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CONFORMALLY CLOSED WEAKLY BERWALD METRICS

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ABSTRACT. A weakly Berwald metric is called a conformally closed weakly Berwald metric if for any conformal change, it remains a weakly Berwald metric. In this paper, we study the conformally closed weakly Berwald metrics and find the necessary and sufficient condition under which a weakly Berwald metric be conformally closed. We show that a Randers metric is a conformally closed weakly Berwald metric if and only if it is a Riemannian metric or the conformal transformation is homothety.

1. INTRODUCTION.

A conformal map, also called a conformal change, is a transformation that preserves local angles. The conformal theory has old history in mathematics. In 1850, Joseph Liouville proved his well-known classical theorem that explains any conformal map from an open subset of Euclidean space $\mathbb{R}^n$ into the same Euclidean space $\mathbb{R}^n$ $(n \geq 3)$ can be composed of three types of transformations: a homothety, an isometry, and a special conformal transformation. Conformal transformations have important applications in cartography, general relativity, Maxwell's equations, and engineering.

The theory of conformal transformations (changes) of Finsler metrics, Riemannian and non-Riemannian curvatures has been studied by many people (for more details, see [1, 3, 4, 6, 7, 9–11]). The Weyl theorem states that the conformal and projective properties of a Finsler metric characterize the metric properties uniquely. Therefore, studying the conformal transformation of a Finsler metric needs extra consideration. Let F and $\tilde{F}$ be two Finsler metrics on a manifold M.

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Then F is called conformal to $\tilde{F}$ if there is a scalar function $\kappa = \kappa(x)$ such that

$$F(x, y) = e^{\kappa} \tilde{F}(x, y).$$

The scalar function κ is called the conformal factor of the conformal transformation. It is remarkable that, Knebelman [4] proved that $\kappa = \kappa(x)$ is a function of position only. Hashiguchi [3] studied the fundamental properties of Finsler conformal transformations and obtained the relation between the Riemannian curvature and some non-Riemannian curvatures of two conformal metrics. He found some transformation formulas and obtained some conformal invariants. Their investigations show that the conformal transformations do not preserve the Riemannian and non-Riemannian curvatures in Finsler geometry.

In Finsler geometry, there are many non-Riemannian quantities, which have important impact on each other and Riemannian curvature, also. Among them, the Berwald curvature, the mean Berwald curvature, and the S-curvature have a direct relation to each other. Let $F = F(x, y)$ be a Finsler metric on a manifold M. The geodesics of F are characterized locally by the ordinary differential equation

$$\ddot{x}^j(t) + 2G^j\big(x, \dot{x}(t)\big) = 0,$$

where $G^j = G^j(x, y)$ are coefficients of a spray $\mathbf{G}$ defined on M. A Finsler metric F is called a Berwald metric if its spray's coefficients are quadratic in $y \in T_xM$ for any $x \in M$, namely,

$$G^j = \frac{1}{2}\Gamma^j_{ik}(x) y^i y^k.$$

In this case, the Berwald curvature of F is vanishing $\mathbf{B} = 0$. Taking a trace of Berwald curvature yields the mean Berwald curvature

$$\mathbf{E} = \mathrm{trace}(\mathbf{B}).$$

A Finsler metric with vanishing mean Berwald curvature $\mathbf{E} = 0$ is called a weakly Berwald metric. Then, Berwald metrics are trivial weakly Berwald metrics. However, the converse is not true [2].

Let $\mathcal{F}^n$ be a set of a special kind of n-dimensional Finsler metrics. If $F \in \mathcal{F}^n$ remains to belong to $\mathcal{F}^n$ by any conformal transformation of metric, then $\mathcal{F}^n$ is called conformally closed. Matsumoto [5] studied conformally closed Berwald metrics and found the necessary and sufficient conditions under which a Berwald metric be conformally closed. Also, he considered conformally closed Douglas metrics. Shen [8] studied S-closed conformal transformations in Finsler geometry. He proved that such transformation must be a homothety unless the Finsler manifold is Riemannian. In this paper, we study the conformally closed weakly Berwald metrics and prove the following result.

