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SUBORDINATION PROBLEMS FOR CERTAIN MEROMORPHIC FUNCTIONS

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Abstract. Let Σ be the class of meromorphic functions f of the form $f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots$ that are analytic in the punctured disk $\mathbb{U}_0$. For $f \in \Sigma$, operators $D^n f$ with $n \in \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ are introduced. Applying differential subordinations for analytic functions in the open unit disc $\mathbb{U}$, some interesting properties of $f \in \Sigma$ with $D^n f$ are discussed, and argument problems of $D^n f$ are given. Also, we consider some simple problems for our results.

1. Introduction

Let Σ be the class of meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k$$

that are analytic in the punctured disk $\mathbb{U}_0 = \{z \in \mathbb{C} : 0 < |z| < 1\}$. For $f \in \Sigma$, Güney, Breaz, and Owa [1] introduced

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = Df(z) = \frac{1}{z}\left(z^2 f(z)\right)' = \frac{1}{z} + \sum_{k=0}^{\infty}(k+2)a_k z^k,$$

$$D^2 f(z) = D(Df(z)) = \frac{1}{z} + \sum_{k=0}^{\infty}(k+2)^2 a_k z^k,$$

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and

$$D^n f(z) = D(D^{n-1} f(z)) = \frac{1}{z} + \sum_{k=0}^{\infty} (k+2)^n a_k z^k,$$

for $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. On the other hand, Güney, Breaz, and Owa [1] considered

$$D^{-1} f(z) = \frac{1}{z^2} \int_0^z t f(t) dt = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{1}{k+2} \right) a_k z^k,$$

$$D^{-2} f(z) = D^{-1}(D^{-1} f(z)) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{1}{k+2} \right)^2 a_k z^k,$$

and

$$D^{-n} f(z) = D^{-1}(D^{-n+1} f(z)) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{1}{k+2} \right)^n a_k z^k$$

for $n \in \mathbb{N}$. With the above definitions, we say that

$$D^n f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} (k+2)^n a_k z^k \tag{1.1}$$

for $n \in \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Let F and G be analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Then F is said to be subordinate to G, written $F \prec G$, if there exists a function w in $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$, and $F(z) = G(w(z))$ for all $z \in \mathbb{U}$. If G is univalent in $\mathbb{U}$, then $F \prec G$ if and only if $F(0) = G(0)$ and $F(\mathbb{U}) \subseteq G(\mathbb{U})$ (see [6, 7]). In the present paper, we would like to discuss some interesting problems for differential subordinations of our operator $D^n f$.

In this paper, we introduce the operators $D^n f$ with $n \in \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ for $f \in \Sigma$. Applying differential subordinations for analytic functions in the open unit disc $\mathbb{U}$, we discuss some interesting properties of $f \in \Sigma$ with $D^n f$ and give argument problems of $D^n f$. Also, we consider some simple problems for our results.

2. Problems for differential subordinations

To discuss our problems, we first introduce the following lemma proved by Suffridge [8].

Lemma 2.1. *If a function p is analytic in $\mathbb{U}$ with $p(0) = 1$ satisfying*

$$zp'(z) \prec h(z) \qquad (z \in \mathbb{U})$$

for some starlike function h, then

$$p(z) \prec \int_0^z \frac{h(t)}{t} dt \qquad (z \in \mathbb{U}).$$

With the above lemma, we have the following theorem.

Theorem 2.2. *Let $D^n f$ be given by (1.1) for $n \in \mathbb{Z}$. Then*

$$zD^{n+1}f(z) - zD^n f(z) \prec \frac{1+(1-2\alpha)z}{1-z}$$

satisfies

$$zD^n f(z) \prec \log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right),$$

where $0 \leq \alpha < 1$.

