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A CLASS OF SEQUENCE SPACES DEFINED BY FRACTIONAL DIFFERENCE OPERATOR AND MODULUS FUNCTION

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Abstract. A class of vector-valued sequence spaces is introduced employing the fractional difference operator $\Delta^{(\alpha)}$, a sequence of modulus functions and a non-negative infinite matrix. Sequence spaces of this class generalize many sequence spaces which are defined by difference operators and modulus functions. It is proved that the spaces of this class are complete paranormed spaces under certain conditions. Some properties of these spaces are studied and it is shown that the spaces are not solid in general.

1. Introduction

The theory of sequence space plays a significant role in the study of functional analysis. A sequence space is a linear subspace of the set w of all sequences of complex numbers. Over the years, many sequence spaces have been introduced by several authors. Ruckle [13] used the concept of modulus function to introduce the FK-space

$$\mathcal{L}(f) = \left\{ x = (x_n) \in w : \sum_{n=1}^{\infty} f(|x_n|) < \infty \right\}. \tag{1.1}$$

Later, Maddox [9] introduced and studied some strongly Cesàro summable spaces such as

$$w_0(f) = \{x = (x_k) \in w : t_n(x) \to 0\} \tag{1.2}$$

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and

$$w_\infty(f) = \left\{ x = (x_k) \in w : \sup_n t_n(x) < \infty \right\}, \tag{1.3}$$

where $t_n(x) = \frac{1}{n} \sum\limits_{k=1}^{n} f(|x_k|)$. Ozturk and Bilgin [12] further generalized $w_0(f)$ with respect to a bounded sequence $p = (p_k)$ of positive real numbers and studied the space

$$w_0(f, p) = \left\{ x = (x_k) \in w : \frac{1}{n} \sum_{k=1}^{n} [f(|x_k|)]^{p_k} \to 0 \right\}. \tag{1.4}$$

In an analogous way, sequence spaces using difference operator are also introduced by several authors, among them Kizmaz [7] was the first to introduce in this direction. For the spaces $\mathcal{X} = l_\infty, c$ and c_0, he defined and studied some Banach spaces $\mathcal{X}(\Delta) = \{x = (x_k) : \Delta x \in \mathcal{X}\}$, where $\Delta x = (x_k - x_{k+1})$. Et and Colak [6] replaced the first order difference operator by an m-th order difference operator in $\mathcal{X}(\Delta)$ and defined $\mathcal{X}(\Delta^m) = \{x = (x_k) : \Delta^m x \in \mathcal{X}\}$, which are BK- spaces. Subsequently, Sahinar [14] used both the concepts, viz. modulus function and difference operator to introduce a more general space

$$\mathcal{B}_g(p, f, q, s) = \left\{ x = (x_k) \in w(\mathcal{X}) : \sum_{k=1}^{\infty} \frac{1}{k^s} [f(q(\Delta x_k))]^{p_k} < \infty \right\}, \tag{1.5}$$

where $s \geq 0$ and $(\mathcal{X}, q)$ is a seminormed linear space over the set of complex numbers. Further generalizations in this direction are

$$l(\Delta^m, f, p, q, s) = \left\{ x = (x_k) \in w(\mathcal{X}) : \sum_{k=1}^{\infty} \frac{1}{k^s} [f(q(\Delta^m x_k))]^{p_k} < \infty \right\}, \tag{1.6}$$

given by Altin et al.[2] and

$$\mathcal{N}_p(\mathcal{E}_k, \Delta^m, f, s) = \left\{ x = (x_k) \in w(\mathcal{E}_k) : \left(\frac{1}{|v_k|^{\frac{s}{p_k}}} f(q_k(\Delta^m x_k)) \right) \in \mathcal{N}_p \right\}, \tag{1.7}$$

given by Srivastava and Kumar [15], where $v = (v_k)$ is such that $1 \leq |v_k| < \infty$. To study more about difference sequence space, one can follow [1], [16], [4], [11], [10] etc. Recently, many authors such as Diaz et al. [5], Baliarsingh [3] and others have studied some fractional difference operators. This motivated us to develop a class of sequence spaces $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$ by the use of the fractional difference operator $\Delta^{(a)}$ and a sequence of modulus functions $f = (f_k)$ that generalizes many sequence spaces.

2. Preliminaries

Before we proceed, let us recall some preliminaries, definitions and results.

Definition 2.1. [13] A function $f : [0, \infty) \to [0, \infty)$ is said to be a modulus function if it satisfies the following conditions:

(1) $f(x) = 0$ if and only if $x = 0$
(2) $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$

(3) f is an increasing function
(4) f is continuous from the right at 0.

