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LIE IDEALS AND GENERALIZED DERIVATIONS IN PRIME RINGS AND BANACH ALGEBRAS

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ABSTRACT. Let R be a prime ring and let L be a noncentral Lie ideal of R. The main purpose of this paper is to describe generalized derivations of R satisfying some algebraic identities on L. Moreover, using a topological approach based on Baire's category theorem and some properties of functional analysis, our results have been extended to Banach algebras.

1. INTRODUCTION

Rings considered in this paper are associative and not necessarily unitary. For a ring R, we shall use $Z(R)$ to stand for the center of R. An ideal P of R is a prime ideal if $xRy \subseteq P$ yields $x \in P$ or $y \in P$. In particular, if the zero ideal of R is *prime*, then R is said to be a *prime ring*. For any $x, y \in R$, we will write $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the *Lie product* and *Jordan product*, respectively. An additive subgroup L of R is said to be a *Lie ideal* of R if $[x, r] \in L$ for all $x \in L$ and $r \in R$. An additive mapping $d : R \longrightarrow R$ is a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $F : R \longrightarrow R$ is a *generalized derivation* associated to a derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. A *Banach algebra* is a normed algebra whose underlying vector space is a Banach space. The closure of a subset X of a Banach algebra $\mathcal{A}$, denoted by $\overline{X}$, is the intersection of all closed subsets of $\mathcal{A}$ containing X. The interior of a subset X of a Banach algebra $\mathcal{A}$, denoted by $\overset{\circ}{X}$, is the largest open set contained in X. Equivalently, $\overset{\circ}{X}$ is the union of all open subsets of $\mathcal{A}$ contained in X.

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Numerous results in literature show how the global structure of a ring R is often tightly linked to the behavior of some special additive mappings defined on R. A popular result in this area is due to Posner [11] who proved that a prime ring equipped with a nonzero centralizing derivation is a commutative integral domain. This remarkable theorem of Posner has been influential, and it has played a key role in the development of various notions. This result was subsequently refined and extended by a number of algebraists. More specifically, they studied the commutativity of rings admitting suitably constrained generalized derivation verifying specific identities.

In [7, Theorem 2.7], it is demonstrated that if R is a prime ring of characteristic different from two, admitting two generalized derivations F_1 and F_2 such that $F_1(x)F_2(x) + F_2(x)F_1(x) = 0$ for all $x \in R$, then $F_1 = 0$ or $F_2 = 0$. An interesting result proved in [9, Theorem 2] by Hvala states that if F_1 and F_2 are two generalized derivations on a prime ring R of characteristic different from two, verifying $[F_1(x), F_2(x)] = 0$ for all $x \in R$, then there exists $\lambda \in C$ such that $F_1(x) = \lambda F_2(x)$ for all $x \in R$. Later, Demir, De Filippis, and Argaç [6] managed to get the same classification by only considering the main identity on a noncentral Lie ideal of a prime ring R, except possibly when R satisfies the standard identity s_4 of degree 4.

In [1, Theorem 3.1], it was shown that if $\mathcal{A}$ is a unital prime Banach algebra, F is a nonzero continuous generalized derivation with associated derivation d, and G_1 and G_2 are two nonvoid open subsets of $\mathcal{A}$ satisfying $F((xy)^m) - x^m y^m \in Z(\mathcal{A})$ or $F((xy)^m) - y^m x^m \in Z(\mathcal{A})$ for all $(x, y) \in G_1 \times G_2$ and $m = m(x, y) > 1$, then $\mathcal{A}$ is commutative under the additional assumption that $d(Z(\mathcal{A})) \neq 0$.

Motivated by the previous results, we here continue this line of investigation by describing generalized derivations satisfying some specific identities on a noncentral Lie ideal of a prime ring. Moreover, as an application, we study continuous generalized derivations satisfying similar algebraic identities locally on nonvoid open subsets of a prime Banach algebra $\mathcal{A}$. Our topological approach is based on Baire's category theorem and some properties of functional analysis.

2. Generalized derivations acting on Lie ideals

Before starting our results, we collect some well-known facts, which will be used frequently.

