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APPLICATIONS OF NON-HOMOGENEOUS CAUCHY-EULER FRACTIONAL q-DIFFERENTIAL EQUATION TO A NEW CLASS OF ANALYTIC FUNCTIONS

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ABSTRACT. We define a new general fractional q-differential operator, and by means of this operator, we introduce a new subclass of analytic functions that are the solutions of the non-homogeneous Cauchy–Euler fractional q-differential equations. Our aim is to determine upper bounds of Taylor–Maclaurin coefficients for functions belong to this class.

1. INTRODUCTION AND PRELIMINARIES

The sets of real numbers, complex numbers, and positive integers will be denoted by

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{C} = \mathbb{C}^{*} \cup \{0\} \quad \text{and} \quad \mathbb{N} = \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}.$$

Also, we need the following basic definitions of the q-calculus, which are used in this paper (see, for details, [10, 11] and also [4]).

For $0 < q < 1$, the q-number and the q-factorial are defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & n \in \mathbb{C}, \\ \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots q^{n-1}, & n \in \mathbb{N}, \end{cases}$$

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and

$$[n]_q! = \begin{cases} 1, & n = 0, \\ \prod_{r=1}^{n} [r]_q, & n \in \mathbb{N}, \end{cases}$$

respectively. As $q \to 1^-$, $[n]_q \to n$, and $[n]_q! \to n!$.

For $\tau, \sigma \in \mathbb{C}$, the q-shifted factorial $(\tau; q)_\sigma$ is defined by (see [3])

$$(\tau; q)_\sigma = \prod_{r=0}^{\infty} \left(\frac{1 - \tau q^r}{1 - \tau q^{\sigma + r}} \right)$$

so that

$$(\tau; q)_n = \begin{cases} 1, & n = 0, \\ \prod_{r=0}^{n-1} (1 - \tau q^r), & n \in \mathbb{N}, \end{cases} \tag{1.1}$$

and

$$(\tau; q)_\infty = \prod_{r=0}^{\infty} (1 - \tau q^r).$$

Furthermore, the q-Gamma function Γ_q is defined by

$$\Gamma_q(z) = \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z} \qquad (z \in \mathbb{C}).$$

From (1.1), we obtain that

$$(q^z; q)_n = \frac{\Gamma_q(z + n)}{\Gamma_q(z)} (1 - q)^n.$$

Thus for $n = 1$, the above equality implies that

$$\Gamma_q(z + 1) = [z]_q \Gamma_q(z) \qquad \text{and} \qquad \Gamma_q(1) = 1.$$

For a function f defined on a subset of $\mathbb{C}$, Jackson's q-derivative $\partial_q f$ is defined by (see [10, 11])

$$\partial_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \tag{1.2}$$

provided that $f'(0)$ exists. Then for a function $g(z) = z^k$, we have

$$\partial_q \left(z^k \right) = [k]_q z^{k-1},$$

$$\lim_{q \to 1^-} \left(\partial_q \left(z^k \right) \right) = k z^{k-1} = g'(z),$$

where g' is the ordinary derivative.

Jackson [10] introduced the q-integral by

$$\int_0^z f(t) d_q t = z (1 - q) \sum_{k=0}^{\infty} q^k f\left(z q^k \right),$$

as long as the series converges. Then for a function $g(z) = z^k$, we obtain

$$\int_0^z g(t)\, d_q t = \int_0^z t^k d_q t = \frac{1}{[k+1]_q} z^{k+1} \qquad (k \neq -1)$$

and

$$\lim_{q \to 1^-} \int_0^z g(t)\, d_q t = \int_0^z g(t)\, dt,$$

where $\int_0^z g(t)\, dt$ is the ordinary integral.

The fractional q-derivative operator of order ρ is defined, for a function f, by (see [13, 14, 18])

$$D_{q,z}^{\rho} f(z) = \frac{1}{\Gamma_q(1-\rho)} \partial_q \int_0^z (z - tq)_q^{-\rho} f(t) d_q t \qquad (0 \leq \rho < 1), \tag{1.3}$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin, and the multiplicity of the q-binomial $(z - tq)_q^{-\rho}$ is single-valued when

$$\left| \arg \left(-\frac{tq^{\rho}}{z} \right) \right| < \pi, \quad \left| \frac{tq^{\rho}}{z} \right| < 1, \quad \text{and} \quad |\arg(z)| < \pi.$$

It readily follows from (1.3) that

$$D_{q,z}^{\rho} z^k = \frac{\Gamma_q(k+1)}{\Gamma_q(k-\rho+1)} z^{k-\rho} \qquad (0 \leq \rho < 1,\ k \in \mathbb{N}).$$

