f;\delta) = \omega_{\rho_0}(f;\delta)$ for $\rho_0(x) = 1+x^2$, and if we choose $\alpha = 2+\lambda$ in the definition of $\overline{\Omega}_\alpha(f;\delta)$, then we obtain

$$\widehat{\Omega}_{\rho\lambda}(f;\delta) = \sup_{0\le x, |h|\le\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^{2+\lambda}}$$

(see [2]).

Finally, Holhoş [25] defined a more general weighted modulus of continuity as

$$\omega_\varphi(f;\delta) = \sup_{0\le x\le y, |\varphi(y)-\varphi(x)|\le\delta} \frac{|f(x)-f(y)|}{\rho(x)+\rho(y)}$$

such that, for $\varphi(x) = x$, this modulus of continuity is equivalent to $\Omega(f;\delta)$ defined in (6.1).

Also, let $C^0_\rho(\mathbb{R})$ be the subspace of all functions in $C_\rho(\mathbb{R})$ such that $\lim_{|x|\to\infty}\frac{f(x)}{\rho(x)}$ exists finitely.

Note also that some remarkable properties about these type of modulus of continuities can be found in [15].

In the light of these definitions, we can give the following theorem.

Theorem 6.1. *Let $f : [0,\infty) \to \mathbb{R}_+$ be continuous. Then for all $x \in [0,\infty)$, satisfying*

$$(n+1)^{\alpha-1} \geq \max\left\{[2(1+x)]^{\alpha-1}, 2^{\alpha+2}x(1+x)^{\alpha-1}\right\},$$

we have the following order of approximation for the operators (3.1) to the function f by means of the weighted modulus of continuity defined in (6.2). Then for each $f \in C^0_{\rho_0}(\mathbb{R}_+)$, we have

$$\frac{\left|H_n^{(M)}(f)(x) - f(x)\right|}{(\rho_0(x))^2} \leq \frac{(1+9x^2)(1+8(1+x)^{2-\frac{1}{\alpha}}x^{\frac{1}{\alpha}})}{(1+x^2)^2}\,\omega_{\rho_0}\left(f; \frac{1}{(n+1)^{1-\frac{1}{\alpha}}}\right), \tag{6.3}$$

for all $n \in \mathbb{N}$, where $\rho_0(x) = 1 + x^2$ and $\alpha = 2, 3, \ldots$.

Proof. By using the properties of $\omega_{\rho_0}(f;\delta)$, (see [27]), we can write

$$\begin{aligned}&\left|H_n^{(M)}(f)(x) - f(x)\right| \\ &\quad\leq \left(1 + (2x + H_n^{(M)}(e_1)(x))^2\right)\left(\frac{1}{\delta}H_n^{(M)}(\varphi_x)(x) + 1\right)\omega_{\rho_0}(f;\delta).\end{aligned} \tag{6.4}$$

In the proof of Theorem 5.1, under the condition

$$(n+1)^{\alpha-1} \geq \max\left\{[2(1+x)]^{\alpha-1}, 2^{\alpha+2}x(1+x)^{\alpha-1}\right\},$$

we obtain

$$H_n^{(M)}(\varphi_x)(x) \leq 8\frac{(1+x)^{2-\frac{1}{\alpha}}x^{\frac{1}{\alpha}}}{(n+1)^{1-\frac{1}{\alpha}}}, \qquad \text{for all } n \in \mathbb{N}. \tag{6.5}$$

On the other hand, since

$$\begin{aligned} H_n^{(M)}(e_1)(x) &= \frac{x\bigvee_{k=1}^{n}\frac{n!}{(n-k)!(k-1)!}x^{k-1}\frac{1}{n+1-k}}{\bigvee_{k=0}^{n}\binom{n}{k}x^k} \\ &= \frac{x\bigvee_{k=0}^{n-1}\binom{n}{k}x^k}{\bigvee_{k=0}^{n}\binom{n}{k}x^k}, \end{aligned}$$

we have

$$H_n^{(M)}(e_1)(x) \leq x. \tag{6.6}$$

So, using the inequalities (6.5) and (6.6) in (6.4) and choosing

$$\delta = \frac{1}{(n+1)^{1-\frac{1}{\alpha}}},$$

the proof is completed. ∎

This theorem allows us to express the following weighted uniform approximation result.

Theorem 6.2. *Let $f : [0, \infty) \to \mathbb{R}_+$ be continuous. Then for all $x \in [0, \infty)$, satisfying*

$$(n+1)^{\alpha-1} \geq \max\left\{[2(1+x)]^{\alpha-1}, 2^{\alpha+2} x (1+x)^{\alpha-1}\right\},$$

we have the following order of approximation for the operators (3.1) to the function f by means of the weighted modulus of continuity defined in (6.2). Then for each $f \in C^0_{\rho_0}(\mathbb{R}_+)$, we have

$$\left\| H_n^{(M)}(f)(x) - f(x) \right\|_{\rho_0^2(x)} \leq 170\ \omega_{\rho_0}\left(f; \frac{1}{(n+1)^{1-\frac{1}{\alpha}}}\right), \tag{6.7}$$

for all $n \in \mathbb{N}$, where $\rho_0(x) = 1 + x^2$ and $\alpha = 2, 3, \ldots$.

Proof. By using the inequalities $\frac{1}{1+x^2} \leq 1$, $\frac{x^2}{1+x^2} \leq 1$, $\frac{x^{\frac{1}{\alpha}}}{1+x^2} \leq 1$, and $\frac{x^{2-\frac{1}{\alpha}}}{1+x^2} \leq 1$, we have

$$\frac{(1+9x^2)(1+8(1+x)^{2-\frac{1}{\alpha}} x^{\frac{1}{\alpha}})}{(1+x^2)^2} \leq 170. \tag{6.8}$$

If we use (6.8) in (6.3), then we obtain the desired result. □

Remark 6.3. In [7], the order of approximation for nonlinear max-product BBH operators was found as $1/\sqrt{n}$ by means of modulus of continuity and the authors claimed that this order of approximation cannot be improving except for some subclasses of functions such as concave functions. So, Theorems 5.1, 6.1, and 6.2 show that the orders of pointwise approximation, weighted approximation, and weighted uniform approximation are $1/(n+1)^{1-\frac{1}{\alpha}}$. For big enough α, $1/(n+1)^{1-\frac{1}{\alpha}}$ tends to $1/(n+1)$. As a result, since $1 - \frac{1}{\alpha} \geq \frac{1}{2}$ for $\alpha = 2, 3, \ldots$, this selection of α improves the order of approximation.

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[1]Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey.

Email address: sezincit@gazi.edu.tr; ogun.dogru@gazi.edu.tr