Khayyam J. Math. 9 (2023), no. 2, 256-262
DOI: 10.22034/KJM.2023.378145.2733

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ON THE BETTI NUMBERS OF MONOMIAL IDEALS AND THEIR POWERS

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Communicated by B. Mashayekhy

Abstract. Let $S = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring over a field $\mathbb{K}$. In this paper, for some families of monomial ideals $I \subset S$, we study the minimal number of generators of I^k. We use these results to find some other Betti numbers of these families of ideals for special choices of n, the number of variables.

1. Introduction and preliminaries

Using the structure of an ideal I in a commutative ring to find the Betti numbers of I and the powers I^k is a complicated problem. In particular, finding $\mu(I)$, the minimal number of generators of a graded polynomial ideal I, and predicting the behavior of the function $\mu(I^k)$ are quite difficult and have been studied a lot (for instance, see [2, 3, 6, 9, 10, 12, 14]). In this paper, we find the minimal generators of some families of equigenerated monomial ideals (monomial ideals generated in a single degree) in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$. Moreover, we find some other Betti numbers of these ideals for special choices of n, the number of variables.

Let $\mathbb{K}$ be a field and let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in the variables $x_1, \ldots, x_n$ over $\mathbb{K}$. Also, let I be a graded ideal in S and let

$$0 \to S^{\beta_n} \to \cdots \to S^{\beta_2} \to S^{\beta_1} \to S \to S/I \to 0$$

be the minimal free resolution of S/I. The numbers $\beta_1, \ldots, \beta_n$ are called the Betti numbers of S/I.

Date: Received: 25 December 2022; Revised: 25 June 2023; Accepted: 26 June 2023.

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2020 *Mathematics Subject Classification.* Primary: 13F20; Secondary: 05E40.

Key words and phrases. Monomial ideal, Betti number, t-spread monomial, ideal of Veronese type.

An equigenerated monomial ideal I with the minimal set of generators $G(I)$ is called a *polymatroidal ideal* if for any pairs of monomial $x_1^{a_1} \dots x_n^{a_n}$ and $x_1^{a'_1} \dots x_n^{a'_n}$ in $G(I)$ with the property that $a_i > a'_i$ for some i, there exists j such that $a_j < a'_j$ and $(x_j/x_i)(x_1^{a_1} \dots x_n^{a_n}) \in G(I)$. We say that the ideal I has a d-linear resolution if the graded minimal free resolution of S/I^k is of the form

$$0 \to S(-d-s)^{\beta_s} \to \cdots \to S(-d-1)^{\beta_2} \to S(-d)^{\beta_1} \to S \to S/I^k \to 0.$$

Let I be a polymatroidal ideal. Since all powers of a polymatroidal ideal are polymatroidal (see [8, Theorem 12.6.3]) and a polymatroidal ideal has a linear resolution (see [13]), the minimal free resolution of S/I^k is of the form

$$0 \to S(-(kd+n-1))^{\beta_n^k} \to \cdots \to S(-(kd+1))^{\beta_2^k} \to S(-kd)^{\beta_1^k} \to S \to S/I^k \to 0,$$

where $\beta_i^k = \beta_i(S/I^k)$.

An important class of polymatroidal ideals is the class of ideals of Veronese type. Fix integer d and the integer vector $\mathbf{a} = (a_1, \dots, a_n)$ with $d \geq a_1 \geq \cdots \geq a_n \geq 1$. An *ideal of Veronese type* is an ideal $I_{\mathbf{a},n,d}$ with the following minimal set of generators

$$G(I_{\mathbf{a},n,d}) = \{x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \mid \sum_{i=1}^{n} b_i = d \text{ and } b_i \leq a_i \text{ for } i = 1, \dots, n\}.$$

In Section 2, we find the minimal set of generators of all powers of ideals of Veronese type. Also, we use this result to find the minimal number of generators (β_1) of some other classes of equigenerated monomial ideals. In Section 3, we use β_1 to find some other Betti numbers of these families of ideals for special choices of n.

