Khayyam J. Math. 9 (2023), no. 2, 263-287
DOI: 10.22034/KJM.2023.386608.2779

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

# BOUNDEDNESS OF THE HARDY–LITTLEWOOD MAXIMAL OPERATOR, FRACTIONAL INTEGRAL OPERATORS, AND CALDERÓN–ZYGMUND OPERATORS ON GENERALIZED WEIGHTED MORREY SPACES

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Communicated by A. Jiménez-Vargas

ABSTRACT. In this paper, we investigate the boundedness of classical operators, namely the Hardy–Littlewood maximal operator, fractional integral operators, and Calderón–Zygmund operators, on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces. We prove that each of the three operators is bounded on these function spaces under some assumptions.

## 1. INTRODUCTION

We shall discuss the boundedness of three classical operators, namely the Hardy–Littlewood maximal operator, fractional integral operators, and Calderón–Zygmund operators, on generalized weighted Morrey spaces. Throughout this paper, we denote by $B(a,r)$ an open ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$. For a set $E$ in $\mathbb{R}^n$, we denote by $E^{\mathrm{c}}$ the complement of $E$. Moreover, if $E$ is a measurable set in $\mathbb{R}^n$, then $|E|$ denotes the Lebesgue measure of $E$. The Hardy–Littlewood maximal operator $M$ and fractional maximal operator $M_\alpha$, where $0 \le \alpha < n$, are defined by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy, \quad x \in \mathbb{R}^n,$$

*Date*: Received: 20 February 2023; Revised: 2 June 2023; Accepted: 26 June 2023.

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2020 *Mathematics Subject Classification.* Primary 42B25; Secondary 42B20.

*Key words and phrases.* Classical operators, generalized weighted Morrey spaces, generalized weighted weak Morrey spaces, $A_p$ weights, $A_{p,q}$ weights.

and

$$M_\alpha f(x) := \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\alpha}{n}}} \int_{B(x,r)} |f(y)|dy, \quad x \in \mathbb{R}^n,$$

for locally integrable functions $f$ on $\mathbb{R}^n$. It is well known that $M$ is bounded on Lebesgue spaces $L^p = L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$, and from $L^1$ to the weak Lebesgue space $WL^1 = WL^1(\mathbb{R}^n)$; see, for example, [11, 27, 29].

For $0 < \alpha < n$, we also know the Riesz potential or the fractional integral operator $I_\alpha$, which is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad x \in \mathbb{R}^n,$$

for suitable functions $f$ on $\mathbb{R}^n$. The operator $I_\alpha$ is bounded from $L^p$ to $L^q$ for $1 < p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$; see, for example, [29]. Since from the definitions we have

$$M_\alpha f(x) \leq C_n I_\alpha(|f|)(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $C_n$ is the Lebesgue measure of the unit ball in $\mathbb{R}^n$, it thus follows that the operator $M_\alpha$ is also bounded from $L^p$ to $L^q$ for $1 < p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

The next operator that we discuss is the Calderón–Zygmund operator. Let $T = T_K$ be a linear operator from the Schwartz class $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'$, which is $L^2-$bounded and, for each $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we have

$$Tf(x) := \int_{\mathbb{R}^n} K(x,y) f(y) dy, \quad x \notin \mathrm{supp}(f),$$

where $K = K(\cdot,\cdot)$ is the standard kernel defined on $\mathbb{R}^n \times \mathbb{R}^n$ except for the diagonal $\{(x,x) : x \in \mathbb{R}^n\}$ with the following properties: There exists a constant $A > 0$ for which

$$|K(x,y)| \leq \frac{A}{|x-y|^n}, \quad x \neq y,$$

and, for some $\delta > 0$,

$$|K(x,y) - K(x',y)| \leq \frac{A|x-x'|^\delta}{(|x-y| + |x'-y|)^{n+\delta}}, \quad |x-x'| \leq \frac{1}{2} \max(|x-y|, |x'-y|),$$

and

$$|K(x,y) - K(x,y')| \leq \frac{A|y-y'|^\delta}{(|x-y| + |x-y'|)^{n+\delta}}, \quad |y-y'| \leq \frac{1}{2} \max(|x-y|, |x-y'|).$$

The operator $T$ is called the Calderón–Zygmund operator, which was introduced by Coifman and Meyer [4] in 1979. The operator is bounded on $L^p$ for $1 < p < \infty$ and from $L^1$ to $WL^1$ [11].

Let us now discuss about Morrey spaces that we shall work on. For $1 \leq p < \infty$ and $0 \leq \lambda < n$, the classical Morrey space $\mathcal{M}^{p,\lambda} = \mathcal{M}^{p,\lambda}(\mathbb{R}^n)$, equipped with the following norm

$$\|f\|_{\mathcal{M}^{p,\lambda}} := \sup_{a \in \mathbb{R}^n, r>0} \frac{1}{r^\lambda} \left( \int_{B(a,r)} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty,$$

was first introduced in [18]. The same space may be denoted by $\mathcal{M}_q^p = \mathcal{M}_q^p(\mathbb{R}^n)$, equipped with

$$\|f\|_{\mathcal{M}_p^q} := \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(a,r)} |f(x)|^p dx\right)^{\frac{1}{p}}$$
$$= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}} \|f\|_{L^p(B(a,r))},$$

where $1 \le p \le q < \infty$, as used widely in, for examples, [13, 15, 27]. Note that if we set $p = q$, then $\mathcal{M}_q^p = L^p$. In companion with $\mathcal{M}_q^p$, one may also define the weak Morrey space $W\mathcal{M}_q^p = W\mathcal{M}_q^p(\mathbb{R}^n)$ to be the set of all measurable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{W\mathcal{M}_q^p} := \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}} \|f\|_{WL^p(B(a,r))} < \infty,$$

where $\|f\|_{WL^p(B(a,r)} := \sup_{\gamma>0} \gamma\, |\{x \in B(a,r) : |f(x)| > \gamma\}|^{1/p}$. The last two definitions were used in, for example, [14].

According to [6], $T$ is bounded on $\mathcal{M}_p^q$ for $1 < p \le q < \infty$ and is bounded from $\mathcal{M}_1^q$ to $W\mathcal{M}_1^q$ for $1 \le q < \infty$. In addition, $M$ is bounded on $\mathcal{M}_p^q$ for $1 < p \le q < \infty$ and is bounded from $\mathcal{M}_1^q$ to $W\mathcal{M}_1^q$ for $1 \le q < \infty$ [2]. Moreover, $I_\alpha$ is bounded from one Morrey space to another under certain conditions [1], [25].

In [17, 21], the Morrey space $\mathcal{M}_q^p$ was generalized to $\mathcal{M}_\psi^p = \mathcal{M}_\psi^p(\mathbb{R}^n)$, which consists of all locally integrable functions $f$ on $\mathbb{R}^n$ such that the norm

$$\|f\|_{\mathcal{M}_\psi^p} := \sup_{a\in\mathbb{R}^n} \left(\frac{1}{\psi(B(a,r))} \int_{\mathbb{R}^n} |f(x)|^p dx\right)^{\frac{1}{p}} < \infty.$$

Here $\psi$ is a function from $(0, \mathbb{R}^n \times \infty)$ to $(0, \infty)$ satisfying certain conditions. Moreover, the weak generalized Morrey space $W\mathcal{M}_\psi^p = W\mathcal{M}_\psi^p(\mathbb{R}^n)$, where $1 < p < \infty$ was defined as the set of all functions $f$ for which there exists a constant $C > 0$ such that

$$\frac{\gamma^p}{\psi(B)} |\{x \in B : |f(x)| > \gamma\}| \le C$$

for every ball $B = B(a,r)$ and $\gamma > 0$. We can see that if we set $\psi(B(a,r)) = |B(a,r)|^{1-\frac{p}{q}}$, where $1 \le q < \infty$, then $\mathcal{M}_\psi^p = \mathcal{M}_q^p$. Nakai [21] investigated the sufficient conditions on the function $\psi$ to ensure the boundedness of the operator $M, T$, and $I_\alpha$ on these spaces. Similar results are obtained by Mizuhara [17], where $\psi$ was assumed to be a growth function satisfying doubling condition with a doubling constant $1 \le D = D(\psi) < 2^n$.

Guliyev [12] defined generalized Morrey spaces $\mathcal{M}_\phi^p = \mathcal{M}_\phi^p(\mathbb{R}^n)$ with the norm

$$\|f\|_{\mathcal{M}_\phi^p} := \sup_{a\in\mathbb{R}^n, r>0} \frac{r^{-\frac{n}{p}}}{\phi(a,r)} \|f\|_{L^p(B(a,r))}$$

and also defined generalized weak Morrey spaces $W\mathcal{M}^p_\phi = W\mathcal{M}^p_\phi(\mathbb{R}^n)$ with the norm

$$\|f\|_{W\mathcal{M}^p_\phi} = \sup_{a\in\mathbb{R}^n, r>0} \frac{r^{-\frac{n}{p}}}{\phi(a,r)} \|f\|_{WL^p(B(a,r))}.$$

In contrast with Nakai's approach, Guliyev did not use the doubling condition to prove the boundedness of the operators $M$ and $T$ on these spaces. Furthermore, Guliyev also investigated the boundedness of $M$ and $T$ from $\mathcal{M}^p_{\phi_1}$ to $\mathcal{M}^p_{\phi_2}$ for $1 < p < \infty$, and from $\mathcal{M}^1_{\phi_1}$ to $W\mathcal{M}^1_{\phi_2}$ for $p = 1$, for some functions $\phi_1$ and $\phi_2$ on $\mathbb{R}^n \times (0,\infty)$. To be precise, he obtained the following theorem.