Proof. We consider the functions $p(z) = zD^n f(z)$ and

$$h(z) = \frac{1+(1-2\alpha)z}{1-z}.$$

Then p is analytic in $\mathbb{U}$ with $p(0) = 1$, and h is starlike of order α in $\mathbb{U}$. It follows

$$zp'(z) = z\left(zD^n f(z)\right)' = zD^{n+1}f(z) - zD^n f(z)$$

and

$$\int_0^z \frac{h(t)}{t}dt = \int_0^z \left(\frac{1}{t} - \frac{2(1-\alpha)}{1-t}\right)dt = \log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right).$$

Therefore, applying Lemma 2.1, we prove the theorem. □

Taking $\alpha = \frac{1}{2}$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let $D^n f$ be given by (1.1) for $n \in \mathbb{Z}$. Then*

$$zD^{n+1}f(z) - zD^n f(z) \prec \frac{1}{1-z}$$

satisfies

$$zD^n f(z) \prec \log\left(\frac{z}{1-z}\right).$$

Making $n = 0$ in Theorem 2.2, we have the following corollary.

Corollary 2.4. *If $f \in \Sigma$ satisfies*

$$zf(z) + z^2 f'(z) \prec \frac{1+(1-2\alpha)z}{1-z}, \tag{2.1}$$

then

$$zf(z) \prec \log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right), \tag{2.2}$$

where $0 \leq \alpha < 1$.

Example 2.5. We consider a function $f \in \Sigma$ given by

$$f(z) = \frac{1}{z}\log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right).$$

Then

$$zf(z) + z^2 f'(z) = 1 + 2(1-\alpha)\frac{z}{1-z} = \frac{1+(1-2\alpha)z}{1-z}.$$

Thus f satisfies (2.1) and (2.2).

In 1975, Hallenbeck and Ruscheweyh [3] gave the following lemma.

Lemma 2.6. *If a function p is analytic in $\mathbb{U}$ with $p(0) = 1$ satisfying*

$$p(z) + zp'(z) \prec h(z)$$

for some convex function h, then

$$p(z) \prec \frac{1}{z}\int_0^z \frac{h(t)}{t}dt.$$

Applying the above lemma, we derive the following theorem.

Theorem 2.7. *Let $D^n f$ be given by (1.1) for $n \in \mathbb{Z}$. Then*

$$zD^{n+1}f(z) \prec \log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right) \tag{2.3}$$

satisfies

$$zD^n f(z) \prec \log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right) + \frac{2(1-\alpha)}{z}\log(1-z) + (1-2\alpha),$$

where $0 \leq \alpha < 1$.

Proof. Consider a function p given by $p(z) = zD^n f(z)$. Then p is analytic in $\mathbb{U}$ and $p(0) = 1$. It follows that

$$p(z) + zp'(z) = zD^n f(z) + z\left(D^n f(z) + z(D^n f(z))'\right) = zD^{n+1}f(z).$$

Furthermore, we know that

$$h(z) = \log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right) \qquad (z \in \mathbb{U})$$

satisfies

$$zh'(z) = \frac{1+(1-2\alpha)z}{1-z}.$$

Since zh' is starlike of order α in $\mathbb{U}$, h is convex in $\mathbb{U}$. Therefore, applying Lemma 2.6, we say that if f satisfies the subordination (2.3), then

$$\begin{aligned} zD^n f(z) &\prec \frac{1}{z}\int_0^z \left(\log\left(\frac{t}{(1-t)^{2(1-\alpha)}}\right)\right) dt \\ &= \frac{1}{z}\int_0^z (\log t - 2(1-\alpha)\log(1-t))\, dt \\ &= \log\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right) + \frac{2(1-\alpha)}{z}\log(1-z) + (1-2\alpha) \qquad (z \in \mathbb{U}). \end{aligned}$$

□

Taking $\alpha = \frac{1}{2}$ in Theorem 2.7, we have the following corollary.

Corollary 2.8. *Let $D^n f$ be given by (1.1) for $n \in \mathbb{Z}$. Then*

$$zD^{n+1}f(z) \prec \log\left(\frac{z}{1-z}\right)$$

satisfies

$$zD^n f(z) \prec \log\left(\frac{z}{1-z}\right) + \frac{1}{z}\log(1-z).$$

Next, we have the following theorem.