Let $\mathcal{H}$ be the class of analytic functions in the open unit disk

$$\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}.$$

For two functions $\mathfrak{f}, \mathfrak{g} \in \mathcal{H}$, we say that the function $\mathfrak{f}$ is subordinate to $\mathfrak{g}$ in $\mathbb{D}$, and write

$$\mathfrak{f}(z) \prec \mathfrak{g}(z) \qquad (z \in \mathbb{D})$$

if there exists a Schwarz function $\Theta \in \mathcal{H}$ with

$$\Theta(0) = 0 \qquad \text{and} \qquad |\Theta(z)| < 1 \qquad (z \in \mathbb{D})$$

such that

$$\mathfrak{f}(z) = \mathfrak{g}(\Theta(z)) \qquad (z \in \mathbb{D}).$$

It is known that

$$\mathfrak{f}(z) \prec \mathfrak{g}(z) \quad (z \in \mathbb{D}) \Rightarrow \mathfrak{f}(0) = \mathfrak{g}(0) \qquad \text{and} \qquad \mathfrak{f}(\mathbb{D}) \subset \mathfrak{g}(\mathbb{D}).$$

Lemma 1.1. *[15] Let the function $\mathfrak{G}$ given by*

$$\mathfrak{G}(z) = \sum_{k=1}^{\infty} \mathfrak{B}_k z^k \qquad (z \in \mathbb{D})$$

be convex in $\mathbb{D}$. Also, let the function $\mathfrak{F} \in \mathcal{H}$ be given by

$$\mathfrak{F}(z) = \sum_{k=1}^{\infty} \mathfrak{A}_k z^k \qquad (z \in \mathbb{D}).$$

If

$$\mathfrak{F}(z) \prec \mathfrak{G}(z) \qquad (z \in \mathbb{D}),$$

then

$$|\mathfrak{A}_k| \leq |\mathfrak{B}_1| \qquad (k \in \mathbb{N}).$$

Let $\mathcal{A}_p(n)$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \qquad (p, n \in \mathbb{N}), \tag{1.4}$$

which are analytic and p-valent in the open unit disk $\mathbb{D}$. In particular, we set

$$\mathcal{A}_p(1) := \mathcal{A}_p, \qquad \mathcal{A}_1(1) = \mathcal{A}_1 := \mathcal{A}.$$

For a function $f \in \mathcal{A}_p(n)$ given by (1.4), from (1.2), we deduce that

$$\begin{aligned}
\partial_q^{(1)} f(z) &= [p]_q z^{p-1} + \sum_{k=p+n}^{\infty} [k]_q a_k z^{k-1} =: \partial_q f(z), \\
\partial_q^{(2)} f(z) &= [p]_q [p-1]_q z^{p-2} + \sum_{k=p+n}^{\infty} [k]_q [k-1]_q a_k z^{k-2}, \\
&\vdots \\
\partial_q^{(p)} f(z) &= [p]_q! + \sum_{k=p+n}^{\infty} \frac{[k]_q!}{[k-p]_q!} a_k z^{k-p},
\end{aligned}$$

where $\partial_q^{(p)} f(z)$ is the pth q-derivative of $f(z)$.

Now, using the fractional q-derivative operator $D_{q,z}^{\rho} f$, we define the following q-derivative operator

$$\Omega_{q,p}^{\rho} : \mathcal{A}_p(n) \to \mathcal{A}_p(n)$$

as follows:

$$\begin{aligned}
\Omega_{q,p}^{\rho} f(z) &= \frac{\Gamma_q(p-\rho+1)}{\Gamma_q(p+1)} z^{\rho} D_{q,z}^{\rho} f(z) \qquad (\rho \neq p+1,\ p+2,\ p+3, \ldots) \\
&= z^p + \sum_{k=p+n}^{\infty} \frac{\Gamma_q(k+1)\Gamma_q(p-\rho+1)}{\Gamma_q(p+1)\Gamma_q(k-\rho+1)} a_k z^k,
\end{aligned} \tag{1.5}$$

where the function $f \in \mathcal{A}_p(n)$ is given by (1.4). Note that $\Omega_{q,p}^{0} f(z) = f(z)$.