**Theorem 1.1** ([12])**.** *Let $1 \le p < \infty$ and let the functions $\phi_1(a,r)$ and $\phi_2(a,r)$ satisfy*

$$\int_r^\infty \phi_1(a,t)\frac{dt}{t} \le C\phi_2(a,r)$$

*for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$, where $C > 0$ does not depend on $a$ and $r$. Then $M$ and $T$ are bounded from $\mathcal{M}^p_{\phi_1}$ to $\mathcal{M}^p_{\phi_2}$ for $1 < p < \infty$ and are bounded from $\mathcal{M}^1_{\phi_1}$ to $W\mathcal{M}^1_{\phi_2}$.*

Let $\varphi : (0,\infty) \to (0,\infty)$ be a function and let $\omega$ be a weight, that is, a nonnegative locally integrable function taking value $(0,\infty)$ almost everywhere on $\mathbb{R}^n$. For $0 < p < \infty$, Nakamura [22] introduced the generalized weighted Morrey space $\mathcal{M}^\varphi_p(\omega)$ to be the set of all measurable functions $f$ for which

$$\|f\|_{\mathcal{M}^\varphi_p(\omega)} = \sup_{Q\in\mathcal{Q}} \varphi(l(Q)) \left(\frac{1}{|Q|}\int_Q |f(x)|^p \omega(x)dx\right)^{\frac{1}{p}} < \infty,$$

and the generalized weighted weak Morrey space, denoted by $w\mathcal{M}^\varphi_p(\omega)$, which is equipped with $\|f\|_{w\mathcal{M}^\varphi_p(\omega)}$, where

$$\|f\|_{w\mathcal{M}^\varphi_p(\omega)} = \sup_{Q\in\mathcal{Q},\lambda>0} \varphi(l(Q))\lambda \left(\frac{1}{|Q|}\int_Q \mathcal{X}_{\{|f|>\lambda\}}(x)\omega(x)dx\right)^{\frac{1}{p}} < \infty.$$

Here $\mathcal{Q}$ is a set of all cubes in $\mathbb{R}^n$ that have sides parallel to the coordinate axis and $l(Q)$ denotes the side-length of $Q \in \mathcal{Q}$. By the definition, notation, and the assumption that the weight $\omega$ is in Muckenhoupt class $A_p$ (defined in Section 2), Nakamura obtained the following theorem.

**Theorem 1.2** ([22, Theorems 1.3 and 1.10])**.** *Let $1 \le p < \infty$ and let $\varphi \in \mathcal{G}_p$, that is, $\varphi$ is nondecreasing and the map $t \mapsto t^{-n/p}\varphi(t)$ is nonincreasing. Assume that $\omega \in A_p$ and there is a positive constant $C$ such that for $Q_0 \in \mathcal{Q}$,*

$$\sup_{Q\in\mathcal{Q}, Q\subset Q_0} \varphi(l(Q)) \left(\frac{\omega(Q)}{|Q|}\right)^{\frac{1}{p}} \le C\varphi(l(Q_0)) \left(\frac{\omega(Q_0)}{|Q_0|}\right)^{\frac{1}{p}},$$

*where $\omega(Q) = \int_Q \omega(x)dx$ for $Q \in \mathcal{Q}$. Then $M$ and $T$ are bounded on $\mathcal{M}^\varphi_p(\omega)$ for $1 < p < \infty$. Moreover, the operators are bounded from $\mathcal{M}^\varphi_1(\omega)$ to $w\mathcal{M}^\varphi_1(\omega)$.*

In Theorem 1.2, the author used a function $\varphi$ of one variable, which satisfies some assumptions, on the generalized weighted Morrey spaces and the weak generalized weighted Morrey spaces. Meanwhile, Guliyev used two functions $\phi_1$ and $\phi_2$ of two variables in his results, where the assumptions are different from those in Nakamura's.

In this paper, we use definitions for the generalized weighted Morrey spaces and the weak generalized weighted Morrey space which are more general than the definition introduced by Nakamura. Our aim is then to investigate the boundedness of $M, I_\alpha$, and $T$ on the generalized weighted Morrey spaces and the weak generalized weighted Morrey spaces. Our results, which are presented in Sections 3-5, generalize the previous results obtained by Guliyev. These results are also different from Nakamura's results as we use more general definitions for the generalized weighted Morrey spaces and the weak generalized weighted Morrey spaces.

## 2. $A_p$ weights and generalized weighted Morrey spaces

In this section, we discuss $A_p$ weights, weighted Lebesgue spaces, and weighted Morrey spaces. We also present the definition of generalized weighted Morrey spaces, generalized weighted weak Morrey spaces, and some lemmas, which we shall use to prove the main results about the boundedness of the three classical operators on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces.

A weight $w$ is a nonnegative locally integrable function on $\mathbb{R}^n$ taking values in the interval $(0, \infty)$ almost everywhere. The weight class that we use in this article is the Muckenhoupt class $A_p$ (see, e.g., [10]).

**Definition 2.1.** For $1 < p < \infty$, we denote by $A_p$ the set of all weights $w$ on $\mathbb{R}^n$ for which there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)dx\right)\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leq C$$

for every ball $B(a,r)$ in $\mathbb{R}^n$. For $p = 1$, we denote by $A_1$ the set of all weights $w$ for which there exists a constant $C > 0$ such that

$$\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)dx \leq C\|w\|_{L^\infty(B(a,r))}$$

for every ball $B(a,r)$ in $\mathbb{R}^n$.

*Remark* 2.2. The last inequality is equivalent to the following equation

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)dx\right)\cdot\|w^{-1}\|_{L^\infty(B(a,r))} \leq C$$

for every ball $B(a,r)$ in $\mathbb{R}^n$.

**Theorem 2.3** ([10])**.** *For each $1 \leq p < \infty$ and $w \in A_p$, there exists $C > 0$ such that*

$$\frac{w(B)}{w(E)} \leq C\left(\frac{|B|}{|E|}\right)^p$$

*for every ball $B$ and measurable sets $E \subseteq B$, where $w(B) = \int_B w(x)dx$.*

Associated to a weight $w \in A_p$ with $1 \le p < \infty$, the weighted Lebesgue space $L^{p,w} = L^{p,w}(\mathbb{R}^n)$ is the set of all measurable functions $f$ on $\mathbb{R}^n$ for which

$$\|f\|_{L^{p,w}} := \int_{\mathbb{R}^n} |f(x)|^p w(x)dx < \infty.$$

In addition, $WL^{p,w} = WL^{p,w}(\mathbb{R}^n)$ is the weighted weak Lebesgue space that consists of all measurable functions $f$ on $\mathbb{R}^n$ for which

$$\|f\|_{WL^{p,w}} := \sup_{\gamma>0} \gamma w(\{x \in \mathbb{R}^n : |f(x)| > \gamma\})^{\frac{1}{p}} < \infty.$$

Note that if $w$ is a constant function a.e., then we have that $L^{p,w} = L^p$ and $WL^{p,w} = WL^p$. We note from [10] that $w \in A_p$ if and only if $M$ is bounded on $L^{p,w}$ for $1 < p < \infty$ and $w \in A_1$ if and only if $M$ is bounded from $L^{1,w}$ to $WL^{1,w}$.

Related to the fractional integral operator $I_\alpha$, we have another class of weights $A_{p,q}$.

**Definition 2.4** ([20, 27])**.** Let $1 < p < q < \infty$ and let $p'$ satisfy $1/p + 1/p' = 1$. We denote by $A_{p,q}$ the collection of all weight functions $w$ satisfying

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)^q dx\right)^{\frac{1}{q}} \left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)^{-p'} dx\right)^{\frac{1}{p'}} \le C$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$, where $C$ is a constant independent of $a$ and $r$. For $p = 1$ and $q > 1$, we denote by $A_{1,q}$ the collection of weight functions $w$ for which there exists a constant $C > 0$ such that for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$,

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)^q dx\right)^{1/q} \le C\|w\|_{L^\infty(B(a,r))}.$$

One may observe that $w \in A_{p,q}$ if and only if $w^q \in A_{q/p'+1}$ for $1 \le p < q < \infty$ (see [5]). Moreover, we have the following proposition.

**Proposition 2.5** ([16, Theorem 3.2.2])**.** *Let $1 \le p < q < \infty$. If $w \in A_{p,q}$, then $w^p \in A_p$ and $w^q \in A_q$.*

We rewrite the following results of Muckenhoupt [19] for $M$, Muckenhoupt and Wheeden [20] for $I_\alpha$, Coifman and Fefferman [3], Garcia-Cuerva and Rubio de Francia [10], and Sawyer [28] for $T$ on weighted Lebesgue spaces.