Theorem 2.9. *Let $D^n f$ be given by* (1.1) *for $n \in \mathbb{Z}$. Then*

$$zD^{n+1}f(z) \prec \frac{1+z}{1-z}$$

satisfies

$$zD^n f(z) \prec \frac{2}{z}\log\left(\frac{1}{1-z}\right) - 1.$$

Proof. We consider functions $p(z) = zD^n f(z)$ and

$$h(z) = \frac{1+z}{1-z}.$$

Then p is analytic in $\mathbb{U}$ with $p(0) = 1$, and h is a convex function in $\mathbb{U}$. Noting that

$$\begin{aligned}\frac{1}{z}\int_0^z h(t)dt &= \frac{1}{z}\int_0^z \frac{1+t}{1-t}dt \\ &= \frac{2}{z}\log\left(\frac{1}{1-z}\right) - 1,\end{aligned}$$

we completes the proof by Lemma 2.6. □

Letting $n = 0$ in Theorem 2.9, we have the following corollary.

Corollary 2.10. *If $f \in \Sigma$ satisfies*

$$z(2f(z) + zf'(z)) \prec \frac{1+z}{1-z},$$

then

$$zf(z) \prec \frac{2}{z}\log\left(\frac{1}{1-z}\right) - 1. \tag{2.4}$$

Example 2.11. We consider a function f given by

$$f(z) = \frac{1}{z^2}\left(2\log\left(\frac{1}{1-z}\right) - z\right).$$

Then f satisfies the subordination (2.4). Also, we have

$$z(2f(z) + zf'(z)) = \frac{1+z}{1-z} \qquad (z \in \mathbb{U}).$$

Therefore, f is the function satisfying Corollary 2.10.

3. Applications of Miller and Mocanu lemma

We need to introduce the following lemma for differential subordinations by Miller and Mocanu [5].

Lemma 3.1. *Let* $\beta_0 = 1.21872...$ *be the solution of* $\beta\pi = \frac{3}{2}\pi - \tan^{-1}\beta$ *and let* $\alpha = \alpha(\beta) = \beta + 2\tan^{-1}\left(\frac{\beta}{\pi}\right)$ *for* $0 < \beta < \beta_0$. *If* p *is analytic in* $\mathbb{U}$ *with* $p(0) = 1$, *then*

$$p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}$$

implies that

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

Remark 3.2. If $\beta = 1$ in Lemma 3.1, then $\alpha = \alpha(1) = \frac{3}{2}$. Thus Lemma 3.1 says that if p satisfies

$$p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^{\frac{3}{2}},$$

then

$$p(z) \prec \frac{1+z}{1-z}.$$

Viewing the proof of Lemma 3.1 done by Miller and Mocanu, we say the following result.

Lemma 3.3. *Let* $\beta_0 = 1.21872...$ *be the solution of* $\beta\pi = \frac{3}{2}\pi - \tan^{-1}\beta$ *and let* $\alpha = \alpha(\beta) = \beta + 2\tan^{-1}\left(\frac{\beta}{\pi}\right)$ *for* $0 < \beta < \beta_0$. *If* p *is analytic in* $\mathbb{U}$ *with* $p(0) = 1$, *then*

$$p(z) + zp'(z) \prec \left(\frac{1+(1-2\gamma)z}{1-z}\right)^{\alpha}$$

implies that

$$p(z) \prec \left(\frac{1+(1-2\gamma)z}{1-z}\right)^{\beta},$$

where $0 \leq \gamma < 1$.

Remark 3.4. Let us take $\gamma = \frac{1}{2}$ in Lemma 3.3. If p satisfies

$$p(z) + zp'(z) \prec \left(\frac{1}{1-z}\right)^{\alpha},$$

then

$$p(z) \prec \left(\frac{1}{1-z}\right)^{\beta},$$

where α and β are given in Lemma 3.3.