Fact 2.1. ([4, Main Theorem]). Let R be a prime ring of characteristic different from 2, let L be a noncentral Lie ideal of R, and let F be a generalized derivation of R such that $F(x) \in Z(R)$ for any $x \in L$. Then either $F = 0$ or R embeds in a 2×2 matrix ring over a field.

Fact 2.2. Let R be a noncommutative prime ring of characteristic different from 2 and let F be a generalized derivation of R such that $F(x) \in Z(R)$ for any $x \in R$. Then $F = 0$.

Fact 2.3. ([2, Lemma 2]). Let R be a prime ring of characteristic different from 2, let L be a Lie ideal of R, and let $C_R(L) = \{a \in R : [a, x] = 0 \ \forall x \in L\}$. If L is not central, then $C_R(L) = Z(R)$.

Fact 2.4. ([6]). Let R be a prime ring of characteristic different from 2, let U be its right Utumi quotient ring, let C be its extended centroid, and let L be a noncentral Lie ideal of R. Let $F : R \to R$ and $G : R \to R$ be nonzero generalized derivations on R. If $[F(u), G(u)] = 0$ for all $u \in L$, then one of the following conditions holds:

(1) There exists $\lambda \in C$ such that, for any $x \in R$, $G(x) = \lambda F(x)$;
(2) R satisfies s_4, the standard identity of degree 4.

Lemma 2.5. *Let R be a prime ring of characteristic different from 2, let L be a noncentral Lie ideal of R, and let F and G be generalized derivations of R such that $F(x)y + yG(x) = 0$ for all $x, y \in L$. Then either $F = G = 0$ or R embeds in a 2×2 matrix ring over a field.*

Proof. Suppose that R does not embed in a 2×2 matrix ring over a field and that

$$F(x)y + yG(x) = 0 \qquad \text{for all } x, y \in L. \tag{2.1}$$

By [2, Lemma 1], there exists a nonzero ideal I of R such that $[I, R] \subseteq L$. Replacing y by $[u, r]$ in (2.1) with $u \in I, r \in R$, we have

$$F(x)[u, r] + [u, r]G(x) = 0 \qquad \text{for all } u \in I, x \in L, r \in R. \tag{2.2}$$

Substituting u by ur, we get

$$F(x)[u, r]r + [u, r]rG(x) = 0 \qquad \text{for all } u \in I, x \in L, r \in R. \tag{2.3}$$

Right multiplying (2.2) by r and subtracting it from (2.3), we get $[u, r][G(x), r] = 0$ for all $u \in I, x \in L, r \in R$. Taking ut instead of u with $t \in I$, we get

$$[u, r]I[G(x), r] = 0 \qquad \text{for all } u \in I, x \in L, r \in R.$$

Using the primeness of R, we obtain $[G(x), r] = 0$ for all $x \in L$, $r \in R$, that is, $G(L) \subseteq Z(R)$. Applying Fact 2.1, we get $G = 0$, in which case, (2.1) yields $F = 0$. □

Theorem 2.6. *Let R be a prime ring of characteristic different from 2 and let L be a noncentral Lie ideal of R. If F_1, F_2, and F_3 are generalized derivations of R such that*

$$[F_1(x), y] = F_2(x)y + yF_3(x) \qquad \text{for all } x, y \in L,$$

then one of the following conditions holds:

(1) *There exist $\lambda, \mu \in C$ such that $F_2(x) = \lambda F_1(x)$, $F_3(x) = \mu F_1(x)$ for any $x \in R$;*
(2) *R embeds in a 2×2 matrix ring over a field.*

Proof. We are given that

$$[F_1(x), y] = F_2(x)y + yF_3(x) \qquad \text{for all } x, y \in L. \tag{2.4}$$

Assume that R does not embed in a 2×2 matrix ring over a field.

Firstly we point out that, if $F_1(L) \subseteq Z(R)$, then Fact 2.1 implies $F_1 = 0$ and relation (2.4) reduces to

$$F_2(x)y + yF_3(x) = 0 \qquad \text{for all } x, y \in L.$$

Invoking Lemma 2.5, it follows that $F_2 = F_3 = 0$.

