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OSTROWSKI'S INEQUALITY FOR FUNCTIONS WHOSE FIRST DERIVATIVES ARE s-PREINVEX IN THE SECOND SENSE

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Abstract. In this paper, we establish some new Ostrowski type inequalities for functions whose first derivatives are s-preinvex in the second sense.

1. Introduction

In 1938 A.M. Ostrowski proved an interesting integral inequality which can be stated as follows

Theorem 1.1. [10] *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping in the interior I° of I, and $a, b \in I^\circ$ with $a < b$. If $|f'| \leq M$ for all $x \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall x \in [a, b] \tag{1.1}$$

Inequality (1.1) has attracted a great deal of interest for many researchers due to its diversity of applications in numerical analysis, probability theory, and other fields. The literature in this context is abundant. We can easily find papers dealing generalizations, extensions and variants of such type of inequality. We refer readers to [5, 6, 7, 8, 9, 15, 17] and the references cited therein.

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. Hanson [4] introduced a new class of generalized convex functions, called invex functions. In [1], the authors gave the concept of preinvex functions which is special case of invexity. Pini [14], Noor [11, 12],

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Yang and Li [19] and Weir [18] have studied their basic properties and roles in optimization, variational inequalities and equilibrium problems.

Set et al. [15] established the following Ostrowski's inequalities for differentiable s-convex functions

Theorem 1.2. [15, Theorem 7] *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{(s+1)(s+2)}$$
$$\times \left\{ \left[2(s+1)\left(\frac{b-x}{b-a}\right)^{s+2} - (s+2)\left(\frac{b-x}{b-a}\right)^{s+1} + 1 \right] |f'(a)| \right.$$
$$\left. + \left[2(s+1)\left(\frac{x-a}{b-a}\right)^{s+2} - (s+2)\left(\frac{x-a}{b-a}\right)^{s+1} + 1 \right] |f'(b)| \right\}$$

holds for each $x \in [a,b]$.

Theorem 1.3. [15, Theorem 8] *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leq \frac{b-a}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}$$
$$\left\{ \left(\frac{b-x}{b-a}\right)^{1+\frac{1}{p}} \left[\left(\frac{b-x}{b-a}\right)^{s+1} |f'(a)|^q + \left[1 - \left(\frac{x-a}{b-a}\right)^{s+1} \right] |f'(b)|^q \right] \right.$$
$$\left. + \left(\frac{x-a}{b-a}\right)^{1+\frac{1}{p}} \left[\left[1 - \left(\frac{b-x}{b-a}\right)^{s+1} \right] |f'(a)|^q + \left(\frac{x-a}{b-a}\right)^{s+1} |f'(b)|^q \right] \right\}$$

holds for each $x \in [a,b]$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.4. [15, Theorem 11] *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q \geq 1$, then the following inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq (b-a)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times \left\{ \left(\frac{b-x}{b-a}\right)^{2\left(1-\frac{1}{q}\right)} \left[\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2} |f'(a)|^q \right.\right.$$
$$\left. + \left(\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2} - \frac{1}{s+1}\left(\frac{x-a}{b-a}\right)^{s+1} + \frac{1}{(s+1)(s+2)} \right) |f'(a)|^q \right]^{\frac{1}{q}}$$
$$+ (b-a)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{ \left(\frac{x-a}{b-a}\right)^{2\left(1-\frac{1}{q}\right)} \left[\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2} |f'(a)|^q \right.\right.$$
$$\left.\left. + \left(\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2} - \frac{1}{s+1}\left(\frac{b-x}{b-a}\right)^{s+1} + \frac{1}{(s+1)(s+2)} \right) |f'(a)|^q \right]^{\frac{1}{q}} \right\}$$

holds for each $x \in [a,b]$.

Işcan [5] established the following Ostrowski's inequalities for functions whose derivatives are preinvex

Theorem 1.5. [5, Theorem 2.2] *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a,b \in A$ with $a < a+\eta(b,a)$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function and $|f'|$ is preinvex function on A. If f' is integrable on $[a, a+\eta(b,a)]$, then the following inequality*

$$\left| f(x) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)du \right| \leq \frac{\eta(b,a)}{6}$$
$$\times \left\{ \left[3\left(\frac{x-a}{\eta(b,a)}\right)^2 - 2\left(\frac{x-a}{\eta(b,a)}\right)^3 + 2\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^3 \right] |f'(a)| \right.$$
$$\left. + \left[1 - 3\left(\frac{x-a}{\eta(b,a)}\right)^2 + 4\left(\frac{x-a}{\eta(b,a)}\right)^3 \right] |f'(b)| \right\}$$

holds for each $x \in [a, a+\eta(b,a)]$.