2. The minimal number of generators of some monomial ideals

Let $n, d \geq 1$ and $t \geq 0$ be fixed integers. The following notations are obtained from [5]. We denote by $\mathcal{A}_{n,d}$ the set of all multisets $A \subset [n]$ with $|A| = d$. A multiset $\{i_1 \leq i_2 \leq \cdots \leq i_d\} \subset [n]$ is called *t-spread* if $i_{j+1} - i_j \geq t$ for all j. The set of all t-spread multisets in $\mathcal{A}_{n,d}$ is denoted by $\mathcal{A}_{n,d,t}$. Let $A \in \mathcal{A}_{n,d,t}$ be a t-spread multiset. A subset $B \subset A$ is called a *block of size q* if $B = \{i_j, i_{j+1}, \dots, i_{j+q-1}\}$ with $i_{k+1} - i_k = t$ for all k. Let c be a positive integer. The set of all multisets $A \in \mathcal{A}_{n,d,t}$ such that $|B| \leq c$ for each block $B \subset A$, is denoted by $\mathcal{A}_{c,(n,d,t)}$.

Let $\mathbf{a} = (a_1, \dots, a_n)$ be a vector of integers such $d \geq a_1 \geq \cdots \geq a_n \geq 1$. For the integer vector $\mathbf{c} = (c_1, \dots, c_n)$, we write $\mathbf{c} \leq \mathbf{a}$ if $c_i \leq a_i$ for all i.

Let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in the variables $x_1, \dots, x_n$ over a field $\mathbb{K}$. We fix some notations for the following classes of monomial ideals.

Notation 2.1.
- We denote by $I_{\mathbf{a},n,d}$ the ideal of S generated by all monomials of degree d whose exponent vectors are bounded by $\mathbf{a}$. In other words,

$$G(I_{\mathbf{a},n,d}) = \{x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n} \mid \sum_{i=1}^{n} b_i = d \text{ and } b_i \leq a_i \text{ for } i = 1,\dots,n\}.$$

Moreover, $I_{\mathbf{a},n,d}$ is called an ideal of Veronese type.

- $I_{c,(n,d,t)} := (\mathbf{x}_A \mid A \in \mathcal{A}_{c,(n,d,t)})$. The ideal $I_{c,(n,d,t)}$ is called a *c-bounded t-spread Veronese ideal*. Note that $I_{c,(n,d,0)} = I_{\mathbf{c},n,d}$, where $\mathbf{c} = (c,\dots,c) \in \mathbb{Z}^n$.
- We denote by $I_{n,d,t}$ the ideal generated by all t-spread monomials in S of degree d. The ideal $I_{n,d,t}$ is called a *t-spread Veronese ideal* of degree d. One can easily see that $I_{n,d,t} = I_{d,(n,d,t)}$.
- The ideal generated by all square free monomials of degree d is called a *square free Veronese ideal* of degree d and is denoted by $I_{n,d}$. Recall that a monomial $x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n} \in S$ is called square free if $b_i \leq 1$ for all i. Therefore, $I_{n,d} = I_{\mathbf{e},n,d}$, where $\mathbf{e} = (1,\dots,1) \in \mathbb{Z}^n$, and hence $I_{n,d} = I_{1,(n,d,0)}$.

In this section, we use the structure of the ideals introduced in Notation 2.1 to compute their minimal number of generators (and of their powers). We denote by $\mu(I)$ the minimal number of generators of a graded ideal $I \in S$.

Let $\mathbf{a} = (a_1,\dots,a_n)$ be a vector of integers such that $d \geq a_1 \geq \cdots \geq a_n \geq 1$. For $k \geq 1$, $0 \leq l \leq n$, and $1 \leq i \leq \binom{n}{l}$, set $\alpha_{1,0}^k = 0$ and $\alpha_{i,l}^k = \sum_{r\in J}(ka_r + 1)$, where $J \subseteq \{1,\dots,n\}$ and $|J| = l$.

Theorem 2.2. *Let $I = I_{\mathbf{a},n,d}$ be an ideal of Veronese type with $\mathbf{a} = (a_1,\dots,a_n)$. Then*

$$\mu(I^k) = \sum_{j=0}^{n}\left[(-1)^j \sum_{i=1}^{\binom{n}{j}} \binom{kd+n-1-\alpha_{i,j}^k}{n-1}\right]$$

for all $k \geq 1$.