**Theorem 2.6** ([10])**.** *Let $1 < p < \infty$. Then $M$ is bounded on $L^{p,w}$ if $w \in A_p$. Moreover, $M$ is bounded from $L^{1,w}$ to $WL^{1,w}$ if $w \in A_1$.*

**Theorem 2.7** ([20])**.** *Let $0 < \alpha < n, 1 < p < n/\alpha, 1/q = 1/p - \alpha/n$, and $w \in A_{p,q}$. Then, the operator $I_\alpha$ is bounded from $L^{p,w^p}$ to $L^{q,w^q}$. Moreover, if $w \in A_{1,p}$ with $1/q = 1 - \alpha/n$, then $I_\alpha$ is bounded from $L^{1,w}$ to $WL^{q,w^q}$.*

**Theorem 2.8** ([3, 10, 28])**.** *Let $1 \le p < \infty$. Then, $T$ is bounded on $L^{p,w}$ if $w \in A_p$ and $1 < p < \infty$. Moreover, $T$ is bounded from $L^{1,w}$ to $WL^{1,w}$ if $w \in A_1$.*

We now present the definition of the generalized weighted Morrey spaces and the generalized weighted weak Morrey spaces, which will become the spaces of our interest in this article.

**Definition 2.9.** Let $1 \le p < \infty$, let $w \in A_p$, and let $\psi$ be a positive function on $\mathbb{R}^n \times (0, \infty)$. The *generalized weighted Morrey space* $\mathcal{M}_\psi^{p,w} = \mathcal{M}_\psi^{p,w}(\mathbb{R}^n)$ is the set of all functions $f \in L_{\text{loc}}^{p,w}$ such that

$$\begin{aligned}\|f\|_{\mathcal{M}_\psi^{p,w}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \left( \frac{1}{w(B(a,r))} \int_{B(a,r)} |f(x)|^p w(x)dx \right)^{1/p} \\ &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,r))} < \infty.\end{aligned}$$

**Definition 2.10.** Let $1 \le p < \infty$, let $w \in A_p$, and let $\psi$ be a positive function defined on $\mathbb{R}^n \times (0, \infty)$. The *generalized weighted weak Morrey space* $W\mathcal{M}_\psi^{p,w} = W\mathcal{M}_\psi^{p,w}(\mathbb{R}^n)$ is the set of all functions $f \in L_{\text{loc}}^{p,w}$ such that

$$\begin{aligned}\|f\|_{W\mathcal{M}_\psi^{p,w}} &= \sup_{a\in\mathbb{R}^n, r>0} \sup_{\gamma>0} \frac{1}{\psi(a,r)} \frac{\gamma}{w(B(a,r))^{\frac{1}{p}}} w(\{x \in \mathbb{R}^n : |f(x)| > \gamma\})^{\frac{1}{p}} \\ &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{p}}} \|f\|_{WL^{p,w}(B(a,r))} < \infty.\end{aligned}$$

*Remark* 2.11. There are some other variations of generalized weighted Morrey spaces and generalized weighted weak Morrey spaces and their specific conditions, as well as their relations with some classical operators such as in [7–9, 23, 24].

With Definitions 2.9 and 2.10, we shall investigate the boundedness of the classical operators: the Hardy–Littlewood maximal operator, the fractional integral operators, the fractional maximal operators, and the Calderón–Zygmund operators on those spaces in the next section. We end this section with lemmas, which will be used later in proving our main theorems.

**Lemma 2.12.** *Let $\varphi$ be a nonnegative function on $\mathbb{R}^n \times (0, \infty)$ such that the map $r \mapsto \varphi(a, r)$ is increasing for each $a \in \mathbb{R}^n$. Let $w \in A_p$, where $1 \le p < \infty$. Then, for every ball $B(a, r)$, we have*

$$\varphi(a,r) \le Cw(B(a,r))^{\frac{1}{p}} \sup_{r<s<\infty} \frac{1}{w(B(a,s))^{\frac{1}{p}}} \varphi(a,s)$$

*and*

$$\varphi(a,r) \le Cw(B(a,r))^{\frac{1}{p}} \int_r^\infty \frac{1}{w(B(a,s))^{\frac{1}{p}}} \varphi(a,s) \frac{ds}{s},$$

*where $C > 0$ is independent of the function $\varphi$, $a \in \mathbb{R}^n$ and $r > 0$.*

*Proof.* Let $a \in \mathbb{R}^n$ and let $r > 0$. By Theorem 2.3 and the fact that the map $s \mapsto \varphi(a,s)$ is increasing for each $a \in \mathbb{R}^n$, we have

$$\begin{aligned} w(B(a,r))^{\frac{1}{p}} \sup_{r<s<\infty} \frac{1}{w(B(a,s))^{\frac{1}{p}}}\varphi(a,s) &= \sup_{r<s<\infty} \frac{w(B(a,r))^{\frac{1}{p}}}{w(B(a,s))^{\frac{1}{p}}}\varphi(a,s) \\ &\geq C \sup_{r<s<\infty} \frac{r^n}{s^n}\varphi(a,r) \\ &= C\varphi(a,r). \end{aligned}$$

Moreover,

$$\begin{aligned} w(B(a,r))^{\frac{1}{p}} \int_r^\infty \frac{1}{w(B(a,s))^{\frac{1}{p}}}\varphi(a,s)\frac{ds}{s} &= \int_r^\infty \frac{w(B(a,r))^{\frac{1}{p}}}{w(B(a,s))^{\frac{1}{p}}}\varphi(a,s)\frac{ds}{s} \\ &\geq \int_r^\infty C\frac{|B(a,r)|}{|B(a,s)|}\varphi(a,s)\frac{ds}{s} \\ &\geq C\int_r^{2r} \frac{r^n}{(2r)^n}\varphi(a,r)\frac{ds}{2r} \\ &= C\varphi(a,r). \end{aligned}$$

Therefore,

$$\varphi(a,r) \leq Cw(B(a,r))^{\frac{1}{p}} \sup_{r<s<\infty} \frac{1}{w(B(a,s))^{\frac{1}{p}}}\varphi(a,s)$$

and

$$\varphi(a,r) \leq Cw(B(a,r))^{\frac{1}{p}} \int_r^\infty \frac{1}{w(B(a,s))^{\frac{1}{p}}}\varphi(a,s)\frac{ds}{s},$$

which proves the lemma. □

**Lemma 2.13.** *Let $1 \leq p < \infty$ and let $w \in A_p$. Then, there is a constant $C > 0$ such that for each $(a,s) \in \mathbb{R}^n \times (0,\infty)$,*

$$\frac{1}{|B(a,s)|}\int_{B(a,s)} |f(y)|dy \leq \frac{C}{w(B(a,s))^{\frac{1}{p}}}\|f\|_{L^{p,w}(B(a,s))}, \quad f \in L^{p,w}_{\mathrm{loc}}.$$

*Proof.* First, we assume that $1 < p < \infty$. Thus, by using Hölder's inequality, we have for every $a \in \mathbb{R}^n$ and $s > 0$,

$$\begin{aligned} &\frac{1}{|B(a,s)|}\int_{B(a,s)} |f(y)|dy \\ &= \frac{w(B(a,s))^{\frac{1}{p}}}{|B(a,s)|}\frac{1}{w(B(a,s))^{\frac{1}{p}}}\int_{B(a,s)} |f(y)|\frac{w(y)^{\frac{1}{p}}}{w(y)^{\frac{1}{p}}}dy \\ &\leq \frac{w(B(a,s))^{\frac{1}{p}}}{|B(a,s)|}\frac{1}{w(B(a,s))^{\frac{1}{p}}}\left(\int_{B(a,s)} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}\left(\int_{B(a,s)} w(y)^{-\frac{p'}{p}}dy\right)^{\frac{1}{p'}}. \end{aligned}$$

From the previous inequality and using the assumption that $w \in A_p$, we have the following inequality:

$$
\begin{aligned}
&\frac{1}{|B(a,s)|}\int_{B(a,s)}|f(y)|dy\\
&=\left(\frac{1}{|B(a,s)|}\int_{B(a,s)}w(y)dy\right)^{\frac{1}{p}}\left(\frac{1}{|B(a,s)|}\int_{B(a,s)}w(y)^{-\frac{p'}{p}}dy\right)^{\frac{1}{p'}}\\
&\quad\cdot\frac{1}{w(B(a,s))^{\frac{1}{p}}}\left(\int_{B(a,s)}|f(y)|^p w(y)dy\right)^{\frac{1}{p}}\\
&\le\frac{C}{w(B(a,s))^{\frac{1}{p}}}\left(\int_{B(a,s)}|f(y)|^p w(y)dy\right)^{\frac{1}{p}}\\
&=\frac{C}{w(B(a,s))^{\frac{1}{p}}}\|f\|_{L^{p,w}(B(a,s))}.
\end{aligned}
$$