Theorem 1.6. [5, Theorem 2.8] *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a,b \in A$ with $a < a+\eta(b,a)$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function and $|f'|^q$ is preinvex function on $[a, a+\eta(b,a)]$ for some* *fixed* $q \geq 1$. *If* f' *is integrable on* $[a, a+\eta(b,a)]$, *then the following inequality*

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right| \leq \eta(b,a) \left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\times \left\{ \left(\frac{x-a}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)} \left[\frac{(x-a)^2 (3\eta(b,a) - 2x + 2a)}{6\eta^3(b,a)} |f'(a)|^q \right.\right.$$

$$\left. + \frac{1}{3}\left(\frac{x-a}{\eta(b,a)}\right)^3 |f'(b)|^q \right]^{\frac{1}{q}} + \left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)}$$

$$\times \left[\frac{1}{3}\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^3 |f'(a)|^q \right.$$

$$\left.\left. + \left(\frac{1}{6} + \frac{(x-a)^2 (2x - 3\eta(b,a) - 2a)}{6\eta^3(b,a)}\right) |f'(b)|^q \right]^{\frac{1}{q}} \right\}$$

holds for each $x \in [a, a+\eta(b,a)]$.

Kirmaci [6] established the following midpoint inequalities for differentiable convex functions

Theorem 1.7. [6, Theorem 2.2] *Let* $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $a, b \in I^\circ$ *(*I° *is the interior of* I*) with* $a < b$. *If* $|f'|$ *is convex on* $[a, b]$, *then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} \left(|f'(a)| + |f'(b)|\right).$$

Theorem 1.8. [6, Theorem 2.3] *Let* $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $a, b \in I^\circ$ *(*I° *is the interior of* I*) with* $a < b$ *and let* $p > 1$. *If* $|f'|^{\frac{p}{p-1}}$ *is convex on* $[a, b]$, *then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{\frac{1}{p}}$$

$$\times \left(\left(3|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \left(|f'(a)|^{\frac{p}{p-1}} + 3|f'(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \right).$$

Theorem 1.9. [6, Theorem 2.4] *Let* $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $a, b \in I^\circ$ *(*I° *is the interior of* I*) with* $a < b$ *and let* $p > 1$. *If* $|f'|^{\frac{p}{p-1}}$ *is convex on* $[a, b]$, *then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{4} \left(\frac{4}{p+1}\right)^{\frac{1}{p}} \left(|f'(a)| + |f'(b)|\right).$$

Kirmaci et al. [7] gave a variant of Theorem 2.4 from [6] as follows

Theorem 1.10. [7, Theorem 2.1] *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ (I° is the interior of I) with $a < b$ and let $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \leq \left(\frac{3^{1-\frac{1}{p}}}{8}\right)(b-a)\left(|f'(a)| + |f'(b)|\right).$$

Wang et al. [17] established the following midpoint inequalities for functions whose power of the absolute value of the first derivatives are preinvex.

Theorem 1.11. [17, Theorem 3.1] *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a differentiable function. If $|f'|^q$ is preinvex on A for $q \geq 1$, then for every $a, b \in A$ with $\eta(b, a) \neq 0$ we have*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)du - f(\tfrac{2a+\eta(b,a)}{2})\right| \leq \frac{|\eta(b,a)|}{8}\left(\left(\frac{|f'(a)|^q + 2|f'(b)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{2|f'(a)|^q + |f'(b)|^q}{3}\right)^{\frac{1}{q}}\right).$$

Theorem 1.12. [17, Corollary 3.2] *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a differentiable function. If $|f'|$ is preinvex on A, then for every $a, b \in A$ with $\eta(b, a) \neq 0$ we have*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)du - f(\tfrac{2a+\eta(b,a)}{2})\right| \leq \frac{|\eta(b,a)|}{8}\left(|f'(a)| + |f'(b)|\right).$$

Motivated by these results, in this paper we establish some new Ostrowski's inequalities for functions whose first derivatives in absolute value are s-preinvex in the second sense.

2. Preliminaries

In this section, we recall some concepts of convexity that are well known in the literature. Throughout this section I is an interval of $\mathbb{R}$.

Definition 2.1. [13] A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Definition 2.2. [2] A nonnegative function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0, 1]$, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Let K be a subset in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions.

Definition 2.3. [18] A set K is said to be invex at x with respect to η, if

$$x + t\eta\left(y, x\right) \in K$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

K is said to be invex set with respect to η if K is invex at each $x \in K$.

Definition 2.4. [18] A function f on the invex set K is said to be preinvex with respect to η, if

$$f\left(x + t\eta\left(y, x\right)\right) \leq \left(1 - t\right) f\left(x\right) + tf(y)$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Definition 2.5. [16] A nonnegative function f on the invex set $K \subseteq [0, \infty)$ is said to be s-preinvex in the second sense with respect to η, for some fixed $s \in (0, 1]$ if

$$f\left(x + t\eta\left(y, x\right)\right) \leq \left(1 - t\right)^s f\left(x\right) + t^s f(y)$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Definition 2.6. [3] The incomplete beta function is defined as follows:

$$B_x(\alpha, \beta) = \int\limits_0^x t^{\alpha-1}(1 - t)^{\beta-1} dx$$

where $x \in [0, 1]$ and $\alpha, \beta > 0$.