Thus we may assume that $F_1(L) \nsubseteq Z(R)$ and consider $u_0 \in L$ such that $F_1(u_0) \notin Z(R)$. Setting $a = F_1(u_0)$, $b = F_2(u_0)$ and $c = F_3(u_0)$, then relation (2.4) yields

$$[a, y] = by + yc \qquad \text{for all } y \in L.$$

In particular,

$$[a, [l, r]] = b[l, r] + [l, r]c \qquad \text{for all } l \in L, r \in R. \tag{2.5}$$

Substituting r by rl, we get

$$[a, [l, r]]l + [l, r][a, l] = b[l, r]l + [l, r]lc \qquad \text{for all } l \in L, r \in R. \tag{2.6}$$

Right multiplying (2.5) by l and comparing with (2.6), we find that $[l, r][a, l] = [l, r][l, c]$ in such a way that

$$[l, r][a + c, l] = 0 \qquad \text{for all } l \in L, r \in R.$$

Accordingly, $a + c \in Z(R)$. In particular, this implies that $[F_1(u_0), F_3(u_0)] = 0$.

Furthermore, setting lr instead of r in (2.5), we get

$$[a, l][l, r] + l[a, [l, r]] = bl[l, r] + l[l, r]c \qquad \text{for all } l \in L, r \in R. \tag{2.7}$$

Left multiplying (2.5) by l and subtracting it from (2.7), we obtain $[a-b, l][l, r] = 0$, which assures that $a - b \in Z(R)$, and therefore $[F_1(u_0), F_2(u_0)] = 0$. Hence, in all cases, we have

$$[F_1(u), F_2(u)] = 0 \qquad \text{and} \qquad [F_1(u), F_3(u)] = 0 \qquad \text{for any } u \in L.$$

Applying Fact 2.4, there exist $\lambda, \mu \in C$ such that $F_2(x) = \lambda F_1(x)$ and $F_3(x) = \mu F_1(x)$ for any $x \in R$. $\square$

Remark 2.7. In the case where $[F_1(x), y] = F_2(x)y + yF_3(x)$ for all $x, y \in R$, one might expect R to be commutative. Instead, this would be false, as the following example proves:

For K a field, let $R = M_2(K)$, and let F be a generalized derivation of R. It is obvious that $F_1 = F_2 = F$ and $F_3 = -F$ verify the above relation although the ring is not commutative.

Proposition 2.8. *Let R be a noncommutative prime ring of characteristic different from 2. If F_1, F_2, and F_3 are generalized derivations of R such that*

$$[F_1(x), y] = F_2(x)y + yF_3(x), \qquad \text{for all } x, y \in R,$$

then $F_2 = F_1$ and $F_3 = -F_1$.

Proof. Assume that

$$[F_1(x), y] = F_2(x)y + yF_3(x) \qquad \text{for all } x, y \in R. \tag{2.8}$$

Taking yz instead of y in (2.8), we get

$$[F_1(x), y]z + y[F_1(x), z] = F_2(x)yz + yzF_3(x) \qquad \text{for all } x, y, z \in R. \tag{2.9}$$

Right multiplying relation (2.8) by z and subtracting it from (2.9), we obtain

$$y[F_1(x) + F_3(x), z] = 0, \qquad \text{for all } x, y, z \in R,$$

which assures that $(F_1 + F_3)(R) \subset Z(R)$. Hence, Fact 2.2 forces $F_1 = -F_3$. Therefore, (2.8) becomes $(F_1(x) - F_2(x))y = 0$ for all $x, y \in R$, which forces $F_1 = F_2$. □

Theorem 2.9. *Let R be a prime ring of characteristic different from 2 and let L be a noncentral Lie ideal of R. If F_1, F_2, and F_3 are generalized derivations of R satisfying*

$$F_1(x) \circ y = F_2(x)y + yF_3(x), \qquad \text{for all } x, y \in L,$$

then one of the following conditions holds:

(1) *There exist $\lambda, \mu \in C$ such that $F_2(x) = \lambda F_1(x)$, $F_3(x) = \mu F_1(x)$ for any $x \in R$;*
(2) *R embeds in a 2×2 matrix ring over a field.*

Proof. Assume that R does not embed in a 2×2 matrix ring over a field and that

$$F_1(x) \circ y = F_2(x)y + yF_3(x) \qquad \text{for all } x, y \in L. \tag{2.10}$$

If $F_1(L) \subseteq Z(R)$, then Fact 2.1 forces $F_1 = 0$, in which case, (2.10) reduces to

$$F_2(x)y + yF_3(x) = 0 \qquad \text{for all } x, y \in L.$$

Invoking Lemma 2.5, we conclude that $F_2 = F_3 = 0$.