Remark 1.2. (i) If we let $q \to 1^-$, then we have the operator $\Omega_p^{\rho} : \mathcal{A}_p(n) \to \mathcal{A}_p(n)$ introduced by Bulut [5].
(ii) For $n = 1$, we get the operator $\Omega_{q,p}^{\rho} : \mathcal{A}_p \to \mathcal{A}_p$ introduced by Selvakumaran et al. [17].
(iii) For $q \to 1^-$ and $p = n = 1$, we get the operator $\Omega^{\rho} : \mathcal{A} \to \mathcal{A}$ introduced by Owa and Srivastava [12].

Now by considering the operator $\Omega_{q,p}^{\rho}$ given by (1.5), we define the general fractional q-differential operator $\mathfrak{D}_{q,\lambda,l,p}^{m,\rho}$ as follows:

$$\begin{aligned}
\mathfrak{D}_{q,\lambda,l,p}^{m,0} f(z) &= f(z), \\
\mathfrak{D}_{q,\lambda,l,p}^{1,\rho} f(z) &= \frac{[p]_q - \lambda [p]_q + l}{[p]_q + l} \Omega_{q,p}^{\rho} f(z) + \frac{\lambda}{[p]_q + l} z\partial_q \left(\Omega_{q,p}^{\rho} f(z)\right) \\
&= \mathfrak{D}_{q,\lambda,l,p}^{\rho} f(z) \qquad (\lambda, l \geq 0,\ 0 \leq \rho < 1), \qquad (1.6)\\
\mathfrak{D}_{q,\lambda,l,p}^{2,\rho} f(z) &= \mathfrak{D}_{q,\lambda,l,p}^{\rho} \left(\mathfrak{D}_{q,\lambda,l,p}^{1,\rho} f(z)\right), \\
&\ \vdots \\
\mathfrak{D}_{q,\lambda,l,p}^{m,\rho} f(z) &= \mathfrak{D}_{q,\lambda,l,p}^{\rho} \left(\mathfrak{D}_{q,\lambda,l,p}^{m-1,\rho} f(z)\right) \qquad (m \in \mathbb{N}). \qquad (1.7)
\end{aligned}$$

If f is given by (1.4), then by (1.5), (1.6), and (1.7), we see that

$$\mathfrak{D}_{q,\lambda,l,p}^{m,\rho} f(z) = z^p + \sum_{k=p+n}^{\infty} \Psi_{q,k,m}(\rho, \lambda, l, p)\, a_k z^k \qquad (m \in \mathbb{N}_0), \qquad (1.8)$$

where

$$\Psi_{q,k,m}(\rho, \lambda, l, p) = \left[\frac{\Gamma_q(k+1)\Gamma_q(p-\rho+1)}{\Gamma_q(p+1)\Gamma_q(k-\rho+1)} \frac{[p]_q + \lambda\left([k]_q - [p]_q\right) + l}{[p]_q + l}\right]^m. \qquad (1.9)$$

Remark 1.3. (i) If we let $q \to 1^-$, then we obtain the operator $\mathfrak{D}_{\lambda,l,p}^{m,\rho}$ introduced by Bulut [5]. The operator $\mathfrak{D}_{\lambda,l,p}^{m,\rho}$ is a comprehensive generalization some known operators (see [1,2,16]).
(ii) For $l = 0$ and $n = 1$ in (1.8), we obtain the operator $\mathfrak{D}_{q,\lambda,p}^{m,\rho}$ introduced by Selvakumaran et al. [17].
(iii) For $l = 0$, $\lambda = 1$ and $\alpha = 0$ in (1.8), we obtain p-valent q-Sălăgean operator defined by El-Qadeem and Mamon [8]. In addition for $p = 1$ and $n = 1$, we get q-Sălăgean operator introduced by Govindaraj and Sivasubramanian [9].

By means of the fractional q-differential operator $\mathfrak{D}_{q,\lambda,l,p}^{m,\rho}$, we introduce a new subclass of analytic and p-valent functions.

Definition 1.4. Let $\varphi : \mathbb{D} \to \mathbb{C}$ be a convex function such that

$$\varphi(0) = 1 \qquad \text{and} \qquad \Re(\varphi(z)) > 0 \qquad (z \in \mathbb{D}). \qquad (1.10)$$

