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SOME COHOMOLOGICAL PROPERTIES OF GENERALIZED MODULE EXTENSION BANACH ALGEBRAS

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Abstract. Let A and X be Banach algebras such that X is a Banach algebraic A-module. The generalized module extension $A \bowtie X$, which is a strongly splitting Banach algebra extension of X by A, was recently introduced and studied. Many known Banach algebras, such as module extension, Lau product, (generalized) semidirect product, and also the direct product of Banach algebras, have this general framework. We first investigate the n-weak amenability of $A \bowtie X$, and then we characterize its cyclic amenability improving some known results. We also examine our results to some special generalized module extension algebras. Furthermore, we characterize the cyclic amenability of some concrete Banach algebras related to locally compact groups.

1. Introduction and preliminaries

Let A and X be Banach algebras such that X is a Banach A-module. We say that X is an algebraic Banach A-module if for every $x, y \in X$ and $a \in A$,

$$a(xy) = (ax)y, \ (xy)a = x(ya), \ (xa)y = x(ay),$$

and $\|ax\| \leq \|a\|\|x\|$ and $\|xa\| \leq \|a\|\|x\|$. Then the direct sum $A \times X$ under the multiplication

$$(a, x)(b, y) = (ab, ay + xb + xy) \quad (a, b \in A, x, y \in X)$$

and the norm $\|(a, x)\| = \|a\| + \|x\|$ is a Banach algebra, which is called the generalized module extension of A by X and denoted by $A \bowtie X$. This type of product was first introduced by the second author and Barootkoob [17], where

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the structure of n-dual valued derivations and the n-weak amenability for this Banach algebra have been investigated. This Banach algebra also was named the amalgamated duplication of A along X in [11]. From the homological algebra point of view, $A \bowtie X$ is a strongly splitting Banach algebra extension of X by A, which means that X is a closed two-sided ideal of $A \bowtie X$, and the quotient $(A \bowtie X)/X$ is isometrically isomorphic to A.

Recently the generalized module extension Banach algebras has attracted the attention of many authors and various notions of them have been studied from different points of view; one may refer the reader to [7,8,11,14,15] and references therein. Generalized module extension Banach algebras appear in the study of many classes of Banach algebras. For example, every measure algebra has a decomposition to generalized module extension of Banach algebras. In [3], using the structure of this decomposition, the authors have studied the amenability and weak amenability of measure algebras; see also [4].

Generalized module extension $A \bowtie X$ is also a generalization of important classes of Banach algebras, which may obtained by considering different module actions on X and algebra multiplications on A. Module extension algebra, (generalized) semidirect product algebra, θ-Lau product, and also the direct product of Banach algebras are some important examples of Banach algebras that can be recognized as generalized module extensions. Some notions of amenability and many other aspects of the mentioned special cases have been investigated by many authors; see, for example, [5, 9, 13, 15, 16, 19, 20] and references therein.

The structure of n-dual valued derivations and the n-weak amenability for $A \bowtie X$ were investigated in [11, 17]. The cyclic amenability of $A \bowtie X$ was studied also in [11], where the authors raised some open questions. Question (6) of [11] has already been answered by the second author and Barootkoob in [17]. In this paper, we improve some results of [17], answer Question 5 of [11] and provide a necessary and sufficient condition for $A \bowtie X$ to be cyclic amenable.

This paper is organized as follows. In Section 2, we turn our attention to the concept of n-weak amenability of $A \bowtie X$. The main result of this section (Theorem 2.2), which is an improvement of [17, Proposition 2.3] and [11, Proposition 5.1], asserts that under certain condition on X, the $(2n+1)$-weak amenability of $A \bowtie X$ and that of A and X are equivalent. This result not only answers Question 5 of [11], but also is a generalization of some recent results such as [9, Theorem 4.1], [5, Proposition 2.4], and [16, Theorem 2.9]. Section 3 is devoted to the cyclic amenability of $A \bowtie X$. We give a general necessary and sufficient condition for $A \bowtie X$ to be cyclic amenable and show that there is a close relation between the cyclic amenability of $A \bowtie X$ and that of A and X. We also apply our results to extend [11, Theorem 5.3], characterize the cyclic amenability of direct product, and also improve [9, Theorem 5.1] about the cyclic amenability of Lau product Banach algebra $A\ {}_\theta\!\times X$. Furthermore, for a locally compact group G, we show that the cyclic amenability of $M_c(G) \bowtie M_c(G)$ as well as $M(G)\ {}_\theta\!\times M_c(G)$ is equivalent to the discreteness of G.