Now, let us fix an element $u_0 \in L$ such that $F_1(u_0) \notin Z(R)$ and set $a = F_1(u_0)$, $b = F_2(u_0)$, and $c = F_3(u_0)$. In light of relation (2.10), we have

$$a \circ y = by + yc \qquad \text{for all } y \in L.$$

Therefore

$$a \circ [l, r] = b[l, r] + [l, r]c \qquad \text{for all } l \in L, r \in R. \tag{2.11}$$

Replacing r by rl, we get

$$(a \circ [l, r])l - [l, r][a, l] = b[l, r]l + [l, r]lc \qquad \text{for all } l \in L, r \in R. \tag{2.12}$$

Right multiplying (2.11) by l and subtracting it from (2.12), we arrive at $[l, r][a, l] = [l, r][l, c]$, which yields that

$$[l, r][a + c, l] = 0 \qquad \text{for all } l \in L, r \in R.$$

Accordingly, $a + c \in Z(R)$, and therefore $[F_1(u_0), F_3(u_0)] = 0$.

On the other hand, taking lr instead of r in (2.11), we get

$$l(a \circ [l, r]) + [a, l][l, r] = bl[l, r] + l[l, r]c \qquad \text{for all } l \in L, r \in R. \tag{2.13}$$

Left multiplying (2.11) by l and subtracting it from (2.13), we obtain $[a - b, l][l, r] = 0$, which assures that $a - b \in Z(R)$, and thus $[F_1(u_0), F_2(u_0)] = 0$. Hence, in all cases we find that

$$[F_1(u), F_2(u)] = 0 \qquad \text{and} \qquad [F_1(u), F_3(u)] = 0 \qquad \text{for any } u \in L.$$

Using Fact 2.4, there exist $\lambda, \mu \in C$ such that $F_2(x) = \lambda F_1(x)$ and $F_3(x) = \mu F_1(x)$ for any $x \in R$. □

Using similar arguments as in the proof of Proposition 2.8, we get the following result.

Proposition 2.10. *Let R be a noncommutative prime ring of characteristic different from 2. If F_1, F_2, and F_3 are generalized derivations of R satisfying*

$$F_1(x) \circ y = F_2(x)y + yF_3(x), \qquad \text{for all } x, y \in R,$$

then $F_1 = F_2 = F_3$.

3. Applications on prime Banach Algebras

Throughout this section, $\mathcal{A}$ denotes a real or complex Banach algebra. To prove our main results we need the following lemma.

Lemma 3.1 ([3]). *Let $\mathcal{A}$ be a Banach algebra, if $P(t) = \sum_{k=0}^{n} b_k t^k$ is a polynomial in the real variable t with coefficients in $\mathcal{A}$, and if for an infinite set of real values of t, $P(t) \in M$, where M is a closed linear subspace of $\mathcal{A}$, then every b_k lies in M.*

Theorem 3.2. *Let $\mathcal{A}$ be a prime Banach algebra, let O_1 and O_2 be nonvoid open subsets of $\mathcal{A}$, let F_1, F_2, and F_3 be nonzero continuous generalized derivations of $\mathcal{A}$, and let n be a fixed positive integer. If*

$$[F_1(x), y]^n = F_2(x)y + yF_3(x), \qquad \text{for all } (x, y) \in O_1 \times O_2,$$

then $\mathcal{A}$ is commutative.

Proof. Assume that

$$[F_1(x), y]^n - F_2(x)y - yF_3(x) = 0 \qquad \text{for all } (x, y) \in O_1 \times O_2. \tag{3.1}$$

Let $u \in \mathcal{A}$ and let $x \in O_1$. Then $x + tu \in O_1$ for a sufficiently small real t.