Theorem 3.5. *Let* $\beta_0 = 1.21872...$ *be the solution of* $\beta\pi = \frac{3}{2}\pi - \tan^{-1}\beta$ *and let* $\alpha = \alpha(\beta) = \beta + 2\tan^{-1}\left(\frac{\beta}{\pi}\right)$ *for* $0 < \beta < \beta_0$. *If* p *is analytic in* $\mathbb{U}$ *with* $p(0) = 1$, *then*

$$p(z) + zp'(z) \prec \delta + (1-\delta)\left(\frac{1+(1-2\gamma)z}{1-z}\right)^{\alpha}$$

implies that

$$p(z) \prec \delta + (1-\delta)\left(\frac{1+(1-2\gamma)z}{1-z}\right)^{\beta},$$

where $0 \leq \gamma < 1$ *and* $0 \leq \delta < 1$.

Proof. We consider a function F given by

$$F(z) = \frac{p(z) - \delta}{1 - \delta} \qquad (z \in \mathbb{U}).$$

Then F is analytic in $\mathbb{U}$ with $F(0) = 1$ and

$$zF'(z) = \frac{zp'(z)}{1 - \delta}.$$

Using Lemma 3.3, we know that

$$F(z) + zF'(z) = \frac{p(z) + zp'(z) - \delta}{1 - \delta} \prec \left(\frac{1 + (1 - 2\gamma)z}{1 - z}\right)^{\alpha} \qquad (z \in \mathbb{U}) \tag{3.1}$$

implies that

$$F(z) = \frac{p(z) - \delta}{1 - \delta} \prec \left(\frac{1 + (1 - 2\gamma)z}{1 - z}\right)^{\beta}. \tag{3.2}$$

From the subordinations (3.1) and (3.2), we have

$$p(z) + zp'(z) \prec \delta + (1 - \delta)\left(\frac{1 + (1 - 2\gamma)z}{1 - z}\right)^{\alpha}$$

implies that

$$p(z) \prec \delta + (1 - \delta)\left(\frac{1 + (1 - 2\gamma)z}{1 - z}\right)^{\beta}.$$

□

Taking $\gamma = \frac{1}{2}$ in Theorem 3.5, we have the following corollary.

Corollary 3.6. *Let α and β be defined in Theorem 3.5. If p is analytic in $\mathbb{U}$ with $p(0) = 1$, then*

$$p(z) + zp'(z) \prec \delta + (1 - \delta)\left(\frac{1}{1 - z}\right)^{\alpha}$$

implies that

$$p(z) \prec \delta + (1 - \delta)\left(\frac{1}{1 - z}\right)^{\beta},$$

where $0 \leq \delta < 1$.

Remark 3.7. If we take $\gamma = 0$ in Theorem 3.5, then we have the result proved by Güney,Breaz and Owa [2].

Corollary 3.8. *Let $D^n f$ be given by (1.1) for $n \in \mathbb{Z}$. Let α and β be as in Theorem 3.5. If f satisfies*

$$zD^{n+1}f(z) \prec \delta + (1 - \delta)\left(\frac{1 + (1 - 2\gamma)z}{1 - z}\right)^{\alpha},$$

then

$$zD^{n}f(z) \prec \delta + (1 - \delta)\left(\frac{1 + (1 - 2\gamma)z}{1 - z}\right)^{\beta},$$

where $0 \leq \gamma < 1$ and $0 \leq \delta < 1$.

Furthermore, taking $n = 1$ in Corollary 3.8, we see the following corollary.

Corollary 3.9. *Let α and β be as in Theorem 3.5. If $f \in \Sigma$ satisfies*

$$z(2f(z) + zf'(z)) \prec \delta + (1-\delta)\left(\frac{1+(1-2\gamma)z}{1-z}\right)^{\alpha},$$

then

$$zf(z) \prec \delta + (1-\delta)\left(\frac{1+(1-2\gamma)z}{1-z}\right)^{\beta},$$

where $0 \leq \gamma < 1$ and $0 \leq \delta < 1$.

Letting $\beta = 1$ in Theorem 3.5, we have the following corollary.