We denote by $\mathcal{SK}_{q,\rho,\lambda,l}^{m,p,n}(\delta, \gamma; \varphi)$ the class of functions $f \in \mathcal{A}_p(n)$ satisfying

$$1 + \frac{1}{\gamma}\left(\frac{1}{[p]_q} \frac{z\partial_q\left(\frac{1}{1+\delta([p]_q-1)}\left[\delta z\partial_q\left(\mathfrak{D}_{\lambda,l,p}^{m,\rho} f(z)\right) + (1-\delta)\,\mathfrak{D}_{\lambda,l,p}^{m,\rho} f(z)\right]\right)}{\frac{1}{1+\delta([p]_q-1)}\left[\delta z\partial_q\left(\mathfrak{D}_{q,\lambda,l,p}^{m,\rho} f(z)\right) + (1-\delta)\,\mathfrak{D}_{q,\lambda,l,p}^{m,\rho} f(z)\right]} - 1\right) \in \varphi(\mathbb{D}),$$

where $z \in \mathbb{D}$, $\gamma \in \mathbb{C}^*$, $0 \leq \delta \leq 1$, and $\mathfrak{D}_{q,\lambda,l,p}^{m,\rho}$ is given by in (1.8).

Remark 1.5. If the function φ satisfying the condition (1.10) is chosen as

$$\varphi(z) = \frac{1 + (1 - (1+q)\beta)\, z}{1 - qz} \qquad (0 \leq \beta < 1;\ z \in \mathbb{D}),$$

then we obtain the class $\mathcal{SK}^{m,p,n}_{q,\rho,\lambda,l}(\delta,\gamma;\beta)$ that consists of functions $f\in\mathcal{A}_p(n)$ satisfying

$$\Re\left\{1+\frac{1}{\gamma}\left(\frac{1}{[p]_q}\frac{z\partial_q\left(\frac{1}{1+\delta([p]_q-1)}\left[\delta z\partial_q\left(\mathfrak{D}^{m,\rho}_{\lambda,l,p}f(z)\right)+(1-\delta)\,\mathfrak{D}^{m,\rho}_{\lambda,l,p}f(z)\right]\right)}{\frac{1}{1+\delta([p]_q-1)}\left[\delta z\partial_q\left(\mathfrak{D}^{m,\rho}_{q,\lambda,l,p}f(z)\right)+(1-\delta)\,\mathfrak{D}^{m,\rho}_{q,\lambda,l,p}f(z)\right]}-1\right)\right\}>\beta. \tag{1.11}$$

The class

$$\lim_{q\to1^-}\mathcal{SK}^{m,p,n}_{q,\rho,\lambda,l}(\delta,\gamma;\beta)=\mathcal{SK}^{m}_{\rho,\lambda,l}(\delta,\gamma,p,n;\beta)$$

is introduced by Bulut [5]. Furthermore the class

$$\mathcal{SK}^{m}_{0,1,0}(\delta,\gamma,1,1;\beta)=\mathcal{B}(m,\delta,\beta,\gamma)$$

is introduced by Deng [7].

Remark 1.6. For $\delta=0$, the class $\mathcal{SK}^{m,p,n}_{q,\rho,\lambda,l}(\delta,\gamma;\beta)$ reduces to the class $\mathcal{S}^{m,p,n}_{q,\rho,\lambda,l}(\gamma;\beta)$ that consists of functions $f\in\mathcal{A}_p(n)$ satisfying

$$\Re\left\{1+\frac{1}{\gamma}\left(\frac{1}{[p]_q}\frac{z\partial_q\left(\mathfrak{D}^{m,\rho}_{\lambda,l,p}f(z)\right)}{\mathfrak{D}^{m,\rho}_{q,\lambda,l,p}f(z)}-1\right)\right\}>\beta$$

and for $\delta=1$, the class $\mathcal{SK}^{m,p,n}_{q,\rho,\lambda,l}(\delta,\gamma;\beta)$ reduces to the class $\mathcal{K}^{m,p,n}_{q,\rho,\lambda,l}(\gamma;\beta)$ that consists of functions $f\in\mathcal{A}_p(n)$ satisfying

$$\Re\left\{1+\frac{1}{\gamma}\left(\frac{1}{[p]_q}\frac{\partial_q\left(z\partial_q\left(\mathfrak{D}^{m,\rho}_{\lambda,l,p}f(z)\right)\right)}{\partial_q\left(\mathfrak{D}^{m,\rho}_{q,\lambda,l,p}f(z)\right)}-1\right)\right\}>\beta.$$

For $m=0$, the classes $\mathcal{S}^{m,p,n}_{q,\rho,\lambda,l}(\gamma;\beta)$ and $\mathcal{K}^{m,p,n}_{q,\rho,\lambda,l}(\gamma;\beta)$ reduce to the classes

$$\mathcal{S}^*_q(\gamma,p,n;\beta)\quad \textit{and}\quad \mathcal{K}_q(\gamma,p,n;\beta)$$

of p-valently q-starlike function of complex order γ and type β, and p-valently q-convex function of complex order γ and type β, respectively. Furthermore, the classes

$$\lim_{q\to1^-}\mathcal{S}^{m}_{q,\rho,\lambda,l}(\gamma,p,n;\beta)=\mathcal{S}^{m}_{\rho,\lambda,l}(\gamma,p,n;\beta)$$

and

$$\lim_{q\to1^-}\mathcal{K}^{m}_{q,\rho,\lambda,l}(\gamma,p,n;\beta)=\mathcal{K}^{m}_{\rho,\lambda,l}(\gamma,p,n;\beta)$$

are introduced by Bulut [6].