2. n-weak amenability

Let A be a Banach algebra, and let Y be a Banach A-module. Then the dual space Y^* of Y becomes a dual Banach A-module with the module actions defined by

$$(fa)(y) = f(ay), \quad (af)(y) = f(ya),$$

for all $a \in A, y \in Y$ and $f \in Y^*$. Similarly, the nth-dual $Y^{(n)}$ of Y is a Banach A-module. In particular, $A^{(n)}$ is a Banach A-module. A derivation from A into Y is a linear mapping $D : A \to Y$ satisfying

$$D(ab) = D(a)b + aD(b) \quad (a, b \in A).$$

If $y \in Y$, then $\delta_y : A \to Y$ defined by $\delta_y(a) = ay - ya$ is a derivation. A derivation D is inner if there is $y \in Y$ such that $D = \delta_y$.

Recall from [18] that a Banach algebra A is said to be amenable, if every bounded derivation from A into any dual Banach A-module is inner; that is, $H^1(A, Y^*) = \{0\}$, for every Banach A-module Y. It is called contractible if $H^1(A, Y) = \{0\}$, for every Banach A-module Y. These notions was first introduced and studied by Johnson [12]. Amenability (resp., contractibility) has known hereditary properties; see, for example, [18]. In particular, since $A \bowtie X$ is a strongly splitting Banach algebra extension of X by A, we have the following result.

Proposition 2.1. *Let X be an algebraic Banach A-module. Then $A \bowtie X$ is contractible (resp., amenable) if and only if both A and X are contractible (resp., amenable).*

The concept of n-weak amenability was initiated by Dales, Ghahramani and Grønbæk [2], where they presented many important properties of this sort of Banach algebra. A Banach algebra A is said to be n-weakly amenable, for an integer $n \geq 0$, if $H^1(A, A^{(n)}) = \{0\}$, where $A^{(0)} = A$. It is called weakly amenable if it is 1-weakly amenable; see [18] for more details.

Recently, the second author and Barootkoob [17] investigated the n-weak amenability of generalized module extension Banach algebras. In the rest of this section, we will extend and improve some of their results about $(2n+1)$-weak amenability of generalized module extensions. We avoid mentioning details and use the concepts and symbols of that article.

It follows from [17, Proposition 2.3] that if $\langle XX^{(2n)}\rangle$ (resp., $\langle X^{(2n)}X\rangle$) is dense in $X^{(2n)}$, then $(2n+1)$-weak amenability of $A \bowtie X$ follows from $(2n+1)$-weak amenability of both A and X; see also [11, Proposition 5.1]. The converse of this result was left as an open problem in [11, Section 6]. In the following result, we prove the converse in the case where X has a bounded left (resp., right) approximate identity. This result is also a generalization of [5, Proposition 2.4], [9, Theorem 4.1], and [16, Theorem 2.9].