This proves the case for $1<p<\infty$. For $p=1$, by using Hölder's inequality and the assumption that $w\in A_1$, we get

$$
\begin{aligned}
&\frac{1}{|B(a,s)|}\int_{B(a,s)}|f(y)|dy\\
&=\frac{w(B(a,s))}{|B(a,s)|}\frac{1}{w(B(a,s))}\int_{B(a,s)}|f(y)|\frac{w(y)}{w(y)}dy\\
&\le\frac{w(B(a,s))}{|B(a,s)|}\frac{1}{w(B(a,s))}\int_{B(a,s)}|f(y)|w(y)dy\,\|w^{-1}\|_{L^\infty(B(a,s))}\\
&=\frac{1}{|B(a,s)|}\int_{B(a,s)}w(y)dy\,\|w^{-1}\|_{L^\infty(B(a,s))}\cdot\frac{1}{w(B(a,s))}\int_{B(a,s)}|f(y)|w(y)dy\\
&\le\frac{C}{w(B(a,s))}\int_{B(a,s)}|f(y)|w(y)dy\\
&=\frac{C}{w(B(a,s))}\|f\|_{L^{1,w}(B(a,s))},
\end{aligned}
$$

as desired. □

**Corollary 2.14.** *For each $1\le p<\infty$ and $w\in A_p$, there exists $C>0$ such that for every $a\in\mathbb{R}^n$ and $r>0$, we have*

$$
\int_{\mathbb{R}^n\setminus B(a,r)}\frac{|f(y)|}{|a-y|^n}dy\le C\int_r^\infty w(B(a,s))^{-\frac{1}{p}}\|f\|_{L^{p,w}(B(a,s))}\frac{ds}{s},\quad f\in L^{p,w}_{\rm loc}.
$$

*Proof.* Let $a\in\mathbb{R}^n$ and let $r>0$. By Fubini's theorem,

$$
\begin{aligned}
\int_{\mathbb{R}^n\setminus B(a,r)}\frac{f(y)}{|a-y|^n}dy&=\int_{B(a,r)^c}|f(y)|\int_{|a-y|}^\infty\frac{1}{s^n}\frac{ds}{s}dy\\
&=\int_r^\infty\int_{B(a,s)\setminus B(a,r)}|f(y)|dy\frac{1}{s^n}\frac{ds}{s}\\
&\le C\int_r^\infty\frac{1}{|B(a,s)|}\int_{B(a,s)}|f(y)|dy\frac{ds}{s}.
\end{aligned}
$$

Hence, by Lemma 2.13,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B(a,r)} \frac{|f(y)|}{|a-y|^n}dy &\le C\int_r^\infty \frac{1}{|B(a,s)|}\int_{B(a,s)}|f(y)|dy\frac{ds}{s}\\ &\le C\int_r^\infty \frac{1}{w(B(a,s))^{\frac{1}{p}}}\|f\|_{L^{p,w}(B(a,s))}\frac{ds}{s},\end{aligned}$$

as claimed. □

## 3. Hardy–Littlewood maximal operator on generalized weighted Morrey spaces

In this section, we prove the boundedness of the Hardy–Littlewood operator $M$ on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces. Keeping in mind Theorem 2.6, we have the following results.

**Theorem 3.1.** *Let $1 \le p < \infty$ let $w \in A_p$. Then, for every $a \in \mathbb{R}^n$ and $r > 0$,*

$$\|Mf\|_{L^{p,w}(B(a,r))} \le C_1 w(B(a,r))^{\frac{1}{p}} \sup_{r<t<\infty} w(B(a,t))^{-\frac{1}{p}}\|f\|_{L^{p,w}B(a,t)}, \quad f \in L^{p,w}_{\rm loc},$$

*for $1 < p < \infty$, and*

$$\|Mf\|_{WL^{1,w}(B(a,r))} \le C_2 w(B(a,r)) \sup_{r<t<\infty} w(B(a,t))^{-1}\|f\|_{L^{1,w}B(a,t)}, \quad f \in L^{1,w}_{\rm loc},$$

*where $C_1$ and $C_2$ are positive constants that do not depend on $f, a,$ and $r$.*

*Proof.* Let $a \in \mathbb{R}^n$ and let $r > 0$, and write $f$ in the form of $f := f_1 + f_2$, where $f_1 := f \cdot \mathcal{X}_{B(a,2r)}$. Assume that $1 < p < \infty$. Since $w \in A_p$, $M$ is bounded on $L^{p,w}$. Thus,

$$\|Mf\|_{L^{p,w}(B(a,r))} \le \|Mf_1\|_{L^{p,w}(B((a,r))} + \|Mf_2\|_{L^{p,w}(B((a,r))}$$

and

$$\|Mf_1\|_{L^{p,w}(B(a,r))} \le \|Mf_1\|_{L^{p,w}} \le C\|f_1\|_{L^{p,w}} \le C\|f\|_{L^{p,w}(B(a,2r))}.$$

We can see that the map $r \mapsto \|f\|_{L^{p,w}(B(a,2r))}$ is increasing for each $a \in \mathbb{R}^n$. Then, by Theorem 2.3 and Lemma 2.12,

$$\|Mf_1\|_{L^{p,w}(B(a,r))} \le Cw(B(a,r))^{\frac{1}{p}} \sup_{r<t<\infty} \frac{1}{w(B(a,t))^{\frac{1}{p}}}\|f\|_{L^{p,w}(B(a,t))}.$$

Let $x \in B(a,r)$. If $y \in B(x,t) \cap B(a,2r)^{\rm c}$, then $r = 2r - r \le |y-a| - |a-x| \le |y-x| < t$. In other words,

$$\int_{B(x,t)\cap B(a,2r)^c} |f(y)|dy = 0, \quad t \le r.$$

Moreover, $|y-a| \le |y-x| + |x-a| \le t + r < 2t$, which implies that $B(x,t) \cap B(a,2r)^{\rm c} \subseteq B(a,2t)$. It then follows that

$$
\begin{aligned}
&Mf_2(x)\\
&=\sup_{t>0}\frac{1}{|B(x,t)|}\int_{B(x,t)}|f_2(y)|dy\\
&=\max\left(\sup_{t>r}\frac{1}{|B(x,t)|}\int_{B(x,t)}|f_2(y)|dy,\ \sup_{0<t\le r}\frac{1}{|B(x,t)|}\int_{B(x,t)}|f_2(y)|dy\right)\\
&\le\sup_{t>r}\frac{1}{|B(x,t)|}\int_{B(x,t)\cap B(a,2r)^c}|f(y)|dy\\
&\le\sup_{t>r}\frac{1}{|B(x,t)|}\int_{B(a,2t)}|f(y)|dy\\
&=C\sup_{t>2r}\frac{1}{|B(a,t)|}\int_{B(a,t)}|f(y)|dy\\
&\le C\sup_{t>r}\frac{1}{w(B(a,t))^{\frac{1}{p}}}\|f\|_{L^{p,w}(B(a,t))}
\end{aligned}
$$

and

$$
Mf_2(x)\le C\sup_{r<t<\infty}w(B(a,t))^{-\frac{1}{p}}\|f\|_{L^{p,w}(B(a,t))},\quad x\in B(a,r). \tag{3.1}
$$

Hence,

$$
\|Mf_2\|_{L^{p,w}(B(a,r))}\le Cw(B(a,r))^{\frac{1}{p}}\sup_{r<t<\infty}w(B(a,t))^{-\frac{1}{p}}\|f\|_{L^{p,w}B(a,t)},
$$

and so we conclude that

$$
\|Mf\|_{L^{p,w}(B(a,r))}\le C_1w(B(a,r))^{\frac{1}{p}}\sup_{r<t<\infty}w(B(a,t))^{-\frac{1}{p}}\|f\|_{L^{p,w}B(a,t)}.
$$

Assume now that $p=1$. Thus,

$$
\|Mf\|_{WL^{1,w}(B(a,r))}\le 2\left(\|Mf_1\|_{WL^{1,w}(B(a,r)}+\|Mf_2\|_{WL^{1,w}(B(a,r))}\right).
$$

Since $M$ is bounded from $L^{1,w}$ to $WL^{1,w}$, we have

$$
\|Mf_1\|_{WL^{1,w}(B(a,r))}\le\|Mf_1\|_{WL^{1,w}}\le C\|f_1\|_{L^{1,w}}=C\|f\|_{L^{1,w}(B(a,2r))}.
$$

Theorem 2.3 and Lemma 2.12 then imply that

$$
\|Mf_1\|_{WL^{1,w}(B(a,r))}\le Cw(B(a,r))\sup_{r<t<\infty}w(B(a,t))^{-1}\|f\|_{L^{1,w}B(a,t)}.
$$

On the other hand, we can see that 3.1 also holds for $p=1$, which implies the following estimates.

$$
\begin{aligned}
\|Mf_2\|_{WL^{1,w}(B(a,r))} &= \sup_{\gamma>0} \gamma w(\{x \in B(a,r) : |Mf_2(x)| > \gamma\}) \\
&= \sup_{\gamma>0} \gamma \int_{\{x\in B(a,r):|Mf_2(x)|>\gamma\}} w(x)dx \\
&\leq \int_{B(a,r)} |Mf_2(x)|w(x)dx \\
&\leq C \int_{B(a,r)} \sup_{r<t<\infty} w(B(a,t))^{-1}\|f\|_{L^{1,w}(B(a,t))} w(x)dx \\
&= Cw(B(a,r)) \sup_{r<t<\infty} w(B(a,t))^{-1}\|f\|_{L^{1,w}(B(a,t))}.
\end{aligned}
$$

Therefore,

$$
\|Mf\|_{WL^{1,w}(B(a,r))} \leq C_2 w(B(a,r)) \sup_{r<t<\infty} w(B(a,t))^{-1}\|f\|_{L^{p,w}(B(a,t))},
$$

and this proves Theorem 3.1. □

The following theorem is one of our main results.