Lemma 2.7. [5] *Let $A \subset \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $a < a + (b, a)$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If f' is integrable on $[a, a + (b, a)]$, then the following equality*

$$\begin{aligned} f(x) - \frac{1}{\eta\left(b, a\right)} \int\limits_a^{a+\eta(b,a)} f(u)du &= \eta\left(b, a\right) \left(\int\limits_0^{\frac{x-a}{\eta(b,a)}} tf'(a + t\eta\left(b, a\right))dt \right. \\ &\quad \left. + \int\limits_{\frac{x-a}{\eta(b,a)}}^1 \left(t - 1\right) f'(a + t\eta\left(b, a\right))dt \right) \end{aligned}$$

holds for all $x \in [a, a + (b, a)]$.

Also, we recall that the Euler Beta function is defined as follows: for $x, y > 0$

$$B\left(x, y\right) = \int\limits_0^1 t^{x-1}\left(1 - t\right)^{y-1} dt = \frac{\Gamma\left(x\right)\Gamma\left(y\right)}{\Gamma\left(x + y\right)}.$$

3. Main Results

In what follows, we assume that $K \subset [0, \infty)$ is an invex subset with respect to the bifunction η, where $\eta : K \times K \to \mathbb{R}$, and $a, b \in K^{\circ}$ the interior of K such that $[a, a + \eta(b, a)] \subset K$.

Theorem 3.1. *Let $f : K \to \mathbb{R}$ be a differentiable function such that $f' \in L([a, a + \eta(b, a)])$ with $\eta(b, a) > 0$. If $|f'|$ is s-preinvex in the second sense for some fixed $s \in (0, 1]$, then we have the following inequality*

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right| \leq \frac{\eta(b,a)}{(s+1)(s+2)} \left[\Psi_1 |f'(a)| + \Psi_2 |f'(b)| \right], \tag{3.1}$$

where

$$\Psi_1 = 1 - (s+2) \frac{x-a}{\eta(b,a)} \left(1 - \frac{x-a}{\eta(b,a)}\right)^{s+1} + s \left(1 - \frac{x-a}{\eta(b,a)}\right)^{s+2}, \tag{3.2}$$

and

$$\Psi_2 = 1 - (s+2) \left(\frac{x-a}{\eta(b,a)}\right)^{s+1} + 2(s+1) \left(\frac{x-a}{\eta(b,a)}\right)^{s+2}. \tag{3.3}$$

Proof. From Lemma 2.7, and property of modulus, we have

$$\begin{aligned} &\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right| \\ &\leq \eta(b,a) \left(\int_0^{\frac{x-a}{\eta(b,a)}} t \left| f'(a + t\eta(b,a)) \right| dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) \left| f'(a + t\eta(b,a)) \right| dt \right). \end{aligned} \tag{3.4}$$

Since $|f'|$ is s-preinvex function in the second sense, we deduce

$$\begin{aligned}
&\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right| \\
&\leq \eta(b,a) \left(\int_0^{\frac{x-a}{\eta(b,a)}} t(1-t)^s |f'(a)| \right. \\
&\left. + t^{s+1} |f'(b)| dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{s+1} |f'(a)| + (1-t) t^s |f'(b)| dt \right) \\
&= \eta(b,a) \left(|f'(a)| \left(\int_0^{\frac{x-a}{\eta(b,a)}} t(1-t)^s dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{s+1} dt \right) + \right. \\
&\left. + |f'(b)| \left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{s+1} dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) t^s dt \right) \right).
\end{aligned} \tag{3.5}$$

By a simple computation, we easily obtained

$$\begin{aligned}
\int_0^{\frac{x-a}{\eta(b,a)}} t(1-t)^s dt &= -\frac{1}{(s+1)} \frac{x-a}{\eta(b,a)} \left(1 - \frac{x-a}{\eta(b,a)}\right)^{s+1} \\
&\quad + \frac{1}{(s+1)(s+2)} - \frac{1}{(s+1)(s+2)} \left(1 - \frac{x-a}{\eta(b,a)}\right)^{s+2} \\
\int_0^{\frac{x-a}{\eta(b,a)}} t^{s+1} dt &= \frac{1}{s+2} \left(\frac{x-a}{\eta(b,a)}\right)^{s+2} \\
\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{s+1} dt &= \frac{1}{s+2} \left(1 - \frac{x-a}{\eta(b,a)}\right)^{s+2} \\
\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) t^s dt &= \frac{1}{(s+1)(s+2)} - \frac{1}{s+1} \left(\frac{x-a}{\eta(b,a)}\right)^{s+1} + \frac{1}{s+2} \left(\frac{x-a}{\eta(b,a)}\right)^{s+2}.
\end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5), we get the desired result. □

Remark 3.2. In Theorem 3.1, if we put $s = 1$ we obtain Theorem 2.2 from [5]. And if we choose $\eta(b,a) = b - a$ we obtain Theorem 7 from [15].