Corollary 3.10. *If p is analytic in $\mathbb{U}$ with $p(0) = 1$, and*

$$p(z) + zp'(z) \prec \delta + (1-\delta)\left(\frac{1+(1-2\gamma)z}{1-z}\right)^{\frac{3}{2}},$$

then

$$p(z) \prec \frac{1+(1-2(\gamma+\delta-\gamma\delta))z}{1-z}$$

and

$$Re p(z) > \gamma + \delta - \gamma\delta \qquad (z \in \mathbb{U}),$$

where $0 \leq \gamma < 1$ and $0 \leq \delta < 1$.

Taking $p(z) = zD^n f(z)$ in Corollary 3.10, we obtain the following corollary.

Corollary 3.11. *Let $D^n f$ be given by (1.1) for $n \in \mathbb{Z}$. If $f \in \Sigma$ satisfies*

$$zD^{n+1}f(z) \prec \delta + (1-\delta)\left(\frac{1+(1-2\gamma)z}{1-z}\right)^{\frac{3}{2}},$$

then

$$zD^n f(z) \prec \frac{1+(1-2(\gamma+\delta-\gamma\delta))z}{1-z}$$

and

$$Re(zD^n f(z)) > \gamma + \delta - \gamma\delta \qquad (z \in \mathbb{U}),$$

where $0 \leq \gamma < 1$ and $0 \leq \delta < 1$.

Next, we introduce the following lemma due to Miller and Mocanu [6] (also, due to Jack [4]).

Lemma 3.12. *Let w be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then*

$$z_0 w'(z_0) = m w(z_0)$$

and

$$Re\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq m,$$

where $m \geq 1$.

Theorem 3.13. *Let $D^n f$ be given by (1.1) for $n \in \mathbb{Z}$. If $f \in \Sigma$ satisfies*

$$zD^n f(z) \prec \frac{\alpha(1+z)}{\alpha+(2-\alpha)z} \tag{3.3}$$

for some real $\alpha > 1$, then

$$\left|zD^n f(z) - \frac{\alpha}{2}\right| < \frac{\alpha}{2}.$$

This means that

$$0 < Re(zD^n f(z)) < \alpha \qquad (z \in \mathbb{U}).$$

Proof. We consider a function w defined by

$$zD^n f(z) = \frac{\alpha(1+w(z))}{\alpha+(2-\alpha)w(z)} \qquad (z \in \mathbb{U}). \tag{3.4}$$

Then w is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ by (3.3). It follows from (3.4) that

$$|w(z)| = \left|\frac{\alpha(zD^n f(z) - 1)}{\alpha - (2-\alpha)zD^n f(z)}\right| < 1 \qquad (z \in \mathbb{U}).$$

This gives us that

$$2\left|zD^n f(z)\right|^2 - \alpha\left(zD^n f(z) + \overline{zD^n f(z)}\right) < 0 \qquad (z \in \mathbb{U})$$

and that

$$\left|zD^n f(z) - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}).$$

□

Taking $n = 0$ in Theorem 3.13, we have the following result.

Corollary 3.14. *If $f \in \Sigma$ satisfies*

$$zf(z) \prec \frac{\alpha(1+z)}{\alpha+(2-\alpha)z}$$

for some real $\alpha > 1$, then

$$\left|zf(z) - \frac{\alpha}{2}\right| < \frac{\alpha}{2}.$$

Theorem 3.15. *Let $D^n f$ be given by (1.1) for $n \in \mathbb{Z}$. If $f \in \Sigma$ satisfies*

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) < 1 + \frac{\alpha - 1}{2\delta} \qquad (z \in \mathbb{U}) \tag{3.5}$$

for some real α $(1 < \alpha \leq 2)$ or

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) < 1 + \frac{1}{2\delta(\alpha - 1)} \qquad (z \in \mathbb{U}) \tag{3.6}$$

for some real α $(\alpha > 2)$, then

$$\left|(zD^n f(z))^\delta - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}) \tag{3.7}$$

where $0 < \delta \leq 1$.