**Theorem 3.2.** *Let $1 \leq p < \infty$, let $w \in A_p$, and let $M$ be the Hardy–Littlewood maximal operator. Suppose that $\psi_1$ and $\psi_2$ are two positive functions on $\mathbb{R}^n \times (0,\infty)$ satisfying*

$$
\sup_{r<t<\infty} \psi_1(a,t) \leq C\psi_2(a,r)
$$

*for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$, where $C$ is a positive constant that does not depend on $a$ and $r$. Then*

(1) *$M$ is bounded from $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$ for $1 < p < \infty$.*

(2) *$M$ is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{1,w}_{\psi_2}$.*

*Proof.* First, assuming that $1 < p < \infty$, let $f \in \mathcal{M}^{p,w}_{\psi_1}$. By using Theorem 3.1 and the hypothesis about $\psi_1$ and $\psi_2$, we get

$$
\begin{aligned}
\|Mf\|_{\mathcal{M}^{p,w}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w(B(a,r))^{-\frac{1}{p}}\|Mf\|_{L^{p,w}(B(a,r))} \\
&\leq C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} w(B(a,t))^{-\frac{1}{p}}\|f\|_{L^{p,w}B(a,t)} \\
&= C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} \frac{\psi_1(a,t)}{\psi_1(a,t)} w(B(a,t))^{-\frac{1}{p}}\|f\|_{L^{p,w}(B(a,t))} \\
&\leq C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} \psi_1(a,t)\|f\|_{\mathcal{M}^{p,w}_{\psi_1}} \\
&= C\|f\|_{\mathcal{M}^{p,w}_{\psi_1}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} \psi_1(a,t) \leq C\|f\|_{\mathcal{M}^{p,w}_{\psi_1}}.
\end{aligned}
$$

Therefore, we conclude that $M$ is bounded from $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$. Next, we assume that $p = 1$, and let $f \in \mathcal{M}^{1,w}_{\psi_1}$. By using Theorem 3.1 and the hypothesis concerning $\psi_1$ and $\psi_2$, we get

$$
\begin{aligned}
\|Mf\|_{W\mathcal{M}^{1,w}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w(B(a,r))^{-1} \|Mf\|_{WL^{1,w}(B(a,r))} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} w(B(a,t))^{-1} \|f\|_{L^{1,w}B(a,t)} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} \psi_1(a,t) \|f\|_{\mathcal{M}^{1,w}_{\psi_1}} \\
&= C \|f\|_{\mathcal{M}^{1,w}_{\psi_1}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} \psi_1(a,t) \\
&\le C \|f\|_{\mathcal{M}^{1,w}_{\psi_1}}.
\end{aligned}
$$

Therefore, $M$ is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{1,w}_{\psi_2}$, and this completes the proof of the theorem. □

Consider the function $(a,t) \mapsto \psi_1(a,t)$ on $\mathbb{R}^n \times (0,\infty)$ by $\psi_1(a,t) = n$, where $t = 1/n$ for some $n \in \mathbb{N}$ and $\psi_1(a,t) = te^{-t}$ for otherwise. We also consider the function $(a,t) \mapsto \psi_2(a,t)$ by $\psi_2(a,t) = e^{-t}$ for every $a \in \mathbb{R}^n$ and $t > 0$. We can see that

$$\int_r^\infty \psi_1(a,t) \frac{dt}{t} \le \psi_2(a,r)$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$, but there is no $C > 0$ such that

$$\sup_{r<t<\infty} \psi_1(a,t) \le \psi_2(a,r), \quad (a,r) \in \mathbb{R}^n \times (0,\infty).$$

Moreover, we also consider the constant functions $(a,t) \mapsto \psi_1(a,t)$ and $(a,t) \mapsto \psi_2(a,t)$, where $\psi_2(a,t) = 2\psi_1(a,t) := 2$ for $a \in \mathbb{R}^n$ and $t > 0$. It is easy to see that

$$\sup_{r<t<\infty} \psi_1(a,t) \le \psi_2(a,r), \quad r > 0,$$

but there is no $C > 0$ such that

$$\int_r^\infty \psi_1(a,t) \frac{dt}{t} \le C\psi_2(a,r), \quad r > 0,$$

since the left-hand side is not convergent. Hence, we also investigate the condition

$$\int_r^\infty \psi_1(a,t) \frac{dt}{t} \le C\psi_2(a,r), \quad (a,r) \in \mathbb{R}^n \times (0,\infty)$$

for the boundedness of the Hardy–Littlewood maximal operator and obtain the following results.

**Theorem 3.3.** *Let $1 \le p < \infty$ and let $w \in A_p$. Suppose that $\psi_1$ and $\psi_2$ are positive functions on $\mathbb{R}^n \times (0,\infty)$ satisfying*

$$\int_r^\infty \psi_1(a,t) \frac{dt}{t} \le C\psi_2(a,r)$$

*for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$, where $C$ is a positive constant that does not depend on $a$ and $r$. Then*

(1) *$M$ is bounded from $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$ for $1<p<\infty$.*

(2) *$M$ is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{1,w}_{\psi_2}$.*

Before we present the proof of Theorem 3.3, we prove the following theorem.

**Theorem 3.4.** *Let $1 \le p < \infty$ and let $w \in A_p$. Then, for every $a \in \mathbb{R}^n$ and $r>0$,*

$$\|Mf\|_{L^{p,w}(B(a,r))} \le C_1 w(B(a,r))^{\frac{1}{p}} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}, \quad f \in L^{p,w}_{\mathrm{loc}},$$

*for $1<p<\infty$, and*

$$\|Mf\|_{WL^{1,w}(B(a,r))} \le C_2 w(B(a,r)) \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}, \quad f \in L^{1,w}_{\mathrm{loc}},$$

*where $C_1$ and $C_2$ are positive constants that are independent of $f$, $a$, and $r$.*

*Proof.* Given $a \in \mathbb{R}^n$ and $r>0$, we write $f := f_1 + f_2$, where $f_1 := f \cdot \mathcal{X}_{B(a,2r)}$. Then, by Theorems 2.3 and 2.6 and Lemma 2.12,

$$\begin{aligned}\|Mf_1\|_{L^{p,w}(B(a,r))} &\le C\|f\|_{L^{p,w}(B(a,r))}\\ &\le C w(B(a,r))^{\frac{1}{p}} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}\end{aligned}$$

for $1<p<\infty$. Meanwhile, for $p=1$, we have

$$\begin{aligned}\|Mf_1\|_{WL^{1,w}(B(a,r))} &\le C\|f\|_{L^{1,w}(B(a,r))}\\ &\le C w(B(a,r)) \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.\end{aligned}$$

Regarding $f_2$, for every $x \in B(a,r)$, we have

$$Mf_2(x) \le C \sup_{t>r} \frac{1}{|B(a,2t)|} \int_{B(a,2t)} |f(y)|dy,$$

so that we get

$$Mf_2(x) \le C \sup_{t>2r} \frac{1}{|B(a,t)|} \int_{B(a,t)} |f(y)|dy.$$

Hence,

$$Mf_2(x) \le C \sup_{t>r} \frac{1}{w(B(a,t))^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,t))}$$

for $1 \le p < \infty$. Therefore,

$$\begin{aligned}Mf_2(x) &\le C \sup_{t>r} \int_t^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}\\ &\le \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s},\end{aligned}$$

for every $x \in B(a,r)$. It thus follows that

$$\|Mf_2\|_{L^{p,w}(B(a,r))} \le C_1 w(B(a,r))^{\frac{1}{p}} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}$$

for $1 < p < \infty$ and

$$\|Mf_2\|_{WL^{1,w}(B(a,r))} \le C_2 w(B(a,r)) \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.$$

This proves Theorem 3.4. $\square$

Now, we are ready to prove Theorem 3.3.