Corollary 3.3. *In Theorem 3.1, if we take $\eta(b,a) = b - a$ and $s = 1$ we obtain the following inequality*

$$\begin{aligned}&\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \\ \leq\ & \frac{1}{6(b-a)^2}\left\{\left((b-a)^3 - 3(b-a)(b-x)^2 + 4(b-x)^3\right)|f'(a)|\right. \\ &\left.+\left((b-a)^3 - 3(b-a)(x-a)^2 + 4(x-a)^3\right)|f'(b)|\right\}.\end{aligned}$$

Corollary 3.4. *In Theorem 3.1, if we choose $x = \frac{2a+\eta(b,a)}{2}$, we obtain the following midpoint inequality*

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)du\right| \leq \frac{\eta(b,a)}{(s+1)(s+2)}\left[1 - \frac{1}{2^{s+1}}\right]\left[|f'(a)| + |f'(b)|\right].$$

Remark 3.5. Corollary 3.4 will be reduced to Corollary 3.2 from [17], if we put $s = 1$. And if we choose $\eta(b,a) = b - a$, we obtain Corollary 1 from [15]. Moreover, if we take $\eta(b,a) = b - a$, and $s = 1$ we obtain Theorem 2.2 from [6].

Theorem 3.6. *Let $f : K \to \mathbb{R}$ be a differentiable function such that $f' \in L([a, a+\eta(b,a)])$ with $\eta(b,a) > 0$, and let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $|f'|^q$ is s-preinvex function in the second sense for some fixed $s \in (0,1]$, then the following inequality*

$$\begin{aligned}&\left|f(x) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)du\right| \leq \frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}}\right. \\ &\times\left(\left(1 - \left(1 - \frac{x-a}{\eta(b,a)}\right)^{s+1}\right)|f'(a)|^q + \left(\frac{x-a}{\eta(b,a)}\right)^{s+1}|f'(b)|^q\right)^{\frac{1}{q}} \\ &+\left(1 - \frac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}} \\ &\left.\times\left(\left(1 - \frac{x-a}{\eta(b,a)}\right)^{s+1}|f'(a)|^q + \left(1 - \left(\frac{x-a}{\eta(b,a)}\right)^{s+1}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right) \quad (3.7)\end{aligned}$$

holds for all $x \in [a, a+\eta(b,a)]$.

Proof. From Lemma 2.7, property of modulus, and Hölder's inequality, we have

$$\left|f(x)-\frac{1}{\eta(b,a)}\int\limits_{a}^{a+\eta(b,a)}f(u)du\right|\leq\eta(b,a)\left(\left(\int\limits_{0}^{\frac{x-a}{\eta(b,a)}}t^{p}dt\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int\limits_{0}^{\frac{x-a}{\eta(b,a)}}\left|f'(a+t\eta(b,a))\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int\limits_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{p}dt\right)^{\frac{1}{p}}$$

$$\left.\times\left(\int\limits_{\frac{x-a}{\eta(b,a)}}^{1}\left|f'(a+t\eta(b,a))\right|^{q}dt\right)^{\frac{1}{q}}\right)$$

$$=\frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{p+1}{p}}\left(\int\limits_{0}^{\frac{x-a}{\eta(b,a)}}\left|f'(a+t\eta(b,a))\right|^{q}dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{p+1}{p}}\left(\int\limits_{\frac{x-a}{\eta(b,a)}}^{1}\left|f'(a+t\eta(b,a))\right|^{q}dt\right)^{\frac{1}{q}}\right). \tag{3.8}$$

Since $|f'|^{q}$ is s-preinvex function in the second sense, we deduce

$$\left|f(x)-\frac{1}{\eta(b,a)}\int\limits_{a}^{a+\eta(b,a)}f(u)du\right|\leq\frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}}$$

$$\times\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{p+1}{p}}\left(\int\limits_{0}^{\frac{x-a}{\eta(b,a)}}\left((1-t)^{s}\left|f'(a)\right|^{q}+t^{s}\left|f'(b)\right|^{q}\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\frac{p+1}{p}}\left(\int\limits_{\frac{x-a}{\eta(b,a)}}^{1}\left((1-t)^{s}\left|f'(a)\right|^{q}+t^{s}\left|f'(b)\right|^{q}\right)dt\right)^{\frac{1}{q}}\right)$$

$$= \frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{p+1}{p}}\left(|f'(a)|^q\int_0^{\frac{x-a}{\eta(b,a)}}(1-t)^s dt+|f'(b)|^q\int_0^{\frac{x-a}{\eta(b,a)}}t^s dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{p+1}{p}}\left(|f'(a)|^q\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^s dt+|f'(b)|^q\int_{\frac{x-a}{\eta(b,a)}}^{1}t^s dt\right)^{\frac{1}{q}}\right). \tag{3.9}$$