Theorem 1.1. *Let (M, F) be a Finsler manifold. Then F is a conformally closed weakly Berwald metric if and only if the following statement holds:*

$$F^2 I^m_{,k,l} + 2g_{kl} I^m + 2y_l I^m_{,k} + 2y_k I^m_{,l} = 0, \tag{1.1}$$

where "," denotes the vertical derivation, $y_i := F F_{y^i}$, and $\mathbf{g} = g_{ij} dx^i \otimes dx^j$ and $\mathbf{I} = I_i dx^i$ are the fundamental form and the mean Cartan torsion of F, respectively.

The class of Randers metrics forms the simplest non-Riemannian Finsler metrics, which are defined by

$$F = \alpha + \beta,$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a positive-definite Riemmanian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M. They were founded from the general relativity and have been widely applied in many areas of natural science. Here, we study conformally closed weakly Berwald Randers metrics and prove the following theorem.

Theorem 1.2. *A Randers metric is a conformally closed weakly Berwald metric if and only if it is a Riemannian metric or the conformal transformation is homothety.*

2. Preliminaries

A Finsler structure on a manifold M is a function $F : TM \to [0, \infty)$ with the following properties: (i) F is C^∞ on $TM_0 := TM - \{0\}$; (ii) F is positively 1-homogeneous on T_xM, that is, $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda > 0$; (iii) The quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is positive-definite on T_xM, where

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \qquad u, v \in T_xM.$$

Then, the pair (M, F) is called a Finsler manifold.

Let $x \in M$ and let $F_x := F|_{T_xM}$. For $y \in T_xM_0$, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u, v)\right]_{t=0}, \qquad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion.

Let (M, F) be a Finsler manifold. For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. Every positive-definite Finsler metric F is Riemannian if and only if $\mathbf{I} = 0$.

For a Finsler manifold (M, F), its induced spray on TM, denoted by $\mathbf{G}$, which in a standard coordinate (x^i, y^i) for TM_0 is defined by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where

$$G^i := \frac{1}{4}g^{il}\left[\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\right].$$

For a vector $y \in T_xM_0$, the Berwald curvature $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ is defined by $\mathbf{B}_y(u, v, w) := B^m_{\ jkl}(y)u^j v^k w^l \partial/\partial x^m|_x$, where

$$B^m_{\ jkl} := \frac{\partial^3 G^m}{\partial y^j \partial y^k \partial y^l}.$$

Then F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

Taking a trace of Berwald curvature $\mathbf{B}$ give us the mean of Berwald curvature $\mathbf{E}$, which is defined by $\mathbf{E}_y : T_xM \times T_xM \to \mathbb{R}$, where

$$\mathbf{E}_y(u,v) := \frac{1}{2}\sum_{i=1}^{n} g^{kl}(y)\mathbf{g}_y\Big(\mathbf{B}_y(u,v,e_k),\ e_l\Big).$$

In local coordinates, $\mathbf{E}_y(u,v) := E_{ij}(y)u^iv^j$, where

$$E_{ij} := \frac{1}{2}B^m{}_{mij}.$$

Then F is called a weakly Berwald metric if $\mathbf{E} = \mathbf{0}$.

3. Conformally Closed Weakly Berwald Metrics

A weakly Berwald metric $F = F(x,y)$ on a manifold M is called a conformally closed weakly Berwald metric if for any conformal change $\bar{F}(x,y) = e^{\kappa}F(x,y)$ remains a weakly Berwald metric, where $\kappa = \kappa(x)$ is a scalar function on M.