*Proof of Theorem 3.3.* First, we assume that $1 < p < \infty$. Given $f \in \mathcal{M}^{p,w}_{\psi_1}$. By using Theorem 3.4 and the assumption concerning $\psi_1$ and $\psi_2$, we get

$$\begin{aligned}
\|Mf\|_{\mathcal{M}^{p,w}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w(B(a,r))^{-\frac{1}{p}} \|Mf\|_{L^{p,w}(B(a,r))} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s} \\
&= C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \frac{\psi_1(a,s)}{\psi_1(a,s)} w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \|f\|_{\mathcal{M}^{p,w}_{\psi_1}} \frac{ds}{s} \\
&= C \|f\|_{\mathcal{M}^{p,w}_{\psi_1}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \frac{ds}{s} \\
&\le C \|f\|_{\mathcal{M}^{p,w}_{\psi_1}}.
\end{aligned}$$

Therefore, we conclude that $M$ is bounded from $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$. Next, we assume that $p = 1$. Given $f \in \mathcal{M}^{1,w}_{\psi_1}$. By using Theorem 3.4 and the assumption concerning $\psi_1$ and $\psi_2$, we get

$$\begin{aligned}
\|Mf\|_{W\mathcal{M}^{1,w}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w(B(a,r))^{-1} \|Mf\|_{WL^{1,w}(B(a,r))} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s} \\
&= C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \frac{\psi_1(a,s)}{\psi_1(a,s)} w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s} \\
&\le C \sup_{a\in\mathbb{R}^n} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \|f\|_{\mathcal{M}^{1,w}_{\psi_1}} \frac{ds}{s} \\
&= C \|f\|_{\mathcal{M}^{1,w}_{\psi_1}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \frac{ds}{s} \\
&\le C \|f\|_{\mathcal{M}^{1,w}_{\psi_1}}.
\end{aligned}$$

Therefore, we conclude that $M$ is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{1,w}_{\psi_2}$. This completes the proof of Theorem 3.3. $\square$

## 4. Fractional integral and fractional maximal operators on generalized weighted Morrey spaces

In this section, we prove the boundedness of the fractional integral operator $I_\alpha$ on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces. The results then imply the boundedness of the fractional maximal operators on those spaces. As an application of Theorem 2.7, we have the following results.

**Theorem 4.1.** *Let $0 < \alpha < n, 1 \leq p < n/\alpha, 1/q = 1/p - \alpha/n$, and $w \in A_{p,q}$. Then*

$$\|I_\alpha f\|_{L^{q,w^q}(B(a,r))} \leq C_1 w^q(B(a,r))^{\frac{1}{q}} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{p,w^p}(B(a,s))} \frac{ds}{s},$$

*for every $a \in \mathbb{R}^n$ $r > 0$, and $f \in L^{p,w}_{\rm loc}$, where $1 < p < \infty$, and*

$$\|I_\alpha f\|_{WL^{q,w^q}(B(a,r))} \leq C_2 w^q(B(a,r))^{\frac{1}{q}} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s},$$

*for every $a \in \mathbb{R}^n$ $r > 0$, and $f \in L^{1,w}_{\rm loc}$. The constants $C_1$ and $C_2$ are positive constants that do not depend on $f, a$, and $r$.*

*Proof.* Given $a \in \mathbb{R}^n$ and $r > 0$, we decompose the function $f$ as $f := f_1 + f_2$, where $f_1 := f\mathcal{X}_{B(a,2r)}$, such that

$$I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x).$$

First, we assume that $1 < p < n/\alpha$. By Theorem 2.7, $I_\alpha$ is bounded from $L^{p,w^p}$ to $L^{q,w^q}$. Hence,

$$\|I_\alpha f_1\|_{L^{q,w^q}(B(a,r))} \leq \|I_\alpha f_1\|_{L^{q,w^q}} \leq C\|f_1\|_{L^{p,w^p}} = C\|f\|_{L^{p,w^p}(B(a,2r))}.$$

Since $w \in A_{p,q}$, it follows from Proposition 2.5 that $w^q \in A_q$. We see that the map $r \mapsto \|f\|_{L^{p,w^p}(B(a,2r))}$ is increasing for each $a \in \mathbb{R}^n$, and so by Theorem 2.3 and Lemma 2.12, we have

$$\|I_\alpha f_1\|_{L^{q,w^q}} \leq C w^q(B(a,r))^{\frac{1}{q}} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{p,w^p}(B(a,s))} \frac{ds}{s}.$$

Next, we obtain the same estimate for $I_\alpha f_2$. For this, we observe that

$$|I_\alpha f_2(x)| \leq \int_{B(a,2r)^c} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy.$$

The inequalities $|a-x| < r$ and $|x-y| \geq 2r$ imply

$$\frac{1}{2}|a-y| \leq |x-y| \leq \frac{3}{2}|a-y|.$$

Then

$$|I_\alpha f_2(x)| \leq C \int_{B(a,2r)^c} \frac{|f(y)|}{|a-y|^{n-\alpha}} dy, \quad x \in B(a,r).$$

By Fubini's theorem,

$$\begin{aligned}
|I_\alpha f_2(x)| &\le C\int_{B(a,2r)^c}\frac{|f(y)|}{|a-y|^{n-\alpha}}dy\\
&= C\int_{B(a,2r)^c}|f(y)|\int_{|a-y|}^{\infty}\frac{1}{s^{n-\alpha}}\frac{ds}{s}dy\\
&= C\int_r^\infty\int_{B(a,s)\setminus B(a,r)}\frac{1}{|B(a,s)|^{1-\frac{\alpha}{n}}}|f(y)|dy\frac{ds}{s}\\
&= \int_r^\infty\frac{1}{|B(a,s)|^{1+\frac{1}{q}-\frac{1}{p}}}\int_{B(a,s)}|f(y)|dy\frac{ds}{s}.
\end{aligned}$$

Next, by the Hölder's inequality and the assumption that $w\in A_{p,q}$, we have

$$\begin{aligned}
&\frac{1}{|B(a,s)|^{1+\frac{1}{q}-\frac{1}{p}}}\int_{B(a,s)}|f(y)|dy\\
&= \frac{1}{|B(a,s)|^{1+\frac{1}{q}-\frac{1}{p}}}\int_{B(a,s)}\frac{|f(y)|w(y)}{w(y)}dy\\
&\le \frac{1}{|B(a,s)|^{1+\frac{1}{q}-\frac{1}{p}}}\left(\int_{B(a,s)}|f(y)|^p w(y)^p dy\right)^{\frac{1}{p}}\left(\int_{B(a,s)}w(y)^{-p'}dy\right)^{\frac{1}{p'}}\\
&= w^q(B(a,s))^{-\frac{1}{q}}\|f\|_{L^{p,w^p}(B(a,s))}\left(\frac{1}{|B(a,s)|}\int_{B(a,s)}w(y)^q dy\right)^{\frac{1}{q}}\\
&\quad\cdot\left(\frac{1}{|B(a,s)|}\int_{B(a,s)}w(y)^{-p'}dy\right)^{\frac{1}{p'}}\\
&\le Cw^q(B(a,s))^{-\frac{1}{q}}\|f\|_{L^{p,w^p}(B(a,s))}.
\end{aligned}$$

Hence,

$$|I_\alpha f_2(x)|\le C\int_r^\infty w^q(B(a,s))^{-\frac{1}{q}}\|f\|_{L^{p,w^p}(B(a,s))}\frac{ds}{s},\quad x\in B(a,r),$$

and this implies that

$$\|I_\alpha f_2\|_{L^{q,w^q}(B(a,r))}\le Cw^q(B(a,r))^{\frac{1}{q}}\int_r^\infty w^q(B(a,s))^{-\frac{1}{q}}\|f\|_{L^{p,w^p}(B(a,s))}\frac{ds}{s}.$$

Therefore,

$$\|I_\alpha f\|_{L^{q,w^q}(B(a,r))}\le C_1 w^q(B(a,r))^{\frac{1}{q}}\int_r^\infty w^q(B(a,s))^{-\frac{1}{q}}\|f\|_{L^{p,w^p}(B(a,s))}\frac{ds}{s}.$$

Next, we assume $p=1$. Note that

$$\|I_\alpha f\|_{WL^{q,w^q}(B(a,r))}\le 2\left(\|I_\alpha f_1\|_{WL^{q,w^q}(B(a,r))}+\|I_\alpha f_2\|_{WL^{q,w^q}(B(a,r))}\right).$$

By Theorem 2.7, we have

$$\|I_\alpha f_1\|_{WL^{q,w^q}(B(a,r))}\le\|I_\alpha f_1\|_{WL^{q,w^q}}\le C\|I_\alpha f_1\|_{L^{1,w}}=C\|I_\alpha f\|_{L^{1,w}(B(a,2r))}.$$

Since $w \in A_{1,q}$, it follows from Proposition 2.5 that $w^q \in A_q$. By the same argument as for the case $p > 1$, we obtain

$$\|I_\alpha f_1\|_{WL^{q,w^q}(B(a,r))} \leq C w^q(B(a,r))^{\frac{1}{q}} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.$$