The main purpose of this paper is to obtain some coefficient bounds for functions belong to the subclass $\mathcal{B}^{m,p,n}_{q,\rho,\lambda,l}(\delta,\gamma;\varphi,u)$ that consists of functions $f\in\mathcal{A}_p(n)$ satisfying the following nonhomogeneous Cauchy–Euler fractional q-differential equation:

$$\begin{aligned} z^2\partial_q^{(2)}f(z)+&\left(1+2u+q^{p-1}\right)\,z\partial_q^{(1)}f(z)+u\,(1+u)\,f(z)\\ &=\left([p]_q+u\right)\left([p]_q+1+u\right)g(z)\\ &\left(f\in\mathcal{A}_p(n)\,;\ g\in\mathcal{SK}^{m,p,n}_{q,\rho,\lambda,l}(\delta,\gamma;\varphi)\,;\ u>-[p]_q\ \ (u\in\mathbb{R})\right). \end{aligned}$$

Remark 1.7. For the function

$$\varphi(z)=\frac{1+(1-(1+q)\beta)z}{1-qz}\qquad(0\leq\beta<1;\ z\in\mathbb{D}),$$

the class $\mathcal{B}_{q,\rho,\lambda,l}^{m,p,n}(\delta,\gamma;\varphi,u)$ reduces to the class $\mathcal{B}_{q,\rho,\lambda,l}^{m,p,n}(\delta,\gamma;\beta,u)$. Also, the classes

$$\lim_{q\to1^-}\mathcal{B}_{q,\rho,\lambda,l}^{m,p,n}(\delta,\gamma;\beta,u)=\mathcal{B}_{\rho,\lambda,l}^{m}(\beta,\gamma,p,n;\delta,u)$$

and

$$\mathcal{B}_{0,1,0}^{m}(\beta,\gamma,1,1;\delta,u)=\mathcal{T}(m,\delta,\beta,\gamma;u)$$

are introduced by Bulut [5] and Deng [7], respectively.

2. Coefficient bounds

Unless otherwise stated, throughout this paper, we assume that $\varphi:\mathbb{D}\to\mathbb{C}$ is a convex function defined in (1.10), that $\mathfrak{D}_{q,\lambda,l,p}^{m,\rho}$ is given by (1.8), that $\Psi_{q,k,m}(\rho,\lambda,l,p)=:\Psi_k$ is given by (1.9), that

$$0<q<1,\quad 0\leq\rho<1,\quad \lambda,l\geq0,\quad 0\leq\delta,\eta\leq1,\quad u>-[p]_q\quad(u\in\mathbb{R}),$$

and that

$$p,n\in\mathbb{N},\quad m\in\mathbb{N}_0,\quad \gamma\in\mathbb{C}^*.$$

Theorem 2.1. *Let the function $f\in\mathcal{A}_p(n)$ be defined by (1.4). If*

$$f\in\mathcal{SK}_{q,\rho,\lambda,l}^{m,p,n}(\delta,\gamma;\varphi),$$

then

$$|a_{p+n}|\leq\chi_{p+n}$$

and

$$|a_k|\leq\chi_k\prod_{j=0}^{k-p-n-1}\left[q^p\,[n+j]_q+[p]_q\,|\gamma|\,|\varphi'(0)|\right]\qquad(k\geq p+n+1),$$

where

$$\chi_k=\frac{\Gamma_q(n)\,\left[1+\delta\left([p]_q-1\right)\right]\,[p]_q\,|\gamma|\,|\varphi'(0)|}{\Gamma_q(k-p+1)\,\left[1+\delta\left([k]_q-1\right)\right]\,q^{(k-p-n+1)p}\,\Psi_k}\qquad(k\geq p+n).\tag{2.1}$$