Theorem 2.2. *Suppose that $\langle XX^{(2n)}\rangle$ (resp., $\langle X^{(2n)}X\rangle$) is dense in $X^{(2n)}$ and that X has a bounded left (resp. right) approximate identity. Then $A \bowtie X$ is $(2n+1)$-weakly amenable if and only if both A and X are $(2n+1)$-weakly amenable.*

Proof. The sufficiency follows from [17, Proposition 2.3]. To prove the necessity, in view of [17, Theorem 2.2], we need to prove X is $(2n+1)$-weakly amenable. For this end, let $S : X \to X^{(2n+1)}$ be a bounded derivation and let $(x_\alpha)_\alpha$ be a bounded left approximate identity for X. Define $T : X \to A^{(2n+1)}$ and $D : A \to X^{(2n+1)}$ by

$$T(x) = w^* - \lim_\alpha (S(x_\alpha)x + x_\alpha S(x)) \qquad (x \in X),$$
$$D(a) = w^* - \lim_\alpha (S(ax_\alpha) - aS(x_\alpha)) \qquad (a \in A).$$

To prove S is inner, we show that these operators satisfy in condition (2) of [17, Theorem 2.2]. To do this, let $x, y \in X$. Then

$$\begin{aligned} T(xy) &= w^* - \lim_\alpha (S(x_\alpha)xy + x_\alpha S(xy)) \\ &= w^* - \lim_\alpha (S(x_\alpha)xy + x_\alpha S(x)y + x_\alpha xS(y)) \\ &= w^* - \lim_\alpha (S(x_\alpha x)y + x_\alpha xS(y)) \\ &= S(x)y + xS(y). \end{aligned}$$

For each $a \in A$ and $x \in X$, we have

$$\begin{aligned} D(a)x &= w^* - \lim_\alpha (S(ax_\alpha)x - aS(x_\alpha)x) \\ &= w^* - \lim_\alpha (S(ax_\alpha x) - ax_\alpha S(x) - aS(x_\alpha)x) \\ &= w^* - \lim_\alpha S(ax_\alpha x) - aS(x_\alpha x) \\ &= S(ax) - aS(x). \end{aligned}$$

From this, we get $xD(a)y = (S(xa) - S(x)a)y$. This implies that $xD(a) = S(xa) - S(x)a$ on $\langle XX^{(2n)} \rangle$. By assumption, we therefore have $xD(a) = S(xa) - S(x)a$ in $X^{(2n+1)}$.

To show that D is a derivation, let $a, b \in A$ and let $x \in X$. Then

$$\begin{aligned} (D(a)b + aD(b))x &= D(a)bx + aD(b)x \\ &= S(abx) - aS(bx) + aS(bx) - abS(x) \\ &= D(ab)x. \end{aligned}$$

Again, from the density of $\langle XX^{(2n)} \rangle$ in $X^{(2n)}$, we get $D(ab) = D(a)b + aD(b)$.

Finally, from $T(xy) = S(x)y + xS(y)$, for each $a \in A$ and $x, y \in X$, we have

$$\begin{aligned} T(axy) - aT(xy) &= S(ax)y + axS(y) - aS(x)y - axS(y) \\ &= S(ax)y - aS(x)y = (S(ax) - aS(x))y \\ &= D(a)xy. \end{aligned}$$

Since $\langle X^2 \rangle$ is dense in X, it follows that $T(ax) = aT(x) + D(a)x$, for all $a \in A, x \in X$. Similarly, we have $T(xa) = T(x)a + xD(a)$. □

As an immediate consequence, we have th following corollary.

Corollary 2.3. *Suppose that X has a bounded left (resp., right) approximate identity. Then $A \bowtie X$ is weakly amenable if and only if both A and X are weakly amenable.*

In the case where X has a left (resp., right) identity, we have the next result, which answers Question 5 of [11, Section 6] and improves [17, Theorem 4.2].

Corollary 2.4. *Suppose that X has a left (resp., right) identity. Then $A \bowtie X$ is $(2n+1)$-weakly amenable if and only if both A and X are $(2n+1)$-weakly amenable.*

3. Cyclic amenability

In this section, we characterize the cyclic amenability of generalized module extension Banach algebra $A \bowtie X$. We start with some preliminaries about cyclic amenability.