Proof. Let us consider a function w by

$$(zD^n f(z))^\delta = \frac{\alpha(1+w(z))}{\alpha+(2-\alpha)w(z)} \qquad (z \in \mathbb{U}) \tag{3.8}$$

for some $0 < \delta \leq 1$. Then we see that w is analytic in $\mathbb{U}$ and $w(0) = 0$. It follows from (3.8) that

$$\frac{D^{n+1}f(z)}{D^n f(z)} - 1 = \frac{zw'(z)}{\delta w(z)} \left(\frac{w(z)}{1+w(z)} - \frac{(2-\alpha)w(z)}{\alpha+(2-\alpha)w(z)} \right).$$

We suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|\leq|z_0|}|w(z)| = |w(z_0| = 1.$$

Then, using Lemma 3.12, we say that

$$z_0 w'(z_0) = m w(z_0) \qquad (m \geq 1)$$

and $w(z_0) = e^{i\theta}$ $(0 \leq \theta < 2\pi)$. This implies that

$$\begin{aligned} \operatorname{Re}\left(\frac{D^{n+1}f(z_0)}{D^n f(z_0)}\right) &= 1 + \frac{m}{\delta}\left(\frac{e^{i\theta}}{1+e^{i\theta}} - \frac{(2-\alpha)e^{i\theta}}{\alpha+(2-\alpha)e^{i\theta}}\right) \\ &= 1 + \frac{m}{\delta}\left(\frac{1}{2} + \frac{(\alpha-2)(2-\alpha+\alpha\cos\theta)}{\alpha^2+(2-\alpha)^2+2\alpha(2-\alpha)\cos\theta}\right). \end{aligned}$$

Let a function g be defined by

$$g(t) = \frac{2-\alpha+\alpha t}{\alpha^2+(2-\alpha)^2+2\alpha(2-\alpha)t} \qquad (t = \cos\theta).$$

Then we have

$$g'(t) = \frac{4\alpha(\alpha-1)}{\left(\alpha^2+(2-\alpha)^2+2\alpha(2-\alpha)t\right)^2} > 0$$

for $\alpha > 1$. Thus we obtain that

$$\operatorname{Re}\left(\frac{D^{n+1}f(z_0)}{D^n f(z_0)}\right) \geq 1 + \frac{m}{\delta}\left(\frac{1}{2} + \frac{\alpha-2}{2}\right) \geq 1 + \frac{\alpha-1}{2\delta} \tag{3.9}$$

for $1 < \alpha \leq 2$ and that

$$\operatorname{Re}\left(\frac{D^{n+1}f(z_0)}{D^n f(z_0)}\right) \geq 1 + \frac{m}{\delta}\left(\frac{1}{2} - \frac{\alpha-2}{2(\alpha-1)}\right) \geq 1 + \frac{1}{2\delta(\alpha-1)} \tag{3.10}$$

for $\alpha > 2$. The inequalities (3.9) and (3.10) contradict our conditions (3.5) and (3.6). Therefore there is no w such that $w(0) = 0$ and $|w(z_0)| = 1$ for $z_0 \in \mathbb{U}$. This means that $|w(z)| < 1$ for all $z \in \mathbb{U}$ and that

$$|w(z)| = \left| \frac{\alpha\left((zD^n f(z))^\delta - 1\right)}{\alpha - (2-\alpha)(zD^n f(z))^\delta} \right| < 1 \qquad (z \in \mathbb{U}).$$

This gives that the inequality (3.7) is satisfied for $f \in \Sigma$. $\square$

Making $\delta = 1$ in Theorem 3.15, we obtain the following corollary.

Corollary 3.16. *Let $D^n f$ be given by (1.1) for $n \in \mathbb{Z}$. If $f \in \Sigma$ satisfies*

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) < \frac{\alpha+1}{2} \qquad (z \in \mathbb{U})$$

for some real α $(1 < \alpha \leq 2)$ or

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) < \frac{2\alpha-1}{2(\alpha-1)} \qquad (z \in \mathbb{U})$$

for some real α $(\alpha > 2)$, then

$$\left|zD^n f(z) - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}).$$

Letting $\delta = \frac{1}{2}$ in Theorem 3.15, we obtain the following corollary.