Clearly, we have

$$\int_0^{\frac{x-a}{\eta(b,a)}}(1-t)^s dt = \frac{1}{s+1}-\frac{1}{s+1}\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1}$$
$$\int_0^{\frac{x-a}{\eta(b,a)}}t^s dt = \frac{1}{s+1}\left(\frac{x-a}{\eta(b,a)}\right)^{s+1}$$
$$\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^s dt = \frac{1}{s+1}\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1}$$
$$\int_{\frac{x-a}{\eta(b,a)}}^{1}t^s dt = \frac{1}{s+1}-\frac{1}{s+1}\left(\frac{x-a}{\eta(b,a)}\right)^{s+1}. \tag{3.10}$$

Substituting (3.10) into (3.9), we obtain the desired result. □

Remark 3.7. Theorem 3.6 will be reduced to Theorem 8 from [15] if we choose $\eta(b,a) = b-a$.

Corollary 3.8. *If we put $s = 1$ in Theorem 3.6, we obtain the following inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{b-a}{2^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}$$
$$\times\left(\left(\frac{x-a}{\eta(b,a)}\right)^2\left(\left(2-\frac{x-a}{\eta(b,a)}\right)|f'(a)|^q+\frac{x-a}{\eta(b,a)}|f'(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(1-\frac{x-a}{\eta(b,a)}\right)^2\left(\left(1-\frac{x-a}{\eta(b,a)}\right)|f'(a)|^q+\left(1+\frac{x-a}{\eta(b,a)}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right).$$

Corollary 3.9. *In Corollary 3.8, if we choose $\eta(b,a) = b-a$, we obtain the following inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{b-a}{2^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}$$
$$\times \left(\left(\frac{x-a}{b-a}\right)^2 \left(\left(1+\frac{b-x}{b-a}\right)|f'(a)|^q + \frac{x-a}{b-a}|f'(b)|^q \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(1-\frac{x-a}{b-a}\right)^2 \left(\frac{b-x}{b-a}|f'(a)|^q + \left(1+\frac{x-a}{b-a}\right)|f'(b)|^q \right)^{\frac{1}{q}} \right).$$

Corollary 3.10. *In Theorem 3.6, if we choose $x = \frac{2a+\eta(b,a)}{2}$, then we obtain the following midpoint inequality*

$$\left| f\left(\tfrac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)du \right| \leq \frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}2^{2+\frac{s}{q}}}$$
$$\times \left(\left(\left(2^{s+1}-1\right)|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}} + \left(|f'(a)|^q + \left(2^{s+1}-1\right)|f'(b)|^q \right)^{\frac{1}{q}} \right).$$

Corollary 3.11. *In Corollary 3.10, if we take $s = 1$ we obtain the following midpoint inequality*

$$\left| f\left(\tfrac{2a+\eta(b,a)}{2}\right) - \tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)du \right| \leq \tfrac{\eta(b,a)}{(p+1)^{\frac{1}{p}}4^{1+\frac{1}{q}}}$$
$$\times \left(\left(3|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}} + \left(|f'(a)|^q + 3|f'(b)|^q \right)^{\frac{1}{q}} \right).$$

Corollary 3.12. *In Corollary 3.10, if we choose $\eta(b,a) = b-a$, we obtain the following midpoint inequality*

$$\left| f\left(\tfrac{a+b}{2}\right) - \tfrac{1}{b-a}\int_a^b f(u)du \right| \leq \tfrac{b-a}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}2^{2+\frac{s}{q}}}$$
$$\times \left(\left(\left(2^{s+1}-1\right)|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}} + \left(|f'(a)|^q + \left(2^{s+1}-1\right)|f'(b)|^q \right)^{\frac{1}{q}} \right).$$

Remark 3.13. Corollary 3.12 will be reduced to Theorem 2.3 from [6] if we take $s = 1$.

Theorem 3.14. *Let $f : K \to \mathbb{R}$ be a differentiable function such that $f' \in L([a, a+\eta(b,a)])$ with $\eta(b,a) > 0$, and let $q > 1$. If $|f'|^q$ is s-preinvex function* *in the second sense for some fixed* $s \in (0,1]$, *then the following inequality*

$$\left|f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du\right| \leq \frac{\eta(b,a)}{2^{1-\frac{1}{q}}(s+1)^{\frac{1}{q}}(s+2)^{\frac{1}{q}}} \left(\left(\frac{x-a}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)}\right.$$
$$\times\left(\left(1-\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+2} - (s+2)\frac{x-a}{\eta(b,a)}\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1}\right)|f'(a)|^q\right.$$
$$\left.+(s+1)\left(\frac{x-a}{\eta(b,a)}\right)^{s+2}|f'(b)|^q\right)^{\frac{1}{q}} + \left(1-\frac{x-a}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)}$$
$$\times\left(\left(1-(s+2)\left(\frac{x-a}{\eta(b,a)}\right)^{s+1} + (s+1)\left(\frac{x-a}{\eta(b,a)}\right)^{s+2}\right)|f'(b)|^q\right.$$
$$\left.\left.+(s+1)\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+2}|f'(a)|^q\right)^{\frac{1}{q}}\right) \tag{3.11}$$

holds for all $x \in [a, a+\eta(b,a)]$.