Proof. Let the function $f\in\mathcal{SK}_{q,\rho,\lambda,l}^{m,p,n}(\delta,\gamma;\varphi)$ be of the form (1.4). Define a function

$$\mathfrak{h}(z)=\frac{1}{1+\delta([p]_q-1)}\left[\delta z\partial_q\left(\mathfrak{D}_{q,\lambda,l,p}^{m,\rho}f(z)\right)+(1-\delta)\,\mathfrak{D}_{q,\lambda,l,p}^{m,\rho}f(z)\right]\qquad(z\in\mathbb{D}).\tag{2.2}$$

We note that the function $\mathfrak{h}$ is of the form

$$\mathfrak{h}(z)=z^p+\sum_{k=p+n}^{\infty}A_kz^k,$$

where

$$A_k = \Psi_k \frac{1+\delta\left([k]_q - 1\right)}{1+\delta\left([p]_q - 1\right)} a_k. \tag{2.3}$$

From (1.11) and (2.2), we obtain that

$$1 + \frac{1}{\gamma}\left(\frac{1}{[p]_q}\frac{z\partial_q \mathfrak{h}(z)}{\mathfrak{h}(z)} - 1\right) \in \varphi(\mathbb{D}) \qquad (z \in \mathbb{D}).$$

Let us define the function $\mathfrak{p}(z)$ by

$$\mathfrak{p}(z) = 1 + \frac{1}{\gamma}\left(\frac{1}{[p]_q}\frac{z\partial_q \mathfrak{h}(z)}{\mathfrak{h}(z)} - 1\right) \qquad (z \in \mathbb{D}). \tag{2.4}$$

Therefore, we get

$$\mathfrak{p}(0) = \varphi(0) = 1 \qquad \text{and} \qquad \mathfrak{p}(z) \in \varphi(\mathbb{D}) \qquad (z \in \mathbb{D}).$$

By Lemma 1.1, we obtain

$$\left|\frac{\mathfrak{p}^{(j)}(0)}{j!}\right| = |c_j| \leq |\varphi'(0)| \qquad (j \in \mathbb{N}), \tag{2.5}$$

where

$$\mathfrak{p}(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots \qquad (z \in \mathbb{D}).$$

Also, from (2.4), we find

$$z\partial_q \mathfrak{h}(z) - [p]_q \mathfrak{h}(z) = [p]_q \gamma(\mathfrak{p}(z) - 1)\mathfrak{h}(z).$$

The last equality implies that

$$\left([k]_q - [p]_q\right) A_k = [p]_q \gamma \left\{c_{k-p} + c_{k-p-n} A_{p+n} + \cdots + c_n A_{k-n}\right\}.$$

Let us set

$$k = p + n + r \quad (r \in \mathbb{N}_0).$$

Then we can write

$$\left([p+n+r]_q - [p]_q\right) A_{p+n+r} = [p]_q \gamma \left(c_{n+r} + c_r A_{p+n} + \cdots + c_n A_{p+r}\right).$$

Applying (2.5), we get

$$|A_{p+n+r}| \leq \frac{[p]_q |\gamma| \, |\varphi'(0)|}{q^p \, [n+r]_q} \left(1 + |A_{p+n}| + \cdots + |A_{p+r}|\right).$$

For $r = 0, 1, 2$, we have

$$\begin{aligned}
|A_{p+n}| &\leq \frac{[p]_q |\gamma| \, |\varphi'(0)|}{q^p \, [n]_q}, \\
|A_{p+n+1}| &\leq \frac{[p]_q |\gamma| \, |\varphi'(0)|}{q^p \, [n+1]_q} \left(1 + |A_{p+n}|\right) \\
&\leq \frac{[p]_q |\gamma| \, |\varphi'(0)| \left[q^p \, [n]_q + [p]_q |\gamma| \, |\varphi'(0)|\right]}{q^{2p} \, [n+1]_q \, [n]_q},
\end{aligned}$$

and

$$\begin{aligned}\left|A_{p+n+2}\right| &\leq \frac{[p]_q\,|\gamma|\,|\varphi'(0)|}{q^p\,[n+2]_q}\left(1+|A_{p+n}|+|A_{p+n+1}|\right)\\ &\leq \frac{[p]_q\,|\gamma|\,|\varphi'(0)|\left[q^p\,[n]_q+[p]_q\,|\gamma|\,|\varphi'(0)|\right]\left[q^p\,[n+1]_q+[p]_q\,|\gamma|\,|\varphi'(0)|\right]}{q^{3p}\,[n+2]_q\,[n+1]_q\,[n]_q},\end{aligned}$$