Since F_1, F_2, and F_3 being continuous, one can obviously see that $F_i(ru) = rF_i(u)$ for all $u \in A$, $r \in \mathbb{R}$, $i \in \{1, 2, 3\}$. Replacing x by $x + tu$ in (3.1), we get

$$\big([F_1(x), y] + [F_1(u), y]t\big)^n - \big(F_2(x)y + yF_3(x) + (F_2(u)y + yF_3(u))t\big) = 0. \tag{3.2}$$

Let $P_{n,m}(u, x, y)$ denote the sum of all monic monomials with n occurrences of $[F_1(x), y]$ and m occurrences of $[F_1(u), y]$. It follows from (3.2) that

$$Q(t) = \sum_{k=0}^{n} P_{n-k,k}(u, x, y)t^k - \big(F_2(x)y + yF_3(x) + (F_2(u)y + yF_3(u))t\big) = 0.$$

Set $Q(t) = \sum_{k=0}^{n} q_k(u, x, y)t^k$ with $q_0(u, x, y) = [F_1(x), y]^n - F_2(x)y - yF_3(x)$, $q_1(u, x, y) = P_{n-1,1}(u, x, y) - F_2(u)y - yF_3(u)$, and $q_k(u, x, y) = P_{n-k,k}(u, x, y)$ for all $k \in \{2, \dots, n\}$. Since (0) is a closed linear subspace of $\mathcal{A}$, then Lemma 3.1 yields $q_k(u, x, y) = 0$ for all $k \in \{0, \dots, n\}$. In particular, $q_n(u, x, y) = 0$, that is,

$$[F_1(u), y]^n = 0 \qquad \text{for all } (u, y) \in \mathcal{A} \times O_2.$$

Similarly, one can show that

$$[F_1(u), v]^n = 0 \qquad \text{for all } u, v \in \mathcal{A}. \tag{3.3}$$

By view of (3.3), equation (3.1) reduces to $F_2(x)y + yF_3(x) = 0$ for all $(x, y) \in O_1 \times O_2$.

Using the same techniques as above, we obviously get

$$F_2(x)y + yF_3(x) = 0 \text{ for all } x, y \in \mathcal{A}.$$

Therefore

$$[F_2(x), r]y = 0 \quad \text{and} \quad y[r, F_3(x)] = 0, \quad \text{for all } r, x, y \in \mathcal{A},$$

proving that $F_2(x) \in Z(R)$ and $F_3(x) \in Z(R)$ for all $x \in \mathcal{A}$. Since F_2 and F_3 being nonzero, it follows from Fact 2.2 that $\mathcal{A}$ is commutative. □

Arguing in a similar manner with slight modifications, we get the following theorem.

Theorem 3.3. *Let $\mathcal{A}$ be a prime Banach algebra, let O_1 and O_2 be nonvoid open subsets of $\mathcal{A}$,let F_1, F_2, and F_3 be nonzero continuous generalized derivations of $\mathcal{A}$, and let n be a fixed positive integer. If*

$$(F_1(x) \circ y)^n = F_2(x)y + yF_3(x), \qquad \text{for all } (x, y) \in O_1 \times O_2,$$

then $\mathcal{A}$ is commutative.

Theorem 3.4. *Let $\mathcal{A}$ be a noncommutative prime Banach algebra, O_1, O_2 nonvoid open subsets of $\mathcal{A}$, F_1, F_2 and F_3 continuous generalized derivations of $\mathcal{A}$. If*

$$[F_1(x^r), y^s] = F_2(x^r)y^s + y^sF_3(x^r), \qquad \text{for all } (x, y) \in O_1 \times O_2,$$

where r and s are nonzero integers depending on the pair of elements x and y, then one of the following conditions holds:

(1) *There exist $\lambda, \mu \in C$ such that $F_2(x) = \lambda F_1(x)$, $F_3(x) = \mu F_1(x)$ for any $x \in \mathcal{A}$;*
(2) *$\mathcal{A}$ embeds in a 2×2 matrix ring over a field.*

Proof. Assume that $\mathcal{A}$ does not embed in a 2×2 matrix ring over a field. Let us fix $x \in O_1$ and set

$$K_{r,s} = \{y \in \mathcal{A} \mid [F_1(x^r), y^s] - F_2(x^r)y^s - y^sF_3(x^r) \neq 0\}.$$

We claim that each $K_{r,s}$ is open in $\mathcal{A}$ or equivalently its complement $K^c_{r,s}$ is closed. For this, we consider a sequence $(y_k)_{k\geq 1} \subset K^c_{r,s}$ converging to y and prove that $y \in K^c_{r,s}$.