A derivations $D : A \to A^*$ is called cyclic if $D(a)(c) + D(c)(a) = 0$, for every $a, c \in A$. Clearly inner derivations are cyclic. A Banach algebra is called cyclic amenable if every continuous cyclic derivations $D : A \to A^*$ is inner. This notion was initiated by Gronbaek [10]. Examples of cyclic amenable Banach algebras include C*-algebras, $\ell^1(G)$ if G is a group and $\ell^1(S)$ if S is the free semigroup on a set. If X is a Banach space with $\dim X > 1$, then X with zero algebra product is not cyclic amenable; see [10] for more details. In this section, among other things, we present nontrivial examples of Banach algebras that are (not) cyclic amenable.

First, as the main result of this section, we gives general necessary and sufficient conditions for $A \bowtie X$ to be cyclic amenable.

Theorem 3.1. *A generalized module extension Banach algebra $A \bowtie X$ is cyclic amenable if and only if the following conditions hold:*

(i) *A is cyclic amenable.*
(ii) *If $d : A \to X^*$ is a derivation such that $xd(a) = 0$ in X^*, for all $a \in A$ and $x \in X$, then there exists an element $g \in X^*$ such that $d = \delta_g$ and $xg = gx$ in X^* for all $x \in X$.*
(iii) *If $S : X \to X^*$ is a cyclic derivation such that there exist a derivation $d : A \to X^*$ satisfying $S(ax) = aS(x) + d(a)x$ and $S(xa) = S(x)a + xd(a)$, for all $a \in A$ and $x \in X$, then S is inner derivation.*

Proof. To prove the necessity, suppose that $A \bowtie X$ is cyclic amenable. Let $d : A \to A^*$ be a cyclic derivation. Then $D : A \bowtie X \to (A \bowtie X)^*$ defined by $D(a, x) = (d(a), 0)$ is a cyclic derivation and so it is inner. Thus, there exists an element $(f, g) \in (A \bowtie X)^*$ such that $D = \delta_{(f,g)}$. Indeed

$$\begin{aligned} \delta_{(f,g)}(a, x) &= (af + xg, xg + ag) - (fa + gx, gx + ga) \\ &= (\delta_f(a) + xg - gx, \delta_g(x) + \delta_g(a)), \end{aligned}$$

for every $a \in A$ and $x \in X$. Now the equality $(d(a), 0) = (\delta_f(a) + xg - gx, \delta_g(x) + \delta_g(a))$, for $x = 0$, implies that d is inner, so A is cyclic amenable.

To prove (ii), let $d : A \to X^*$ be a derivation such that $xd(a) = 0$ for all $a \in A, x \in X$. Define $D : A \bowtie X \to (A \bowtie X)^*$ by $D(a, x) = (-d^*(\widehat{x}), d(a))$, where d^* is the adjoint of d. Then it is easy to check that D is a cyclic derivation, and so $D = \delta_{(f,g)}$, for some $(f, g) \in (A \bowtie X)^*$. Therefore,

$$(-d^*(\widehat{x}), d(a)) = (\delta_f(a) + xg - gx, \delta_g(x) + \delta_g(a)), \quad (a \in A, x \in X).$$

Applying the last equality for $x = 0$, we get $d = \delta_g$, and using it for $a = 0$, we obtain $xg = gx$ in X^* for all $x \in X$. This shows that g satisfies the desired condition.

Let $S : X \to X^*$ be a cyclic derivation and let $d : A \to X^*$ be a derivation such that $S(ax) = aS(x) + d(a)x$ and $S(xa) = S(x)a + xd(a)$ for all $a \in A, x \in X$. We define $D : A \bowtie X \to (A \bowtie X)^*$ by $D(a, x) = (-d^*(\widehat{x}), d(a) + S(x))$. Then a simple computation shows that D is a cyclic derivation. Thus, there exists an element $(f, g) \in (A \bowtie X)^*$ such that $D = \delta_{(f,g)}$. Using the equality

$$(-d^*(\widehat{x}), d(a) + S(x)) = (\delta_f(a) + xg - gx, \delta_g(x) + \delta_g(a)), \quad (a \in A, x \in X),$$

for $a = 0$, we obtain $S = \delta_g$. This proves (iii) and completes the proof of necessity.