Proof. First, we prove the assertion for $k = 1$. In the case that $a_1 = a_2 = \cdots = a_n = d$, the ideal I is the Veronese ideal of S in degree d, that is, the ideal generated by all monomials in S of degree d. Therefore, $\mu(I) = \binom{d+n-1}{n-1}$. Now, we assume that $a_i < d$ for some i. A typical generator of I is in the form $x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$ such that $b_1 + b_2 + \cdots + b_n = d$ and $b_i \leq a_i$ for all i. We must subtract the bad cases $b_i > a_i$. So we subtract the number of solutions of the equation

$$b_1 + b_2 + \cdots + b_{i-1} + (b_i - a_i - 1) + b_{i+1} + \cdots + b_n = d - a_i - 1,$$

which equals to $\binom{d+n-1-(a_i+1)}{n-1}$. Using the inclusion-exclusion principle, we get

$$\begin{aligned}
\mu(I) &= \binom{d+n-1}{n-1} + \sum_{J\subseteq\{1,\dots,n\}} (-1)^{|J|}\binom{d+n-1-\sum_{i\in J}(a_i+1)}{n-1} \\
&= \sum_{j=0}^{n}\left[(-1)^j \sum_{i=1}^{\binom{n}{j}} \binom{d+n-1-\alpha_{i,j}^1}{n-1}\right].
\end{aligned}$$

The assertion for $k \geq 2$ follows from the fact $(I_{\mathbf{a},n,d})^k = I_{k\mathbf{a},n,kd}$ by [11, Lemma 5.1]. $\square$

Remark 2.3. In the case that $a_1 = a_2 = \cdots = a_n = c$, for some positive integer c, it is easy to check that

$$\mu(I^k) = \sum_{j=0}^{\lfloor \frac{kd}{kc+1} \rfloor} (-1)^j \binom{n}{j} \binom{kd+n-1-j(kc+1)}{n-1}.$$

Proposition 2.4. *Let $I = I_{c,(n,d,t)}$ be a c-bounded t-spread Veronese ideal. Then*

$$\mu(I) = \sum_{j=0}^{\lfloor \frac{d}{c+1} \rfloor} (-1)^j \binom{n-(d-1)t}{j} \binom{n-(d-1)(t-1)-j(c+1)}{d}.$$

Proof. The ideals $I_{c,(n,d,t)}$ and $I_{c,(n-(d-1)t,d,0)}$ have the same Betti numbers by [5, Corollary 3.5]. On the other hand, $I_{c,(n-(d-1)t,d,0)} = I_{\mathbf{c},n-(d-1)t,d}$, where $\mathbf{c} = (c, \ldots, c) \in \mathbb{Z}^n$. So, the desired conclusion follows from Remark 2.3. $\square$

Corollary 2.5. *Let $I = I_{n,d,t}$ be a t-spread Veronese ideal of degree d. Then*

$$\mu(I) = \binom{n-(d-1)(t-1)}{d}.$$

Proof. Since $I_{n,d,t} = I_{d,(n,d,t)}$, the assertion results from Proposition 2.4. $\square$

Remark 2.6. An alternative proof for Corollary 2.5 is given in [7, Theorem 2.3(d)].

Proposition 2.7. *Let $I = I_{n,d}$ be a square free Veronese ideal of degree d. Then*

$$\mu(I^k) = \sum_{j=0}^{\lfloor \frac{kd}{k+1} \rfloor} (-1)^j \binom{n}{j} \binom{kd+n-1-j(k+1)}{kd}$$

for all $k \geq 1$.

Proof. The desired conclusion results from Remark 2.3, since $I_{n,d} = I_{\mathbf{e},n,d}$, where $\mathbf{e} = (1, \ldots, 1) \in \mathbb{Z}^n$. $\square$

3. On the other Betti numbers of our ideals and their powers

In the previous section, we computed the minimal number of generators (β_1) of ideals of Veronese type and their powers. It is well known that, for a monomial ideal I in $k[x_1, x_2]$ generated by $\mu(I)$ elements, one has $\beta_2 = \beta_1 - 1 = \mu(I) - 1$ (see [15, Proposition 3.1]). In this section, using β_1, we find the other Betti numbers of ideals of Veronese type and their powers in $K[x_1, x_2, x_3]$. Moreover, for the other classes of monomial ideals, which we studied their first Betti number in Section 2, we find some of their other Betti numbers for particular choices of n.