Since

$$|I_\alpha f_2(x)| \leq C \int_{B(a,2r)^c} \frac{|f(y)|}{|a-y|^{n-\alpha}} dy, \quad x \in B(a,r),$$

by Fubini's theorem, the following estimates hold:

$$\begin{aligned}
|I_\alpha f_2(x)| &\leq C \int_{\mathbb{R}^n \setminus B(a,2r)} \frac{|f(y)|}{|a-y|^{n-\alpha}} dy \\
&= C \int_{B(a,2r)^c} |f(y)| \int_{|a-y|}^\infty \frac{1}{s^{n-\alpha}} \frac{ds}{s} dy \\
&= C \int_r^\infty \int_{B(a,s) \setminus B(a,r)} \frac{1}{|B(a,s)|^{1-\frac{\alpha}{n}}} |f(y)| dy \frac{ds}{s} \\
&= \int_r^\infty \frac{1}{|B(a,s)|^{\frac{1}{q}}} \int_{B(a,s)} |f(y)| dy \frac{ds}{s}.
\end{aligned}$$

By Hölder's inequality combined with the assumption $w \in A_{1,q}$ and the fact that $q > 1$, we obtain

$$\begin{aligned}
&\frac{1}{|B(a,s)|^{\frac{1}{q}}} \int_{B(a,s)} |f(y)| dy \\
&\leq \frac{1}{|B(a,s)|^{\frac{1}{q}}} \int_{B(a,s)} |f(y)| \frac{w(y)}{w(y)} dy \\
&\leq \frac{1}{|B(a,s)|^{\frac{1}{q}}} \left( \int_{B(a,s)} |f(y)| w(y) dy \right) \|w^{-1}\|_{L^\infty(B(a,s))} \\
&= w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{1,w}(B(a,s))} \left( \frac{1}{|B(a,s)|} \int_{B(a,s)} w(y)^q dy \right)^{\frac{1}{q}} \|w^{-1}\|_{L^\infty(B(a,s))} \\
&\leq C w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{1,w}(B(a,s))}.
\end{aligned}$$

Hence, for $x \in B(a,r)$,

$$\begin{aligned}
|I_\alpha f_2(x)| &\leq C \int_r^\infty \frac{1}{|B(a,s)|^{\frac{1}{q}}} \int_{B(a,s)} |f(y)| dy \frac{ds}{s} \\
&\leq C \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s},
\end{aligned}$$

and

$$
\begin{aligned}
\|I_\alpha f_2\|_{WL^{q,w^q}(B(a,r))} &= \sup_{\gamma>0} \gamma w^q(\{x \in B(a,r) : |I_\alpha f_2(x)| > \gamma\})^{\frac{1}{q}} \\
&= \sup_{\gamma>0} \gamma \left( \int_{\{x \in B(a,r) : |I_\alpha f_2(x)| > \gamma\}} w(x)^q dx \right)^{\frac{1}{q}} \\
&\le \sup_{\gamma>0} \left( \int_{\{x \in B(a,r) : |I_\alpha f_2(x)| > \gamma\}} |I_\alpha f_2(y)|^q w(y)^q dy \right)^{\frac{1}{q}} \\
&= \left( \int_{B(a,r)} |I_\alpha f_2(y)|^q w(y)^q dy \right)^{\frac{1}{q}} \\
&\le C w^q(B(a,r))^{\frac{1}{q}} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.
\end{aligned}
$$

Therefore,

$$
\|I_\alpha f\|_{WL^{q,w^q}(B(a,r))} \le C_2 w^q(B(a,r))^{\frac{1}{q}} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.
$$

□

The following theorem is our main results concerning the boundedness of the fractional integrals on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces.

**Theorem 4.2.** *Let $0 < \alpha < n, 1 \le p < n/\alpha, 1/q = 1/p - \alpha/n, w \in A_{p,q}$, and let $I_\alpha$ be the fractional integral operator. Suppose that $\psi_1$ and $\psi_2$ are nonnegative functions on $\mathbb{R}^n \times (0,\infty)$ satisfying*

$$
\int_r^\infty \frac{w^p(B(a,t))^{\frac{1}{p}}}{w^q(B(a,t))^{\frac{1}{q}}} \psi_1(a,t) \frac{dt}{t} \le C\psi_2(a,r)
$$

*for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$, where $C$ is a positive constant that does not depend on $a$ and $r$. Then*

(1) *$I_\alpha$ is bounded from $\mathcal{M}^{p,w^p}_{\psi_1}$ to $\mathcal{M}^{q,w^q}_{\psi_2}$ for $1 < p < \infty$.*
(2) *$I_\alpha$ is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{q,w^q}_{\psi_2}$.*

*Proof.* First, we assume that $1 < p < n/\alpha$, and let $f \in \mathcal{M}_{\psi_1}^{p,w^p}$. By using Theorem 4.1 and the assumption on $\psi_1$ and $\psi_2$, we get

$$\begin{aligned}
\|I_\alpha f\|_{\mathcal{M}_{\psi_2}^{q,w^q}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \left( \frac{1}{w^q(B(a,r))} \int_{B(a,r)} |I_\alpha f(x)|^q w(x)^q dx \right)^{\frac{1}{q}} \\
&= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w^q(B(a,r))^{-\frac{1}{q}} \|I_\alpha f\|_{L^{q,w^q}(B(a,r))} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{p,w^p}(B(a,s))} \frac{ds}{s} \\
&= C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \frac{\psi_1(a,s)}{\psi_1(a,s)} w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{p,w^p}(B(a,s))} \frac{ds}{s} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \frac{w^q(B(a,s))^{-\frac{1}{q}}}{w^p(B(a,s))^{-\frac{1}{p}}} \|f\|_{\mathcal{M}_{\psi_1}^{p,w^p}} \frac{ds}{s} \\
&= C\|f\|_{\mathcal{M}_{\psi_1}^{p,w^p}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_t^\infty \psi_1(a,s) \frac{w^p(B(a,s))^{\frac{1}{p}}}{w^q(B(a,s))^{\frac{1}{q}}} \frac{ds}{s} \\
&\le C\|f\|_{\mathcal{M}_{\psi_1}^{p,w^p}}.
\end{aligned}$$

Therefore, we conclude that $I_\alpha$ is bounded from $\mathcal{M}_{\psi_1}^{p,w^p}$ to $\mathcal{M}_{\psi_2}^{q,w^q}$. Next, we assume that $p = 1$, and let $f \in \mathcal{M}_{\psi_1}^{1,w}$. By using Theorem 4.1 and the assumption on $\psi_1$ and $\psi_2$, we get

$$\begin{aligned}
\|I_\alpha f\|_{W\mathcal{M}_{\psi_2}^{q,w^q}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w^q(B(a,r))^{-\frac{1}{q}} \|I_\alpha f\|_{WL^{1,w}(B(a,r))} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \frac{w^q(B(a,s))^{-\frac{1}{q}}}{w(B(a,s))^{-1}} \|f\|_{\mathcal{M}_{\psi_1}^{1,w}} \frac{ds}{s} \\
&= C\|f\|_{\mathcal{M}_{\psi_1}^{1,w}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_t^\infty \psi_1(a,s) \frac{w(B(a,s))}{w^q(B(a,s))^{\frac{1}{q}}} \frac{ds}{s} \\
&\le C\|f\|_{\mathcal{M}_{\psi_1}^{1,w}}.
\end{aligned}$$

Hence, $I_\alpha$ is bounded from $\mathcal{M}_{\psi_1}^{1,w}$ to $W\mathcal{M}_{\psi_2}^{q,w^q}$, which completes the proof. □

The relation (1.1) and Theorem 4.2 imply the following corollary for $M_\alpha$.

**Corollary 4.3.** *Let $0 < \alpha < n, 1 \le p < n/\alpha, 1/q = 1/p - \alpha/n$, and $w \in A_{p,q}$. Suppose that $\psi_1$ and $\psi_2$ are nonnegative functions on $\mathbb{R}^n \times (0, \infty)$ satisfying*

$$\int_r^\infty \frac{w^p(B(a,t))^{\frac{1}{p}}}{w^q(B(a,s))^{\frac{1}{q}}} \psi_1(a,s) \frac{dt}{t} \le C\psi_2(a,r)$$

*for every* $(a, r) \in \mathbb{R}^n \times (0, \infty)$*, where* $C$ *is a positive constant that does not depend on* $a$ *and* $r$*. Then,* $M_\alpha$ *is bounded from* $\mathcal{M}^{p,w^p}_{\psi_1}$ *to* $\mathcal{M}^{q,w^q}_{\psi_2}$ *for* $1 < p < \infty$ *and* $M_\alpha$ *is bounded from* $\mathcal{M}^{1,w}_{\psi_1}$ *to* $W\mathcal{M}^{q,w^q}_{\psi_2}$*.*

## 5. Calderón–Zygmund operators on generalized weighted Morrey spaces

In this section, we prove the boundedness of the Calderón–Zygmund operators $T = T_K$ on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces. As stated earlier, we have Theorem 2.8 about the boundedness of the Calderón–Zygmund operators on weighted Lebesgue spaces and weighted weak Lebesgue spaces. This means that the Calderón–Zygmund operator, which is initially defined on $\mathcal{S}$, can be extended to $L^{p,w}$ for $1 \leq p < \infty$ and $w \in A_p$. In general, the generalized weighted Morrey space $\mathcal{M}^{p,w}_\psi$ is not equal to $L^{p,w}$ as in the case of the classical Morrey space. Therefore, it is a need to define the Calderón–Zygmund $T$ on $\mathcal{M}^{p,w}_\psi$. To do so, we refer to the method as in [27].