For sufficiency, suppose that $D : A \bowtie X \to (A \bowtie X)^*$ is a cyclic derivation. A direct verification shows that D enjoys the presentation

$$D(a, x) = (D_A(a) - D_X^*(\widehat{x}), D_X(a) + S(x)) \quad ((a, x) \in A \bowtie X),$$

where $D_A : A \to A^*$ and $S : X \to X^*$ are cyclic derivations and $D_X : A \to X^*$ is a derivation satisfying $S(xa) = S(x)a + xD_X(a)$ and $S(ax) = aS(x) + D_X(a)x$ for all $a \in A$ and $x \in X$. By conditions (i) and (iii), D_A and S are inner derivations. Thus there are $f \in A^*$ and $g \in X^*$ such that $D_A = \delta_f$ and $S = \delta_g$. Put $D_1 = D_X - \delta_g$. Then D_1 is a derivation from A into X^*. Moreover $xD_1(a) = 0$, for each $a \in A, x \in X$. Indeed,

$$\begin{aligned} xD_1(a) &= xD_X(a) - xag + xga \\ &= S(xa) - S(x)a - xag + xga = 0. \end{aligned}$$

From condition (ii), it follows that there exists an element $h \in X^*$ such that $D_1 = \delta_h$, and $xh = hx$ in X^*, for all $x \in X$. Therefore, $D_X = \delta_{g+h}$ and $S = \delta_{g+h}$. Moreover, $D_X^*(\widehat{x}) = (g + h)x - x(g + h)$ for all $x \in X$. Thus $D = \delta_{(f,g+h)}$ is inner, as claimed. □

As a consequence of Theorem 3.1, in the following result, we show that, under some mild conditions, the cyclic amenability of $A \bowtie X$ and the cyclic amenability of both A and X are equivalent. This result extends [1, Corollary 2.5] and [11, Theorem 5.3].

Proposition 3.2. *Let X be a Banach algebraic A-module.*

(1) *Suppose that A and X are cyclic amenable. If $\langle X^2 \rangle$ is dense in X, then $A \bowtie X$ is cyclic amenable.*

(2) *Suppose that either X is weakly amenable or has a bounded right or left approximate identity. Then $A \bowtie X$ is cyclic amenable if and only if A and X are cyclic amenable.*

Proof. (1) It follows from the density of $\langle X^2 \rangle$ in X and the cyclic amenability of X, that conditions (ii) and (iii) of Theorem 3.1 are satisfied. Therefore, $A \bowtie X$ is cyclic amenable, since A is cyclic amenable.

(2) Suppose that $A \bowtie X$ is cyclic amenable. The cyclic amenability of A follows from condition (i) of Theorem 3.1. It remains to prove the cyclic amenability of X. If X is weakly amenable, then it trivially is cyclic amenable. So let X have a bounded right approximate identity, say $(x_\alpha)_\alpha$. Let $d : X \to X^*$ be a cyclic derivation, and define $D : A \to X^*$ by $D(a)(x) = \lim_\alpha (d(ax)(x_\alpha) - d(x)(x_\alpha a))$, for all $a \in A$ and $x \in X$. Then D is well-define and $D(a)(xy) = d(ax)(y) - d(x)(ya)$, for all $a \in A$ and $x, y \in X$. So we have

$$\begin{aligned}(aD(b) + D(a)b)(xy) &= D(b)(xya) + D(a)(bxy)\\ &= d(bx)(ya) - d(x)(yab) + d(abx)(y) - d(bx)(ya)\\ &= d(abx)(y) - d(x)(yab)\\ &= D(ab)(xy).\end{aligned}$$

From the density of $\langle X^2 \rangle$ in X we see that D is a derivation. Moreover, since d is cyclic, we have

$$\begin{aligned}(d(x)a + xD(a))(zy) &= d(x)(azy) + D(a)(zyx)\\ &= d(x)(azy) + d(azy)(x) - d(zy)(xa)\\ &= -d(zy)(xa)\\ &= d(xa)(zy).\end{aligned}$$