respectively. Using the mathematical induction, we get

$$\left|A_{p+n+r}\right| \leq \frac{[p]_q\,|\gamma|\,|\varphi'(0)|}{q^{(r+1)p}\,[n+r]_q\,[n+r-1]_q\,[n]_q}\prod_{j=0}^{r-1}\left[q^p\,[n+j]_q+[p]_q\,|\gamma|\,|\varphi'(0)|\right]$$

for $r \in \mathbb{N}$. Thus, we have

$$\left|A_{p+n}\right| \leq \frac{[p]_q\,|\gamma|\,|\varphi'(0)|}{q^p\,[n]_q}$$

and

$$\left|A_k\right| \leq \frac{\Gamma_q(n)}{\Gamma_q(k-p+1)}\frac{[p]_q\,|\gamma|\,|\varphi'(0)|}{q^{(k-p-n+1)p}}\prod_{j=0}^{k-p-n-1}\left[q^p\,[n+j]_q+[p]_q\,|\gamma|\,|\varphi'(0)|\right]$$

for $k \geq p+n+1$. By (2.3), it is clear that

$$a_k = \frac{1+\delta\left([p]_q-1\right)}{1+\delta\left([k]_q-1\right)}\frac{1}{\Psi_k}A_k.$$

Therefore we get

$$\left|a_{p+n}\right| \leq \frac{1+\delta\left([p]_q-1\right)}{1+\delta\left([p+n]_q-1\right)}\frac{1}{\Psi_{p+n}}\frac{[p]_q\,|\gamma|\,|\varphi'(0)|}{q^p\,[n]_q},$$

$$\begin{aligned}\left|a_k\right| &\leq \frac{1+\delta\left([p]_q-1\right)}{1+\delta\left([k]_q-1\right)}\frac{1}{\Psi_k}\frac{\Gamma_q(n)}{\Gamma_q(k-p+1)}\\ &\quad\times\frac{[p]_q\,|\gamma|\,|\varphi'(0)|}{q^{(k-p-n+1)p}}\prod_{j=0}^{k-p-n-1}\left[q^p\,[n+j]_q+[p]_q\,|\gamma|\,|\varphi'(0)|\right]\end{aligned}$$

for $k \geq p+n+1$. □

Setting

$$\varphi(z) = \frac{1+(1-(1+q)\beta)z}{1-qz} \qquad (0 \leq \beta < 1;\ z \in \mathbb{D}),$$

in Theorem 2.1, we get following consequence.

Corollary 2.2. *Let the function $f \in \mathcal{A}_p(n)$ be defined by (1.4). If*

$$f \in \mathcal{SK}_{q,\rho,\lambda,l}^{m,p,n}(\delta,\gamma;\beta),$$

then

$$|a_{p+n}| \leq \chi_{p+n},$$

$$|a_k| \leq \chi_k \prod_{j=0}^{k-p-n-1} \left[q^p \, [n+j]_q + (1+q)\, [p]_q \, |\gamma| \, (1-\beta)\right] \qquad (k \geq p+n+1),$$

where

$$\chi_k = \frac{(1+q)\, \Gamma_q(n) \left[1+\delta\left([p]_q - 1\right)\right] [p]_q \, |\gamma| \, (1-\beta)}{\Gamma_q(k-p+1) \left[1+\delta\left([k]_q - 1\right)\right] q^{(k-p-n+1)p} \, \Psi_k} \qquad (k \geq p+n). \quad (2.6)$$

If we set $\delta = 0$ in Corollary 2.2, then we deduce the following result.

Corollary 2.3. *Let the function $f \in \mathcal{A}_p(n)$ be defined by (1.4). If*

$$f \in \mathcal{S}_{q,\rho,\lambda,l}^{m,p,n}(\gamma;\beta),$$

then

$$|a_{p+n}| \leq \chi_{p+n},$$

$$|a_k| \leq \chi_k \prod_{j=0}^{k-p-n-1} \left[q^p \, [n+j]_q + (1+q)\, [p]_q \, |\gamma| \, (1-\beta)\right] \qquad (k \geq p+n+1),$$

where

$$\chi_k = \frac{(1+q)\, \Gamma_q(n) \, [p]_q \, |\gamma| \, (1-\beta)}{\Gamma_q(k-p+1)\, q^{(k-p-n+1)p} \, \Psi_k} \qquad (k \geq p+n).$$

If we set $\delta = 1$ in Corollary 2.2, then we deduce the following result.