Lemma 3.1. *Let (M,F) be a Finsler manifold. Then F is conformally closed weakly Berwald metric if and only if $\mathcal{A}^m_m := [F^2\kappa^m]_{y^m}$ is a 1-form on M.*

Proof. Let F and $\bar{F}$ be two Finsler metrics on a manifold M. By using the Rapcsák's identity, the following relationship between G^i and $\bar{G}^i$ holds:

$$\bar{G}^i - G^i = \frac{1}{2\bar{F}}\bar{F}_{;m}y^my^i + \frac{\bar{g}^{il}}{2}\Big\{\bar{F}_{;k,l}y^k - \bar{F}_{;l}\Big\}\bar{F}, \tag{3.1}$$

where "$;$" and "$,$" denote the horizontal and vertical derivations with respect to the Berwald connection of F. Suppose that F is conformally related to a $\bar{F}$, namely, $\bar{F} = e^{\kappa}F$, where $\kappa = \kappa(x)$ is a scalar function on M. Let us put

$$\kappa_m := \frac{\partial \kappa}{\partial x^m}.$$

Since $F_{;m} = 0$, then the following conditions hold:

$$\bar{F}_{;m} = \kappa_m e^{\kappa}F, \qquad \bar{F}_{,i} = e^{\kappa}F_{,i}, \qquad \bar{F}_{;m,l} = \kappa_m e^{\kappa}F_{,l}, \tag{3.2}$$

$$\bar{g}_{ij} = e^{2\kappa}g_{ij}, \qquad \bar{g}^{ij} = e^{-2\kappa}g^{ij}. \tag{3.3}$$

By putting (3.2) and (3.3) in (3.1), we get

$$\bar{G}^i = G^i + \kappa_0 y^i - \frac{1}{2}F^2\kappa^i, \tag{3.4}$$

where

$$\kappa_0 := \kappa_m y^m \qquad \text{and} \qquad \kappa^i := g^{im}\kappa_m.$$

Equation (3.4) can be written as follows:

$$\bar{G}^i = G^i + Py^i - Q^i, \tag{3.5}$$

where

$$P := \kappa_k y^k, \qquad Q^i := \frac{1}{2}F^2\kappa^i. \tag{3.6}$$

Let us define

$$G^i_j := \frac{\partial G^i}{\partial y^j}, \qquad G^i_{jk} := \frac{\partial G^i_j}{\partial y^k}, \qquad \bar{G}^i_j := \frac{\partial \bar{G}^i}{\partial y^j}, \qquad \bar{G}^i_{jk} := \frac{\partial \bar{G}^i_j}{\partial y^k},$$
$$Q^i_j := \frac{\partial Q^i}{\partial y^j}, \qquad Q^i_{jk} := \frac{\partial Q^i_j}{\partial y^k}, \qquad Q^i_{jkl} := \frac{\partial Q^i_{jk}}{\partial y^l},$$
$$P_j := \frac{\partial P}{\partial y^j}, \qquad P_{jk} := \frac{\partial P_j}{\partial y^k}.$$

Taking vertical derivations of (3.5) implies that

$$\bar{G}^i_j = G^i_j + P_j y^i + P\delta^i_j - Q^i_j,$$
$$\bar{G}^i_{jk} = G^i_{jk} + P_{jk} y^i + P_j\delta^i_k + P_k\delta^i_j - Q^i_{jk},$$

and

$$\bar{B}^i_{jkl} \;=\; B^i_{jkl} + P_{jkl}y^i + P_{jk}\delta^i_l + P_{jl}\delta^i_k + P_{kl}\delta^i_j - Q^i_{jkl}. \tag{3.7}$$

The following equalities hold:

$$P_i = \kappa_i, \qquad P_{ij} = P_{ijk} = 0. \tag{3.8}$$

By (3.7) and (3.8), we get

$$\bar{B}^i_{jkl} = B^i_{jkl} - Q^i_{jkl}. \tag{3.9}$$

Taking a trace of (3.9) implies that

$$\bar{E}_{ij} = E_{ij} - \frac{1}{2}Q^m_{mij}, \tag{3.10}$$

where

$$Q^m_{mij} := \mathrm{trace}(Q^k_{lij}).$$

By (3.10), we get the proof. □

Lemma 3.2. *A Finsler metric F on a manifold M is conformally closed weakly Berwald metric if and only if $F^2\kappa_m I^m$ is a 1-form on M.*