Corollary 3.17. *Let $D^n f$ be given by (1.1) for $n \in \mathbb{Z}$. If $f \in \Sigma$ satisfies*

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) < \alpha \qquad (z \in \mathbb{U})$$

for some real α $(1 < \alpha \leq 2)$ or

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) < \frac{\alpha}{\alpha-1} \qquad (z \in \mathbb{U})$$

for some real α $(\alpha > 2)$, then

$$\left|\sqrt{zD^n f(z)} - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}).$$

4. Argument problems

In this section, we consider argument problems for $f \in \Sigma$.

Theorem 4.1. *Let $D^n f$ be given by (1.1) for $n \in \mathbb{Z}$. If $f \in \Sigma$ satisfies*

$$\left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| < \frac{\alpha}{2} Re\left(\frac{1+\beta z}{1-z}\right) \qquad (z \in \mathbb{U})$$

for some real α $(0 \leq \alpha < 1)$ and for some real β $(\beta \neq -1)$, then

$$|\arg(zD^n f(z))| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}).$$

Proof. Let us define a function p by $p(z) = zD^n f(z)$. Then p is analytic in $\mathbb{U}$ with $p(0) = 1$ and satisfies

$$\frac{zp'(z)}{p(z)} = \frac{D^{n+1}f(z)}{D^n f(z)} - 1.$$

For such p, we have

$$\begin{aligned}
|\arg p(z)| &= |\mathrm{Im}(\log p(z))| \\
&= \left|\mathrm{Im}\int_0^z \frac{p'(t)}{p(t)}dt\right| = \left|\mathrm{Im}\int_0^r \frac{p'(\rho e^{i\theta})}{p(\rho e^{i\theta})}e^{i\theta}d\rho\right| \\
&\leq \int_0^r \left|\mathrm{Im}\left(\frac{p'(\rho e^{i\theta})}{p(\rho e^{i\theta})}e^{i\theta}\right)\right| d\rho \leq \int_{-r}^r \left|\frac{p'(\rho e^{i\theta})}{p(\rho e^{i\theta})}\right| d\rho \\
&\leq \frac{r}{2}\int_0^{2\pi} \left|\frac{p'(re^{i\theta})}{p(re^{i\theta})}\right| d\theta = \frac{1}{2}\int_0^{2\pi} \left|\frac{re^{i\theta}p'(re^{i\theta})}{p(re^{i\theta})}\right| d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left|\frac{D^{n+1}f(re^{i\theta})}{D^n f(re^{i\theta})} - 1\right| d\theta < \frac{\alpha}{4}\int_0^{2\pi} \mathrm{Re}\left(\frac{1+\beta re^{i\theta}}{1-re^{i\theta}}\right) d\theta \\
&= \frac{\alpha}{4}\int_0^{2\pi} \left\{\frac{1-\beta}{2} + \left(\frac{1+\beta}{2}\right)\frac{1-r^2}{1+r^2-2r\cos\theta}\right\} d\theta = \frac{\alpha}{2}
\end{aligned}$$

using Poisson integral given by

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos\theta}d\theta = 1.$$

Therefore, we have

$$|\arg p(z)| = |\arg(zD^n f(z))| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}).$$

□

Example 4.2. We consider a function $f \in \Sigma$ given by

$$zD^n f(z) = \left(\frac{2}{2-z}\right)^{3\alpha} \qquad (z \in \mathbb{U})$$

for $0 \leq \alpha < 1$. Note that

$$w(z) = \frac{2}{2-z}$$

satisfies

$$\left|w(z) - \frac{4}{3}\right| < \frac{2}{3} \qquad (z \in \mathbb{U})$$

and

$$|\arg w(z)| < \frac{\pi}{6} \qquad (z \in \mathbb{U}).$$

Then we have

$$\left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| = 3\alpha\left|\frac{z}{2-z}\right| < 3\alpha \qquad (z \in \mathbb{U}).$$

Therefore, considering β such that $\beta \leq -11$, we see that

$$\left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| < 3\alpha \leq \frac{\alpha(1-\beta)}{4} < \frac{\alpha}{2}\mathrm{Re}\left(\frac{1+\beta z}{1-z}\right) \qquad (z \in \mathbb{U}).$$

Taking $n = 0$ in Theorem 4.1, we have the following corollary.