Proof. From Lemma 2.7, property of modulus, and power mean inequality, we have

$$\left|f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du\right|$$
$$\leq \eta(b,a)\left(\left(\int_0^{\frac{x-a}{\eta(b,a)}} t dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{x-a}{\eta(b,a)}} t\,|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)\,|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}}\right)$$
$$= \frac{\eta(b,a)}{2^{1-\frac{1}{q}}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)}\left(\int_0^{\frac{x-a}{\eta(b,a)}} t\,|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(1-\frac{x-a}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)}\left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)\,|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}}\right). \tag{3.12}$$

Since $|f'|^q$ is s-preinvex function in the second sense, we deduce

$$\left|f(x)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(u)du\right|\leq\frac{\eta(b,a)}{2^{1-\frac{1}{q}}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)}\right.$$
$$\times\left(|f'(a)|^q\int_0^{\frac{x-a}{\eta(b,a)}}t(1-t)^s dt+|f'(b)|^q\int_0^{\frac{x-a}{\eta(b,a)}}t^{s+1}dt\right)^{\frac{1}{q}}$$
$$+\left(1-\frac{x-a}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)}$$
$$\left.\times\left(|f'(a)|^q\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{s+1}dt+|f'(b)|^q\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)\,t^s dt\right)^{\frac{1}{q}}\right). \tag{3.13}$$

Substituting (3.6) into (3.13), we obtain the desired result. □

Remark 3.15. Theorem 3.14 will be reduced to Theorem 11 from [15] if we choose $\eta(b,a)=b-a$, and to Theorem 2.8 from [5] if we put $s=1$.

Corollary 3.16. *In Theorem 3.14, if we take $\eta(b,a)=b-a$ and $s=1$, we obtain the following inequality.*

$$\left|f(x)-\tfrac{1}{b-a}\int_a^b f(u)du\right|\leq\tfrac{b-a}{2\times3^{\frac{1}{q}}}\left(\left(\tfrac{x-a}{b-a}\right)^{2\left(1-\frac{1}{q}\right)}\right.$$
$$\times\left(\left(1-3\left(\tfrac{b-x}{b-a}\right)^2+2\left(\tfrac{b-x}{b-a}\right)^3\right)|f'(a)|^q+2\left(\tfrac{x-a}{b-a}\right)^3|f'(b)|^q\right)^{\frac{1}{q}}$$
$$+\left(\tfrac{b-x}{b-a}\right)^{2\left(1-\frac{1}{q}\right)}\left(\left(1-3\left(\tfrac{x-a}{b-a}\right)^2+2\left(\tfrac{x-a}{b-a}\right)^3\right)|f'(b)|^q\right.$$
$$\left.\left.+2\left(\tfrac{b-x}{b-a}\right)^3|f'(a)|^q\right)^{\frac{1}{q}}\right).$$

Corollary 3.17. *In Theorem 3.14, if we choose $x=\frac{2a+\eta(b,a)}{2}$, we obtain the following midpoint inequality*

$$\left|f\left(\tfrac{2a+\eta(b,a)}{2}\right)-\tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(u)du\right|$$
$$\leq\frac{\eta(b,a)}{2^{3+\frac{1}{q}}(s+1)^{\frac{1}{q}}(s+2)^{\frac{1}{q}}}\left(\left(\left(1-\tfrac{s+3}{2^{s+2}}\right)|f'(a)|^q+\tfrac{s+1}{2^{s+2}}|f'(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\tfrac{s+1}{2^{s+2}}|f'(a)|^q+\left(1-\tfrac{s+3}{2^{s+2}}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right).$$

Remark 3.18. Corollary 3.17 will be reduced to Theorem 3.1 from [17] if we choose $s = 1$.