Proof. By (3.6), we have

$$Q^i_j = y_j\kappa^i - F^2 C^i{}_{kj}\ \kappa^k. \tag{3.11}$$

Taking a trace of (3.11) yields

$$Q^m_m = (y_m - F^2 I_m)\kappa^m.$$

It is easy to see that $\kappa^m y_m = \kappa_m y^m$ is a 1-form on M. Also, we have

$$\kappa^m F^2 I_m = \kappa_m(x) F^2 I^m.$$

Then by Lemma 3.1, we get the proof. □

Example 3.3. Let F be a scalar function on TM defined by $F = \sqrt[m]{A}$, where A is given by $A := a_{i_1\dots i_m}(x)y^{i_1}y^{i_2}\dots y^{i_m}$, with $a_{i_1\dots i_m}$ symmetric in all its indices. Then F is called an mth root Finsler metric. For an mth root metric $F = \sqrt[m]{\mathfrak{a}_{i_1\dots i_m}(x)y^{i_1}y^{i_2}\dots y^{i_m}}$, we have

$$\det(g_{ij}) = \Big(\frac{-1}{n}\Big)^n.$$

Then we get

$$I_i = \frac{\partial}{\partial y^i}\Big[\ln\sqrt{\det(g_{jk})}\Big] = 0. \tag{3.12}$$

By (3.12), it follows that every mth root Finsler metric is a trivial conformally closed weakly Berwald metric.

Proof of Theorem 1.1. Equation (3.4) is equal to

$$\bar{G}^i - G^i = \kappa_0 y^i - \frac{1}{2}F^2\kappa^i. \tag{3.13}$$

We have

$$\frac{\partial g^{ij}}{\partial y^k} = -2C^{ij}_{\ \ k}.$$

Then, we obtain

$$\frac{\partial \kappa^i}{\partial y^j} = -2\kappa_m C^{mi}_{\ \ \ j}. \tag{3.14}$$

By (3.13) and (3.14), we get

$$\bar{G}^i_{\ j} - G^i_{\ j} = \kappa_j y^i + \kappa_0\delta^i_j - \kappa_m\Big[-F^2C^{im}_{\ \ \ j} + y_j g^{im}\Big],$$

$$\bar{G}^i_{\ jk} - G^i_{\ jk} = \kappa_j\delta^i_k + \kappa_k\delta^i_j - \kappa_m\Big[g_{jk}g^{im} - C^{im}_{\ \ \ j,k}F^2 - 2y_jC^{im}_{\ \ \ k} - 2y_kC^{im}_{\ \ \ j}\Big].$$

Then

$$\bar{B}^i_{\ jkl} - B^i_{\ jkl} = -\kappa_m\Big[2g^{im}C_{jkl} - 2g_{jk}C^{im}_{\ \ \ l} - C^{im}_{\ \ \ j,k,l}F^2 - 2y_lC^{im}_{\ \ \ j,k} - 2g_{jl}C^{im}_{\ \ \ k} - 2y_jC^{im}_{\ \ \ k,l} - 2g_{kl}C^{im}_{\ \ \ j} - 2y_kC^{im}_{\ \ \ j,l}\Big]. \tag{3.15}$$

Taking a trace $i = j$ in (3.15) yields

$$\bar{E}_{kl} - E_{kl} = \kappa_m[F^2I^m_{\ ,k,l} + 2g_{kl}I^m + 2y_lI^m_{\ ,k} + 2y_kI^m_{\ ,l}]. \tag{3.16}$$

By (3.16), we get (1.1). □

4. Proof of Theorem 1.2

In this section, we are going to prove Theorem 1.2. Indeed, we show that a Randers metric $F = \alpha + \beta$ is a conformally closed weakly Berwald metric if and only if it is a Riemannian metric or the conformal transformation is homothety.