As $(y_k)_{k\geq 1} \subset K^c_{r,s}$ then $[F_1(x^r), y_k^s] - F_2(x^r)y_k^s + y_k^sF_3(x^r) = 0$ for all $k \geq 1$. Hence

$$\begin{aligned}\lim_{k\to\infty}[F_1(x^r), y_k^s] - F_2(x^r)y_k^s - y_k^sF_3(x^r) &= [F_1(x^r), (\lim_{k\to\infty} y_k)^s] - F_2(x^r)(\lim_{k\to\infty} y_k)^s \\ &\quad -(\lim_{k\to\infty} y_k)^sF_3(x^r) \\ &= [F_1(x^r), y^s] - F_2(x^r)y^s - y^sF_3(x^r) \\ &= 0.\end{aligned}$$

Therefore $y \in K^c_{r,s}$; thus $K_{r,s}$ is open. Suppose now that all the $K_{r,s}$ are dense in $\mathcal{A}$; then the intersection of $K_{r,s}$ is also dense by Baire category theorem, a contradiction with the fact that $O_2 \neq \emptyset$. Hence there exist some positive integers

p, q depending on x such that $K_{p,q}$ is not dense. Accordingly, there exists a nonvoid open subset O_3 in $K_{p,q}^c$. Therefore

$$[F_1(x^p), y^q] - F_2(x^p)y^q - y^q F_3(x^p) = 0 \qquad \text{for all } y \in O_3. \tag{3.4}$$

Let us consider $z \in O_3$ and $v \in \mathcal{A}$, $z + tv \in O_3$ for all sufficiently small real t.

Replacing y by $z + tv$ in (3.4), we obtain

$$[F_1(x^p), (z + tv)^q] - F_2(x^p)(z + tv)^q - (z + tv)^q F_3(x^p) = 0. \tag{3.5}$$

Let $P_{i,j}(x, v)$ denote the sum of all monic monomials with i occurrences of x and j occurrences of u. Using the fact that

$$(z + tv)^q = P_{q,0}(z, v) + P_{q-1,1}(z, v)t + \cdots + P_{1,q-1}(z, v)t^{q-1} + P_{0,q}(z, v)t^q,$$

(3.5) yields that

$$[F_1(x^p), \sum_{i=0}^{q} P_{q-i,i}(z, v)t^i] - F_2(x^p)\Big(\sum_{i=0}^{q} P_{q-i,i}(z, v)t^i\Big) - \Big(\sum_{i=0}^{q} P_{q-i,i}(z, v)t^i\Big)F_3(x^p) = 0,$$

which implies that

$$Q(t) := \sum_{i=0}^{q}\Big([F_1(x^p), P_{q-i,i}(z, v)] - F_2(x^p)P_{q-i,i}(z, v) - P_{q-i,i}(z, v)F_3(x^p)\Big)t^i = 0.$$

Hence $Q(t) = \sum_{i=0}^{q} a_i(v, x, z)t^i = 0$ with

$$a_i(v, x, z) = [F_1(x^p), P_{q-i,i}(z, v)] - F_2(x^p)P_{q-i,i}(z, v) - P_{q-i,i}(z, v)F_3(x^p).$$

By virtue of Lemma 3.1, we get $a_i(v, x, z) = 0$ for all $i \in \{0, \ldots, q\}$. In particular, $a_q(v, x, z) = 0$ so that $[F_1(x^p), v^q] - F_2(x^p)v^q - v^q F_3(x^p) = 0$. In conclusion, we have proved that for a given $x \in O_1$, there exist some positive integers p and q depending on x, such that

$$[F_1(x^p), v^q] - F_2(x^p)v^q - v^q F_3(x^p) = 0 \qquad \text{for all } v \in \mathcal{A}.$$