Again, since $\langle X^2 \rangle$ is dense in X, we have $d(xa) = d(x)a + xD(a)$. Similarly,

$$\begin{aligned}(ad(x) + D(a)x)(zy) &= d(x)(zya) + D(a)(xzy)\\ &= d(ax)(zy).\end{aligned}$$

Therefore, $d(ax) = ad(x) + D(a)x$. So, by Theorem 3.1(ii), there exists an element $g \in X^*$ such that $d = \delta_g$, and this shows that X is cyclic amenable.

The converse, follows from (1) and this fact that if X is weakly amenable or has a bounded right or left approximate identity, then $\langle X^2 \rangle$ is dense in X. □

Let A be a Banach algebra, which is also considered as a Banach A-module under its own multiplication. Then we have the following characterization of cyclic amenability of $A \bowtie A$. Compare it with the weak amenability of $A \bowtie A$; see [17, Corollary 2.6].

Corollary 3.3. *Let A be a Banach algebra. Then $A \bowtie A$ is cyclic amenable if and only if A is cyclic amenable and $\langle A^2 \rangle$ is dense in A.*

Proof. The sufficiency follows from part (1) of Proposition 3.2. For the necessity, suppose that $A \bowtie A$ is cyclic amenable. By virtue of Theorem 3.1 we only need to show that $\langle A^2 \rangle$ is dense in A. For this, let $f \in A^*$ be such that $f|_{A^2} = 0$. Define $D : A \to A^*$ by $D(a) = f(a)f$, for all $a \in A$. Then D is a bounded derivation such that $xD(a) = 0$ for all $a, x \in A$. So $D = 0$, by condition (ii) of Theorem 3.1. This implies that $f = 0$ and so $\langle A^2 \rangle$ is dense in A, as required. □

Let G be a locally compact group and let $M(G)$ be the measure algebra of G. A measure $\mu \in M(G)$ is called continuous if $\mu(\{s\}) = 0$ for every $s \in G$. By $M_c(G)$ we denote the set of all continuous measures on G. Then $M_c(G)$ is a closed ideal of $M(G)$; see [3]. In the case where G is discrete, $M_c(G) = \{0\}$. If G is not discrete, then $M_c(G) \neq \{0\}$ and $\langle M_c(G)^2 \rangle$ is not dense in $M_c(G)$; see [3, Theorem 2.7]. Thus from Corollary 3.3 we have the following result, which shows that the generalized module extension Banach algebra $M_c(G) \bowtie M_c(G)$ is not cyclic amenable if G is not discrete.

Corollary 3.4. *Let G be a locally compact group. Then $M_c(G) \bowtie M_c(G)$ is cyclic amenable if and only if G is discrete.*

Let $\theta \in \Delta(A)$, the set of all characters of A. It was shown in [9, Theorem 5.1], that if $\langle X^2 \rangle$ is dense in X, then the cyclic amenability of the θ-Lau product Banach algebra $A\,{}_\theta\!\times X$ is equivalent to the cyclic amenability of both A and X. Applying Theorem 3.1, for $A\,{}_\theta\!\times X$, we get the following characterization for cyclic amenability of $A\,{}_\theta\!\times X$, which improves and extends [9, Theorem 5.1]; see also [1, Theorem 2.3]. Before that, we recall that $d \in A^*$ is called a point derivation at $\theta \in \Delta(A)$ if

$$d(ac) = \theta(a)d(c) + \theta(c)d(a) \qquad (a, c \in A).$$

Theorem 3.5. *The θ-Lau product Banach algebra $A\,{}_\theta\!\times X$ is cyclic amenable if and only if the following conditions hold:*

(1) *Both A and X are cyclic amenable;*
(2) *$\langle X^2 \rangle$ is dense in X or there is no nonzero point derivation at θ.*

Proof. Suppose that $A\,{}_\theta\!\times X$ is cyclic amenable. The cyclic amenability of A follows from condition (i) of Theorem 3.1. Let $d : X \to X^*$ be a bounded cyclic derivation. Using condition (iii) of Theorem 3.1 with $S = d$ and $D = 0$, we get d is inner. So X is cyclic amenable. This proves (1).