**Definition 5.1** ([27, Definition 95])**.** Let $T$ be a Calderón-Zymgund operator. Define $T$ as

$$Tf(x) = T[f \cdot \mathcal{X}_{2B}](x) + \int_{(2B)^c} K(x, y)f(y)dy, \quad x \in B, \tag{5.1}$$

for $f \in L^1_{\text{loc}}$ for which the left-hand side converges for every ball $B$.

*Remark* 5.2. Using [11, Poposition 8.2.2], the operator $T$ defined in the definition is consistent to the original Calderón–Zygmund operator.

The definition is well-defined, in the sense that the definition of $Tf$ is independent of the choice of the ball $B$ containing $x$ [27, Proposition 399]. Moreover, the right-hand side of (5.1) converges for every ball $B$ using the properties of Kernel $K$, Theorem 2.8, and Corollary 2.13. We use the definition 5.1 for Calderón–Zygmund operator $T$ on generalized weighted Morrey spaces. As the result, we have the following theorem.

**Theorem 5.3.** *Let* $1 \leq p < \infty, w \in A_p$*, and* $f \in L^{p,w}_{\text{loc}}$*. Then, for every* $a \in \mathbb{R}^n$ *and* $r > 0$*,*

$$\|Tf\|_{L^{p,w}(B(a,r))} \leq C_1 w(B(a, r))^{\frac{1}{p}} \int_r^\infty w(B(a, s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s},$$

*where* $1 < p < \infty$*, and*

$$\|Tf\|_{WL^{1,w}(B(a,r))} \leq C_2 w(B(a, r)) \int_r^\infty w(B(a, s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s},$$

*where* $C_1$ *and* $C_2$ *are positive constants that do not depend on* $f, a,$ *and* $r$*.*

*Proof.* Write $f$ as $f := f_1 + f_2$, where $f_1 := f \cdot \mathcal{X}_{B(a,2r)}$. As in the definition 5.1, we have

$$Tf(x) = Tf_1(x) + \int_{\mathbb{R}^n} K(x, y)f_2(y)dy := Tf_1(x) + T_2(f)(x), \quad x \in B(a, r).$$

First, we consider the case where $1 < p < \infty$. Then, since $w \in A_p$, we know that $T$ is bounded on $L^{p,w}$. Thus, for every $a \in \mathbb{R}^n$ and $r > 0$, we have

$$\|Tf\|_{L^{p,w}(B(a,r))} \leq \|Tf_1\|_{L^{p,w}(B((a,r))} + \|T_2 f\|_{L^{p,w}(B((a,r))}$$

and

$$\|Tf_1\|_{L^{p,w}(B(a,r))} \leq \|Tf_1\|_{L^{p,w}} \leq C\|f_1\|_{L^{p,w}} \leq C\|f\|_{L^{p,w}(B(a,2r))}.$$

By Theorem 2.3 and Lemma 2.12,

$$\|Tf_1\|_{L^{p,w}(B(a,r))} \leq Cw(B(a,r))^{\frac{1}{p}} \int_r^\infty \frac{1}{w(B(a,s))^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}.$$

Note that for $x \in B(a,r)$, we have

$$|T_2 f(x)| \leq C \int_{B(a,2r)^c} \frac{|f(y)|}{|x-y|^n} dy.$$

On other hand, the inequalities $|a - x| < r$ and $|x - y| \geq 2r$ imply that

$$\frac{1}{2}|a-y| \leq |x-y| \leq \frac{3}{2}|a-y|.$$

Then, by using Lemma 2.13,

$$|T_2 f(x)| \leq C \int_{B(a,2r)^c} \frac{|f(y)|}{|a-y|^n} dy \leq C \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}.$$

Hence,

$$\|T_2 f\|_{L^{p,w}(B(a,r))} \leq Cw(B(a,r))^{\frac{1}{p}} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}.$$

Therefore, we conclude that

$$\|Tf\|_{L^{p,w}(B(a,r))} \leq C_1 w(B(a,r))^{\frac{1}{p}} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}.$$

Next, for the case where $p = 1$, we have

$$\|Tf\|_{WL^{1,w}(B(a,r))} \leq 2(\|Tf_1\|_{WL^{1,w}(B(a,r))} + \|T_2 f\|_{WL^{1,w}(B(a,r))})$$

and

$$\|Tf_1\|_{WL^{1,w}(B(a,r))} \leq \|Tf_1\|_{L^{1,w}} \leq C\|f_1\|_{L^{1,w}} \leq C\|f\|_{L^{1,w}(B(a,2r))}$$

for every $a \in \mathbb{R}^n$ and $r > 0$. By the boundedness of $T$ from $L^{1,w}$ to $WL^{1,w}$, we have

$$\|Tf_1\|_{WL^{1,w}(B(a,r))} \leq \|Tf_1\|_{WL^{1,w}} \leq C\|f_1\|_{L^{1,w}} \leq C\|f\|_{L^{1,w}(B(a,2r))}.$$

By Theorem 2.3 and Lemma 2.12,

$$\|Tf_1\|_{WL^{1,w}(B(a,r))} \leq Cw(B(a,r)) \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.$$

As we do before, for $x \in B(a,r)$, we have

$$|T_2 f(x)| \leq C \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.$$

Thus,

$$\begin{aligned}\|T_2 f\|_{WL^{1,w}(B(a,r))} &= \sup_{\gamma>0} \gamma w(\{x \in B(a,r) : |Tf_2(x)| > \gamma\}) \\ &\leq \int_{B(a,r)} |Tf_2(x)| w(x) dx \\ &\leq C w(B(a,r)) \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.\end{aligned}$$

Therefore,

$$\|Tf\|_{WL^{1,w}(B(a,r))} \leq C_2 w(B(a,r)) \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s},$$

and this proves Theorem 5.3. □

The following theorem is our main result concerning the boundedness of the Calderon–Zygmund operators on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces.

**Theorem 5.4.** *Let $1 \leq p < \infty$, let $w \in A_p$, and let $T$ be the Calderon–Zygmund operator. Suppose that $\psi_1$ and $\psi_2$ are functions on $\mathbb{R}^n \times (0,\infty)$ satisfying*

$$\int_r^\infty \psi_1(a,t) \frac{dt}{t} \leq C \psi_2(a,r)$$

*for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$, where $C$ is a positive constant that does not depend on $a$ and $r$. Then*

(1) *$T$ is bounded from $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$ for $1 < p < \infty$.*
(2) *$T$ is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{1,w}_{\psi_2}$.*

*Proof.* First, we assume that $1 < p < \infty$. Let $f \in \mathcal{M}^{p,w}_{\psi_1}$. By using Theorem 5.3 and the assumption concerning $\psi_1$ and $\psi_2$, we get

$$\begin{aligned}\|Tf\|_{\mathcal{M}^{p,w}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w(B(a,r))^{-\frac{1}{p}} \|Tf\|_{L^{p,w}(B(a,r))} \\ &\leq C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s} \\ &\leq C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \|f\|_{\mathcal{M}^{p,w}_{\psi_1}} \frac{ds}{s} \\ &= C \|f\|_{\mathcal{M}^{p,w}_{\psi_1}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \frac{ds}{s} \leq C \|f\|_{\mathcal{M}^{p,w}_{\psi_1}}.\end{aligned}$$

Therefore, $T$ is bounded from $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$. Next, we assume that $p = 1$, and let $f \in \mathcal{M}^{1,w}_{\psi_1}$. By using Theorem 5.3 and the assumption on $\psi_1$ and $\psi_2$, we get

$$\begin{aligned}
\|Tf\|_{W\mathcal{M}^{p,w}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w(B(a,r))^{-1} \|Tf\|_{WL^{1,w}(B(a,r))} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s} \\
&\le C \sup_{a\in\mathbb{R}^n} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \|f\|_{\mathcal{M}^{1,w}_{\psi_1}} \frac{ds}{s} \\
&= C \|f\|_{\mathcal{M}^{1,w}_{\psi_1}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \frac{ds}{s} \le C \|f\|_{\mathcal{M}^{1,w}_{\psi_1}}.
\end{aligned}$$

Therefore, $T$ is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{1,w}_{\psi_2}$, and this completes the proof. $\square$

*Remark* 5.5. By using the results in [26], we can extend Theorem 5.4 by replacing $T$ with $\theta$-type Calderon–Zygmund operators $T_\theta$. The definition of $\theta$-type Calderon–Zygmund Operator $T_\theta$ may be found in [31]. Accordingly, one can obtain a result that is more general than [30].

**Acknowledgement.** The second author is supported by P2MI-ITB 2022 Program. Both authors thank the referee for his/her useful remarks on the earlier version of this paper. We also thank Dr. Denny I. Hakim for his information about a reference for Proposition 2.5.

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