Corollary 2.4. *Let the function $f \in \mathcal{A}_p(n)$ be defined by (1.4). If*

$$f \in \mathcal{K}_{q,\rho,\lambda,l}^{m,p,n}(\gamma;\beta),$$

then

$$|a_{p+n}| \leq \chi_{p+n},$$

$$|a_k| \leq \chi_k \prod_{j=0}^{k-p-n-1} \left[q^p \, [n+j]_q + (1+q)\, [p]_q \, |\gamma| \, (1-\beta)\right] \qquad (k \geq p+n+1),$$

where

$$\chi_k = \frac{(1+q)\, \Gamma_q(n) \, [p]_q^2 \, |\gamma| \, (1-\beta)}{\Gamma_q(k-p+1) \, [k]_q \, q^{(k-p-n+1)p} \, \Psi_k} \qquad (k \geq p+n).$$

Remark 2.5. (i) If we let $q \to 1^-$ in Corollary 2.2, then we get [5, Theorem 2.1]. (ii) If we let $q \to 1^-$ with $\rho = 0, \lambda = 1, l = 1$, and $p = n = 1$ in Corollary 2.2, then we get [7, Theorem 1].

Theorem 2.6. *Let the function $f \in \mathcal{A}_p(n)$ be defined by (1.4). If*

$$f \in \mathcal{B}_{q,\rho,\lambda,l}^{m,p,n}(\delta, \gamma; \varphi, u),$$

then

$$|a_{p+n}| \leq \Lambda_{p+n}\, \chi_{p+n}$$

and

$$|a_k| \leq \Lambda_k\, \chi_k \prod_{j=0}^{k-p-n-1} \left[q^p\, [n+j]_q + [p]_q\, |\gamma|\, |\varphi'(0)| \right] \qquad (k \geq p+n+1),$$

where

$$\Lambda_k = \frac{\left([p]_q + u\right)\left([p]_q + 1 + u\right)}{\left([k]_q + u\right)\left([k]_q + 1 + u\right)} \qquad (k \geq p+n) \tag{2.7}$$

and χ_k is given by (2.1).

Proof. Let the function $f \in \mathcal{B}_{q,\rho,\lambda,l}^{m,p,n}(\delta, \gamma; \varphi, u)$ be given by (1.4). Also, let

$$g(z) = z^p + \sum_{k=p+n}^{\infty} G_k z^k \in \mathcal{SK}_{q,\rho,\lambda,l}^{m,p,n}(\delta, \gamma; \varphi),$$

so that

$$a_k = \frac{\left([p]_q + u\right)\left([p]_q + 1 + u\right)}{\left([k]_q + u\right)\left([k]_q + 1 + u\right)} G_k \qquad \left(k \geq p+n,\ u > -[p]_q\ (u \in \mathbb{R})\right).$$

Thus, by using Theorem 2.1, we obtain

$$|a_{p+n}| \leq \frac{\left([p]_q + u\right)\left([p]_q + 1 + u\right)}{\left([k]_q + u\right)\left([k]_q + 1 + u\right)} \chi_{p+n}$$

and

$$|a_k| \leq \frac{\left([p]_q + u\right)\left([p]_q + 1 + u\right)}{\left([k]_q + u\right)\left([k]_q + 1 + u\right)} \chi_k \prod_{j=0}^{k-p-n-1} \left[q^p\, [n+j]_q + [p]_q\, |\gamma|\, |\varphi'(0)| \right]$$

for $k \geq p+n+1$. Here χ_k is given by (2.1). □

Setting

$$\varphi(z) = \frac{1 + (1 - (1+q)\beta)\, z}{1 - qz} \qquad (0 \leq \beta < 1;\ z \in \mathbb{D}),$$

in Theorem 2.6, we get following consequence.

Corollary 2.7. *Let the function $f \in \mathcal{A}_p(n)$ be defined by (1.4). If*

$$f \in \mathcal{B}_{q,\rho,\lambda,l}^{m,p,n}(\delta, \gamma; \beta, u),$$

then

$$|a_{p+n}| \leq \Lambda_{p+n}\, \chi_{p+n}$$

and

$$|a_k| \leq \Lambda_k \, \chi_k \prod_{j=0}^{k-p-n-1} \left[q^p \, [n+j]_q + (1+q) \, [p]_q \, |\gamma| \, (1-\beta) \right] \qquad (k \geq p+n+1),$$

where Λ_k *and* χ_k *are given by* (2.7) *and* (2.6)*, respectively.*

Remark 2.8. (i) If we let $q \to 1^-$ in Corollary 2.7, then we get [5, Theorem 3.1]. (ii) If we let $q \to 1^-$ with $\rho = 0, \lambda = 1, l = 1$, and $p = n = 1$ in Corollary 2.7, then we get [7, Theorem 2].

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