Corollary 3.19. *In Corollary 3.17, if we choose $\eta(b,a) = b - a$, we obtain the following midpoint inequality*

$$\begin{aligned}
&\left| f\left(\tfrac{a+b}{2}\right) - \tfrac{1}{b-a}\int_a^b f(u)du \right| \\
\leq\ & \tfrac{b-a}{(s+1)^{\frac{1}{q}}(s+2)^{\frac{1}{q}}2^{3\left(1-\frac{1}{q}\right)}} \left(\left(\left(1 - \tfrac{s+3}{2^{s+2}}\right) |f'(a)|^q + \tfrac{s+1}{2^{s+2}} |f'(b)|^q \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\left(1 - \tfrac{s+3}{2^{s+2}}\right) |f'(b)|^q + \tfrac{s+1}{2^{s+2}} |f'(a)|^q\right)^{\frac{1}{q}} \right).
\end{aligned}$$

Corollary 3.20. *Corollary 3.19 will be reduced to Corollary 6 from [15] if we put $s = 1$.*

Theorem 3.21. *Assume that all the assumptions of Theorem 3.14 are satisfied, then we have the following inequality*

$$\begin{aligned}
&\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right| \\
\leq\ & \eta(b,a) \left(\left(\frac{x-a}{\eta(b,a)} \right)^{1-\frac{1}{q}} \left(B_{\frac{x-a}{\eta(b,a)}}(q+1, s+1) |f'(a)|^q \right.\right. \\
&\left. + \frac{1}{s+q+1} \left(\frac{x-a}{\eta(b,a)} \right)^{s+q+1} |f'(b)|^q \right)^{\frac{1}{q}} \\
&+ \left(1 - \frac{x-a}{\eta(b,a)}\right)^{1-\frac{1}{q}} \left(\frac{1}{q+s+1} \left(1 - \frac{x-a}{\eta(b,a)}\right)^{q+s+1} |f'(a)|^q \right. \\
&\left.\left. + \left(B(s+1, q+1) - B_{\frac{x-a}{\eta(b,a)}}(s+1, q+1)\right) |f'(b)|^q \right)^{\frac{1}{q}} \right) \qquad (3.14)
\end{aligned}$$

holds for all $x \in [a, a + \eta(b,a)]$.

Proof. From Lemma 2.7, property of modulus, and power mean inequality, we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} f(u)du\right| \\
\leq\ & \eta(b,a)\left(\left(\int\limits_0^{\frac{x-a}{\eta(b,a)}} dt\right)^{1-\frac{1}{q}}\left(\int\limits_0^{\frac{x-a}{\eta(b,a)}} t^q\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int\limits_{\frac{x-a}{\eta(b,a)}}^1 dt\right)^{1-\frac{1}{q}}\left(\int\limits_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^q\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right) \\
=\ & \eta(b,a)\left(\left(\frac{x-a}{\eta(b,a)}\right)^{1-\frac{1}{q}}\left(\int\limits_0^{\frac{x-a}{\eta(b,a)}} t^q\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(1-\frac{x-a}{\eta(b,a)}\right)^{1-\frac{1}{q}}\left(\int\limits_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^q\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right). \quad (3.15)
\end{aligned}$$

Since $|f'|^q$ is s-preinvex function in the second sense, we deduce

$$\begin{aligned}
&\left|f(x)-\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} f(u)du\right| \leq \eta(b,a)\left(\left(\frac{x-a}{\eta(b,a)}\right)^{1-\frac{1}{q}}\right. \\
&\times\left(|f'(a)|^q\int\limits_0^{\frac{x-a}{\eta(b,a)}} t^q(1-t)^s dt+|f'(b)|^q\int\limits_0^{\frac{x-a}{\eta(b,a)}} t^{s+q}dt\right)^{\frac{1}{q}} \\
&+\left(1-\frac{x-a}{\eta(b,a)}\right)^{1-\frac{1}{q}}\left(|f'(a)|^q\int\limits_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{q+s}dt\right. \\
&\left.+|f'(b)|^q\int\limits_{\frac{x-a}{\eta(b,a)}}^1 t^s(1-t)^q dt\right)
\end{aligned}$$

$$= \eta(b,a)\left(\left(\frac{x-a}{\eta(b,a)}\right)^{1-\frac{1}{q}}\left(B_{\frac{x-a}{\eta(b,a)}}(q+1,s+1)\left|f'(a)\right|^{q}\right.\right.$$
$$\left.+\frac{1}{s+q+1}\left(\frac{x-a}{\eta(b,a)}\right)^{s+q+1}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\left(1-\frac{x-a}{\eta(b,a)}\right)^{1-\frac{1}{q}}\left(\frac{1}{q+s+1}\left(1-\frac{x-a}{\eta(b,a)}\right)^{q+s+1}\left|f'(a)\right|^{q}\right.$$
$$\left.\left.+\left(B(s+1,q+1)-B_{\frac{x-a}{\eta(b,a)}}(s+1,q+1)\right)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right), \tag{3.16}$$

which is the desired result. ∎

Corollary 3.22. *In Theorem 3.21, if we choose $\eta(b,a)=b-a$, we obtain the following inequality*

$$\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(u)du\right|\leq(b-a)\left(\left(\frac{x-a}{b-a}\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(B_{\frac{x-a}{b-a}}(q+1,s+1)\left|f'(a)\right|^{q}+\frac{1}{s+q+1}\left(\frac{x-a}{b-a}\right)^{s+q+1}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\left(\frac{b-x}{b-a}\right)^{1-\frac{1}{q}}\left(\frac{1}{q+s+1}\left(\frac{b-x}{b-a}\right)^{q+s+1}\left|f'(a)\right|^{q}\right.$$
$$\left.\left.+\left(B(s+1,q+1)-B_{\frac{x-a}{b-a}}(s+1,q+1)\right)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right).$$