Corollary 4.3. *If $f \in \Sigma$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{\alpha}{2} Re\left(\frac{1+\beta z}{1-z}\right) \qquad (z \in \mathbb{U})$$

for some real α $(0 \leq \alpha < 1)$ and for some real β $(\beta \neq -1)$, then

$$|\arg(zf(z))| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}).$$

Now, we derive the following theorem.

Theorem 4.4. *Let $D^n f$ be given by (1.1) for $n \in \mathbb{Z}$. If $f \in \Sigma$ satisfies*

$$\left|\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{D^{n+1}f(z)}{D^n f(z)}\right| < \frac{\alpha}{2} Re\left(\frac{1+\beta z}{1-z}\right) \qquad (z \in \mathbb{U})$$

for some real α $(0 \leq \alpha < 1)$ and for some real β $(\beta \neq -1)$, then

$$\left|\arg\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)\right| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}).$$

Proof. We consider a function p by

$$p(z) = \frac{D^{n+1}f(z)}{D^n f(z)}. \tag{4.1}$$

Then p is analytic in $\mathbb{U}$ with $p(0) = 1$. It follows from (4.1) that

$$\begin{aligned}
\frac{zp'(z)}{p(z)} &= \frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \frac{z(D^n f(z))'}{D^n f(z)} \\
&= \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{D^{n+1}f(z)}{D^n f(z)}.
\end{aligned}$$

For such p, we have

$$\begin{aligned}
|\arg p(z)| &= |\mathrm{Im}(\log p(z))| \\
&\leq \left|\mathrm{Im}\int_0^r \frac{p'(\rho e^{i\theta})}{p(\rho e^{i\theta})} e^{i\theta} d\rho\right| \leq \int_{-r}^r \left|\frac{p'(\rho e^{i\theta})}{p(\rho e^{i\theta})}\right| d\rho \\
&\leq \frac{r}{2}\int_0^{2\pi} \left|\frac{p'(re^{i\theta})}{p(re^{i\theta})}\right| d\theta \leq \frac{1}{2}\int_0^{2\pi} \left|\frac{re^{i\theta}p'(re^{i\theta})}{p(re^{i\theta})}\right| d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left|\frac{D^{n+2}f(re^{i\theta})}{D^{n+1}f(re^{i\theta})} - \frac{D^{n+1}f(re^{i\theta})}{D^n f(re^{i\theta})}\right| d\theta \\
&< \frac{\alpha}{4}\int_0^{2\pi} \mathrm{Re}\left(\frac{1+\beta re^{i\theta}}{1-re^{i\theta}}\right) d\theta = \frac{\pi}{2}\alpha.
\end{aligned}$$

This shows us that

$$|\arg p(z)| = \left|\arg\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)\right| < \frac{\alpha}{2} \qquad (z \in \mathbb{U}).$$

$\square$

Example 4.5. We consider a function $f \in \Sigma$ given by

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \left(\frac{2}{2-z}\right)^{3\alpha} \qquad (z \in \mathbb{U})$$

for $0 \leq \alpha < 1$. Since

$$\left|\arg\left(\frac{2}{2-z}\right)\right| < \frac{\pi}{6} \qquad (z \in \mathbb{U})$$

we have

$$\left|\arg\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)\right| < \frac{\pi}{2}\alpha \qquad (z \in \mathbb{U}).$$

Also, we see that

$$\begin{aligned}\left|\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{D^{n+1}f(z)}{D^n f(z)}\right| &= 3\alpha\left|\frac{z}{2-z}\right| \\ &< 3\alpha \leq \frac{\alpha(1-\beta)}{4} < \frac{\alpha}{2}\mathrm{Re}\left(\frac{1+\beta z}{1-z}\right) \qquad (z \in \mathbb{U})\end{aligned}$$

for $\beta \leq -11$.

5. Conflicts of interest

The authors declare that they have no conflict of interest.

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