For a monomial ideal $I \subset S$, we denote by $\dim(I)$, the Krull dimension of S/I. Let $I = I_{\mathbf{a},3,d} \subset \mathbb{K}[x_1, x_2, x_3]$ be an ideal of Veronese type with $\dim(I) = 2$.

So, height$(I) = 1$. Since $a_1 \geq a_2 \geq a_3$, there exists a positive integer d' and a Veronese type ideal J with $\dim(J) = 1$ such that $I = x_1^{d'} J$. Indeed,

$$d' = \max\{\ell : \ x_1^{\ell} | u \ \text{ for all } u \in G(I)\}$$

and $J = I_{\mathbf{b},3,d-d'}$ where $\mathbf{b} = (a_1 - d', a_2, a_3)$. Set $\delta = d - d'$.

Proposition 3.1. *Let $I = I_{\mathbf{a},3,d} \subset \mathbb{K}[x_1, x_2, x_3]$ be an ideal of Veronese type with $\mathbf{a} = (a_1, a_2, a_3)$. Then, for $k \geq 1$, if $\dim(I) = 0$, then*

$$\beta_2(I^k) = (kd)(kd + 2), \quad \beta_3(I^k) = \binom{kd+1}{2}.$$

If $\dim(I) = 1$, then

$$\beta_2(I^k) = 2\beta_1(I^k) - kd - 2, \quad \beta_3(I^k) = \beta_1(I^k) - kd - 1.$$

If $\dim(I) = 2$, then

$$\beta_2(I^k) = \beta_1(I^k) - k\delta - 2, \quad \beta_3(I^k) = \beta_1(I^k) - k\delta - 1.$$

Proof. Sine all powers of I are polymatroidal, the minimal free resolution of I is of the form

$$0 \to S(-dk-2)^{\beta_2^k} \to S(-dk-1)^{\beta_2^k} \to S(-dk)^{\beta_1^k} \to S \to S/I^k \to 0,$$

where $\beta_i^k = \beta_i(S/I^k)$. Therefore, if $\dim(I) = 0$, then I is Cohen–Macaulay. Using [4, Theorem 4.1.15], we get $\beta_2(I^k) = kd(kd+2)$ and $\beta_3(I^k) = (kd)(kd+1)/2 = \binom{kd+1}{2}$.

If $\dim(I) = 1$, then using [16, Theorem 3], we get

$$\begin{pmatrix} \beta_2^k \\ \beta_3^k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1^k - \binom{kd}{0} \\ \beta_1^k - \binom{kd+1}{1} \end{pmatrix} = \begin{pmatrix} 2\beta_1^k - kd - 2 \\ \beta_1^k - kd - 1 \end{pmatrix}.$$

If $\dim(I) = 2$, then we assume that $I = x_1^{d'} J$ with $\dim(J) = 1$ and set $\delta = d - d'$. Since I and J have the same Betti numbers, the assertion follows from the previous case. $\square$

Example 3.2. (a) Let $\mathbf{a} = (2, 2, 2)$ and let $d = 2$. Then $I_{\mathbf{a},3,2} = (x_1, x_2, x_3)^2 \subset \mathbb{K}[x_1, x_2, x_3]$ and so, $\dim(I_{\mathbf{a},3,2}) = 0$. Using Theorem 2.2 and Proposition 3.1, we get $\beta_1(I_{\mathbf{a},3,2}) = 6$, $\beta_2(I_{\mathbf{a},3,2}) = 8$, and $\beta_3(I_{\mathbf{a},3,2}) = 3$, which are confirmed by CoCoA [1].

(b) Let $\mathbf{e} = (1, 1, 1)$ and let $d = 1$. Then $I_{\mathbf{e},3,2} = I_{3,2} = (x_1x_2, x_1x_3, x_2x_3) \subset \mathbb{K}[x_1, x_2, x_3]$ and so, $\dim(I_{\mathbf{e},3,2}) = 1$. Using Corollary 2.7 and Proposition 3.1, we get $\beta_1(I_{\mathbf{e},3,2}) = 3$, $\beta_2(I_{\mathbf{e},3,2}) = 2$, and $\beta_3(I_{\mathbf{e},3,2}) = 0$ (confirmed by CoCoA [1]).