Let us fix $v \in \mathcal{A}$. Using a similar approach, we arrive at

$$[F_1(u^p), v^q] = F_2(u^p)v^q + v^q F_3(u^p) \qquad \text{for all } u, v \in \mathcal{A}.$$

Now let H_1 and H_2 be the additive subgroups generated by $\{a^p \mid a \in \mathcal{A}\}$ and $\{a^q \mid a \in \mathcal{A}\}$, respectively. We have

$$[F_1(x), y] = F_2(x)y + yF_3(x) \qquad \text{for all } (x, y) \in H_1 \times H_2. \tag{3.6}$$

According to [5], (3.6) yields that either H_1 contains a noncentral Lie ideal J_1 or $a^p \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$, in which case, $\mathcal{A}$ is commutative by [12], a contradiction. Consequently, H_1 contains a noncentral Lie ideal J_1. Similarly, H_2 contains also a noncentral Lie ideal J_2. Now let

$$I_k = \{x \in \mathcal{A} \mid [x, \mathcal{A}] \subset J_k\}$$

with $k = 1, 2$. It follows from [8, Lemma 1.4] that I_1 and I_2 are both subrings and Lie ideals of $\mathcal{A}$. Therefore (3.6) becomes

$$[F_1(x), y] = F_2(x)y + yF_3(x) \qquad \text{for all } (x, y) \in [I_1, \mathcal{A}] \times [I_2, \mathcal{A}]. \tag{3.7}$$

As $[I_1, \mathcal{A}]$ and $[I_2, \mathcal{A}]$ are dense submodules of $[\mathcal{A}, \mathcal{A}]$ then by [10, Theorem 2], $[\mathcal{A}, \mathcal{A}]$ satisfies the same identity as $[I_1, \mathcal{A}]$ and $[I_2, \mathcal{A}]$. Hence (3.7) implies that

$$[F_1(x), y] - F_2(x)y - yF_3(x) = 0 \qquad \text{for all } x, y \in [\mathcal{A}, \mathcal{A}]. \tag{3.8}$$

Since $[\mathcal{A}, \mathcal{A}]$ is a noncentral Lie ideal, applying Theorem 2.6, we get the required result. □

Using the same arguments with slight modifications, an application of Theorem 2.9 yields the following result.

Theorem 3.5. *Let $\mathcal{A}$ be a noncommutative prime Banach algebra, let O_1 and O_2 be nonvoid open subsets of $\mathcal{A}$, and let F_1, F_2 and F_3 be continuous generalized derivations of $\mathcal{A}$. If*

$$F_1(x^r) \circ y^s = F_2(x^r)y^s + y^s F_3(x^r), \qquad \text{for all } (x, y) \in O_1 \times O_2,$$

where r and s are nonzero integers depending on the pair of elements x and y, then one of the following conditions holds:

(1) *There exist $\lambda, \mu \in C$ such that $F_2(x) = \lambda F_1(x)$, $F_3(x) = \mu F_1(x)$ for any $x \in \mathcal{A}$;*
(2) *$\mathcal{A}$ embeds in a 2×2 matrix ring over a field.*

The following example shows that the primeness hypothesis in Theorems 2.6 and 2.9 is not superfluous.

Example 3.6. Let us consider the ring $\mathcal{R} = M_2(\mathbb{R}) \times \mathbb{R}$ with operations coordinatewise addition and multiplication. It is obvious that $\mathcal{R}$ is a nonprime ring.

Consider the generalized derivation

$$F_M((A, a)) = (MA + AM, 0), \qquad \text{where} M \in [M_2(\mathbb{R}), M_2(\mathbb{R})],$$

with associated derivation d_M defined by $d_M((A, a)) = (AM - MA, 0)$.

Set $L = [M_2(\mathbb{R}), M_2(\mathbb{R})] \times \mathbb{R}$ along with $F_1 = 0$, $F_2 = F_M$, and $F_3 = -F_M$. A simple computation shows that

$$[F_1((A, a)), (B, b)] = F_1((A, a)) \circ (B, b) = F_2((A, a))(B, b) + (B, b)F_3((A, a)) = 0$$

for all $(A, a), (B, b) \in L$. However, none of the assertions of Theorems 2.6 and 2.9 are satisfied.

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