Corollary 3.23. *In Theorem 3.21, if we put $s=1$, we obtain the following inequality*

$$\left|f(x)-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(u)du\right|\leq\eta(b,a)\left(\left(\frac{x-a}{\eta(b,a)}\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(B_{\frac{x-a}{\eta(b,a)}}(q+1,2)\left|f'(a)\right|^{q}+\frac{1}{q+2}\left(\frac{x-a}{\eta(b,a)}\right)^{q+2}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\left(1-\frac{x-a}{\eta(b,a)}\right)^{1-\frac{1}{q}}\left(\frac{1}{q+2}\left(1-\frac{x-a}{\eta(b,a)}\right)^{q+2}\left|f'(a)\right|^{q}\right.$$
$$\left.\left.+\left(B(2,q+1)-B_{\frac{x-a}{\eta(b,a)}}(2,q+1)\right)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right).$$

Corollary 3.24. *In Corollary 3.23, if we choose* $\eta(b,a)=b-a$*, we obtain the following inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq (b-a)\left(\left(\frac{x-a}{b-a}\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(B_{\frac{x-a}{b-a}}(q+1,2)\left|f'(a)\right|^q+\frac{1}{q+2}\left(\frac{x-a}{b-a}\right)^{q+2}\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{b-x}{b-a}\right)^{1-\frac{1}{q}}\left(\frac{1}{q+2}\left(\frac{b-x}{b-a}\right)^{q+2}\left|f'(a)\right|^q\right.$$
$$\left.\left.+\left(B(2,q+1)-B_{\frac{x-a}{b-a}}(2,q+1)\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right).$$

Corollary 3.25. *In Theorem 3.21, if we choose* $x=\frac{2a+\eta(b,a)}{2}$*, we obtain the following midpoint inequality*

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(u)du\right|\leq\frac{\eta(b,a)}{2^{1-\frac{1}{q}}}$$
$$\times\left(\left(B_{\frac{1}{2}}(q+1,s+1)\left|f'(a)\right|^q+\frac{1}{(s+q+1)2^{s+q+1}}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{1}{(s+q+1)2^{s+q+1}}\left|f'(a)\right|^q\right.$$
$$\left.\left.+\left(B(s+1,q+1)-B_{\frac{1}{2}}(s+1,q+1)\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right).$$

Corollary 3.26. *In Corollary 3.25, if we take* $s=1$*, we obtain the following midpoint inequality*

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(u)du\right|\leq\frac{\eta(b,a)}{2^{1-\frac{1}{q}}}$$
$$\times\left(\left(B_{\frac{1}{2}}(q+1,2)\left|f'(a)\right|^q+\frac{1}{(q+2)2^{q+2}}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{1}{(q+2)2^{q+2}}\left|f'(a)\right|^q\right.$$
$$\left.\left.+\left(B(2,q+1)-B_{\frac{1}{2}}(2,q+1)\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right).$$

Corollary 3.27. *In Corollary 3.25, if we choose $\eta(b,a) = b-a$, we obtain the following midpoint inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{b-a}{2^{1-\frac{1}{q}}}$$
$$\times \left(\left(B_{\frac{1}{2}}(q+1,s+1)\left|f'(a)\right|^q + \frac{1}{(s+q+1)\,2^{s+q+1}}\left|f'(b)\right|^q \right)^{\frac{1}{q}} \right.$$
$$+ \left(\frac{1}{(s+q+1)\,2^{s+q+1}}\left|f'(a)\right|^q \right.$$
$$\left. \left. + \left(B(s+1,q+1) - B_{\frac{1}{2}}(s+1,q+1) \right)\left|f'(b)\right|^q \right)^{\frac{1}{q}} \right).$$

Corollary 3.28. *In Corollary 3.27, if we put $s = 1$, we obtain the following midpoint inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{b-a}{2^{1-\frac{1}{q}}}$$
$$\times \left(\left(B_{\frac{1}{2}}(q+1,2)\left|f'(a)\right|^q + \frac{1}{(q+2)\,2^{q+2}}\left|f'(b)\right|^q \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\frac{1}{(q+2)\,2^{q+2}}\left|f'(a)\right|^q + \left(B(2,q+1) - B_{\frac{1}{2}}(2,q+1) \right)\left|f'(b)\right|^q \right)^{\frac{1}{q}} \right).$$

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