(c) Let $\mathbf{c} = (8, 2, 1)$ and let $d = 8$. Then $I_{\mathbf{c},3,8} = (x_1^8, x_1^7x_2, x_1^7x_3, x_1^6x_2x_3, x_1^6x_2^2, x_1^5x_2^2x_3) \subset \mathbb{K}[x_1, x_2, x_3]$ and so, $\dim(I_{\mathbf{c},3,8}) = 2$. Note that $I_{\mathbf{c},3,8} = x_1^5 I_{\mathbf{b},3,3}$, where $\mathbf{b} = (3, 2, 1)$, and hence $\delta = 8 - 5 = 3$. Using Theorem 2.2 and Proposition 3.1, we get $\beta_1(I_{\mathbf{c},3,3}) = 6$, $\beta_2(I_{\mathbf{c},3,3}) = 7$, and $\beta_3(I_{\mathbf{c},3,3}) = 2$ (approved by CoCoA [1]).

Let $I_{c,(n,d,t)}$ be a c-bounded t-spread Veronese ideal with $\dim(I) = 2$ such that $n-(d-1)t = 3$. By [5, Corollary 3.5] we have $\beta_i(I_{c,(n,d,t)}) = \beta_i(I_{c,(n-(d-1)t,d,0)})$ for all i, and $\mathrm{height}(I_{c,(n,d,t)}) = \mathrm{height}(I_{c,(n-(d-1)t,d,0)})$ by [5, Proposition 3.7 (a)]. On the other hand, we know that $I_{c,(n-(d-1)t,d,0)} = I_{\mathbf{c},n-(d-1)t,d} = I_{\mathbf{c},3,d}$, where $\mathbf{c} = (c,\dots,c) \in \mathbb{Z}^n$. Hence, $\dim(I_{\mathbf{c},3,d}) = 2$. Since $a_1 \geq a_2 \geq a_3$, it follows that $I_{\mathbf{c},n-(d-1)t,d} = x_1^{d'}J$ for a positive integer d' and an ideal of Veronese type J with $\dim(J) = 1$. Set $\delta = d - d'$. So, we obtain the following corollary from Proposition 3.1.

Corollary 3.3. *Let $I = I_{c,(n,d,t)}$ be a c-bounded t-spread Veronese ideal such that $n-(d-1)t = 3$. If $\dim(I) = 0$, then*

$$\beta_2(I) = d(d+2), \quad \beta_3(I) = \binom{d+1}{2}.$$

If $\dim(I) = 1$, then

$$\beta_2(I) = 2\beta_1(I) - d - 2, \quad \beta_3(I) = \beta_1(I) - d - 1.$$

If $\dim(I) = 2$, then

$$\beta_2(I) = \beta_1(I) - \delta - 2, \quad \beta_3(I) = \beta_1(I) - \delta - 1.$$

We also obtain the following corollary from Corollary 3.3 and the fact that $I_{n,d,t} = I_{d,(n,d,t)}$.

Corollary 3.4. *Let $I = I_{n,d,t}$ be a t-spread Veronese ideal of degree d such that $n-(d-1)t = 3$. If $\dim(I) = 0$, then*

$$\beta_2(I) = d(d+2), \quad \beta_3(I) = \binom{d+1}{2}.$$

If $\dim(I) = 1$, then

$$\beta_2(I) = 2\beta_1(I) - d - 2, \quad \beta_3(I) = \beta_1(I) - d - 1.$$

If $\dim(I) = 2$, then

$$\beta_2(I) = \beta_1(I) - \delta - 2, \quad \beta_3(I) = \beta_1(I) - \delta - 1.$$

Acknowledgement. This research was in part supported by a grant from IPM ((No.1401130016). The first author would like to thank the Institute for Research in Fundamental Sciences (IPM), Tehran, Iran. Also, the authors would like to thank the referee for a careful reading of this paper.

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