To prove (2), take a nonzero element $f \in X^*$ with $f|_{X^2} = 0$ and let $d \in A^*$ be a continuous point derivation at θ. Define $D : A \to X^*$ by $D(a) = d(a)f$, for $a \in A$. Since $ax = xa = \theta(a)x$, for all $a \in A, x \in X$, the map D is a bounded derivation with $xD(a) = 0$ for $a \in A, x \in X$. It follows from condition (ii) of Theorem 3.1 that $D = \delta_g$ for some $g \in X^*$. So $D = 0$. This implies that $d = 0$, as required.

For the converse, we need to show that conditions (ii) and (iii) in Theorem 3.1 hold. Condition (iii) follows trivially from (1). For condition (ii), let $D : A \to X^*$ be a bounded derivation such that $xD(a) = 0$ for all $a \in A, x \in X$. Since the module actions given by $ax = xa = \theta(a)x$, for all $a \in A, x \in X$, it follows that $F \circ D$ is a continuous point derivation at θ, for any $F \in X^{**}$. If there is no nonzero continuous point derivation at θ, then $F \circ D = 0$ for any $F \in X^{**}$. That implies that $D = 0$. So $D = \delta_g$ for $g = 0$. In the case where $\langle X^2 \rangle$ is dense in X, since $xD(a) = 0$ for each $a \in A, x \in X$, we also arrive to $D = 0$, and this completes the proof. $\square$

Let G be a locally compact group. The subalgebra of $M(G)$ consisting of all discrete measures on G can be identified with $\ell^1(G)$. Thus

$$\ell^1(G) \simeq \left\{ \mu \in M(G); \quad \mu = \sum_{s\in G} \alpha_s \delta_s \ \text{ with } \ \sum_{s\in G} |\alpha_s| < \infty \right\},$$

where δ_s denote the point mass measure at $s \in G$. Then $M(G) = \ell^1(G) \bowtie M_c(G)$, [4]. Define $\tau : M(G) \to \mathbb{C}$ by

$$\tau(\mu) = \sum_{s\in G} \mu_d(s), \qquad (\mu = (\mu_d, \mu_c) \in \ell^1(G) \bowtie M_c(G)).$$

Then $\tau \in \Delta(M(G))$, which is called the discrete augmentation character on $M(G)$; see [3]. It is proved in [3, Theorem 3.2] that if G is not discrete, then there is a nonzero continuous point derivation at τ. It is also known that the group algebra $L^1(G)$ of a locally compact group G is always cyclic amenable; see [10]. So as a consequence of Theorem 3.5, we have the following result.

Corollary 3.6. *Let G be a locally compact group and let τ be the discrete augmentation character on $M(G)$. Then $M(G) \ {}_\tau\!\times M_c(G)$ is cyclic amenable if and only if G is discrete.*

If we apply Theorem 3.5 for $\theta = 0$, then we get the following result.

Theorem 3.7. *The direct sum $A \times_1 X$ of two Banach algebras A and X is cyclic amenable if and only if*

(1) *Both A and X are cyclic amenable,*
(2) *$\langle A^2 \rangle$ is dense in A or $\langle X^2 \rangle$ is dense in X.*

Remark 3.8. The notion of approximate cyclic amenability for a Banach algebra was introduced and studied by Esslamzadeh and Shojaee [6]. A Banach algebra A is called approximately cyclic amenable if every continuous cyclic derivation $D : A \to A^*$ is approximately inner. Using similar argument, one can obtain the same results about approximate cyclic amenability of $A \bowtie X$ and special cases.

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