Proof of Theorem 1.2. Let $F = \alpha + \beta$ be a conformally closed weakly Berwald metric. For a Randers metric $F = \alpha + \beta$ on an n-dimensional manifold M, we have

$$g_{ij} = \frac{F}{\alpha}\Big[a_{ij} - \frac{y_i}{\alpha}\frac{y_j}{\alpha} + \frac{\alpha}{F}\Big(b_i + \frac{y_i}{\alpha}\Big)\Big(b_j + \frac{y_j}{\alpha}\Big)\Big],$$
$$g^{ij} = \frac{\alpha}{F}a^{ij} + \frac{b^2\alpha + \beta}{F^3}y^i y^j - \frac{\alpha}{F^2}(b^i y^j + b^j y^i),$$
$$I_i = \frac{1}{2}(n+1)F^{-1}\alpha^{-2}(\alpha^2 b_i - \beta y_i),$$

which yields

$$I^j = g^{ji}I_i = \frac{n+1}{2F\alpha^2}\left(\frac{\alpha}{F}a^{ij} - \frac{\alpha}{F^2}(b^i y^j + b^j y^i) + \frac{1}{F^3}(b^2\alpha + \beta)y^i y^j\right)(\alpha^2 b_i - \beta y_i).$$

Then

$$F^2 I^j = \frac{n+1}{2}\left[\alpha b^j - \frac{b^2\alpha + \beta}{\alpha + \beta}y^j\right] = \frac{n+1}{2}\left[\alpha b^j + \frac{(1-b^2)\alpha}{\alpha+\beta}y^j - y^j\right],$$

where $b := \|\beta\|_\alpha = \sqrt{b^i b_i} < 1$. It follows that

$$F^2\kappa_j I^j = \frac{n+1}{2}\left[\alpha\kappa_j b^j + \frac{(1-b^2)\alpha}{\alpha+\beta}\kappa_j y^j - \kappa_j y^j\right].$$

By the assumption, F is conformally closed weakly Berwald metric. Then, by Lemma 3.2, $F^2\kappa_j I^j$ is a 1-form on M. Then, the following equality holds:

$$\alpha\kappa_j b^j + \frac{(1-b^2)\alpha}{\alpha+\beta}\kappa_j y^j = \theta, \tag{4.1}$$

where $\theta = \theta_i(x)y^i$ is a 1-form on M. Simplifying (4.1) give us

$$\kappa_j b^j \alpha^2 + \big[\kappa_j b^j \beta + (1-b^2)\kappa_0 - \theta\big]\alpha - \theta\beta = 0. \tag{4.2}$$

By (4.2), we get

$$\kappa_j b^j \alpha^2 = \theta\beta, \tag{4.3}$$
$$\kappa_j b^j \beta + (1-b^2)\kappa_0 - \theta = 0. \tag{4.4}$$

It is easy to see that $\kappa_j b^j$ is a function on M. By (4.3), we get $\beta = 0$ or $\theta = 0$. In the first case, F is Riemannian. Suppose that F is not a Riemannian metric. Then, by putting $\theta = 0$ in (4.4), we obtain

$$\kappa_j b^j \beta + (1-b^2)\kappa_0 = 0. \tag{4.5}$$

Taking a vertical derivation of (4.5) yields

$$\kappa_j b^j b_m + (1-b^2)\kappa_m = 0. \tag{4.6}$$

Contracting (4.6) with b^m implies

$$\kappa_m b^m = 0. \tag{4.7}$$

Putting (4.7) in (4.6) and considering $b < 1$ give us

$$\kappa_m = 0.$$

Then $\kappa = \textit{constant}$ and the conformal transformation is homothety. □

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