Definition 2.2. (Paranorm): A paranorm g is a real function defined on a linear space X such that for all x, y in X and for all scalars β, it satisfies the following conditions:

(1) $g(\theta) = 0$, where θ is the zero element of X
(2) $g(-x) = g(x)$
(3) $g(x) \geq 0$
(4) $g(x+y) \leq g(x) + g(y)$ and
(5) If (β_n) is a sequence of scalars with $\beta_n \to 0$ as $n \to \infty$ and (x_n) is a sequence in X such that $g(x_n - x) \to 0$ as $n \to \infty$ for some $x \in X$, then $g(\beta_n x_n - \beta x) \to 0$ as $n \to \infty$.

Definition 2.3. A sequence space λ is called normal or solid if and only if it contains all such sequences $y = (y_k)$ corresponding to each of which there is a sequence $x = (x_k) \in \lambda$ such that $|y_k| \leq |x_k|$ for all non-negative integers k.

Definition 2.4. (fractional difference operator)[3] Let $x = (x_k) \in w$ and α be a real number, then the fractional difference operator $\Delta^{(\alpha)}$ is defined by

$$\Delta^{(\alpha)} x_k = \sum_{i=0}^{k} \frac{(-\alpha)_i}{i!} x_{k-i}, \tag{2.1}$$

where $(\alpha)_i$ denotes the Pochhammer symbol defined as:

$$(\alpha)_i = \begin{cases} 1, \text{ if } \alpha = 0 \text{ or } i = 0, \\ \alpha(\alpha+1)(\alpha+2)...(\alpha+i-1), \text{ otherwise.} \end{cases} \tag{2.2}$$

Lemma 2.5. [8] *For a complex number β, we have the inequalities $|\beta|^{p_k} \leq \max\left(1, |\beta|^H\right)$ and $|a_k + b_k|^{p_k} \leq C\big(|a_k|^{p_k} + |b_k|^{p_k}\big)$, where $H = \sup_k p_k < \infty$ and $C = \max\left(1, 2^{H-1}\right)$.*

Remark 2.6. Let f be a modulus function. Then for a non-negative integer n and for a real number $a \in [0, \infty)$, we have

(1) $f(nx) \leq n f(x)$.
(2) $f(ax) \leq (1 + \lfloor a \rfloor) f(x)$, where $\lfloor . \rfloor$ denotes the greatest integer function.

Throughout this article we assume $x_{-k} = 0$ for all non negative integer k i.e all such term which has negative suffix is considered to be zero.

3. The sequence space $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$

Let $(\mathcal{E}_k, q_k)$ be a sequence of seminormed spaces such that $\mathcal{E}_{k-1} \subseteq \mathcal{E}_k$ for all non-negative integers k. We define $w(\mathcal{E}_k) = \big\{x = (x_k) \in w : x_k \in \mathcal{E}_k$ *for all non-negative integers* $k\big\}$. We see that $w(\mathcal{E}_k)$ is a linear space with respect to the operations $x + y = (x_k + y_k)$ and $ax = (ax_k)$, where $a \in \mathbb{C}$. Let N be a normal sequence space, $f = (f_k)$ be a sequence of modulus functions, $A = (a_{nk})$ be a non-negative infinite matrix, i.e. $a_{nk} \geq 0$ for all non-negative integers, α be any real number and $p = (p_k)$ be a bounded sequence of positive real numbers such that $0 < \inf_k p_k \leq \sup_k p_k = H < \infty$. We introduce a space $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$ of sequences as follows:

$$\begin{aligned} &N\left[A, f, q, \Delta^{(\alpha)}, p\right] \\ &= \left\{ x = (x_k) \in w(\mathcal{E}_k) : \left(\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} x_k \right) \right) \right]^{p_k} \right)_n \in N \right\}. \end{aligned} \tag{3.1}$$

This space is a paranormed space with the paranorm g defined by

$$g(x) = \sup_n \left(\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} x_k \right) \right) \right]^{p_k} \right)^{\frac{1}{M}}, \tag{3.2}$$

where $M = \max\{1, H\}$. Moreover, this space is a complete paranormed space with respect to g under some suitable conditions.

3.1. **Special cases.** The sequence space (3.1) generalizes many known sequence spaces. Some of them are as follows:

- For $\mathcal{E}_k = \mathbb{C}, A = I$, the unit matrix of infinite order, $f = (f_k \equiv f), \alpha = 0, N = l_\infty, p = (1)$, the class $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$ reduces to $\mathcal{L}(f)$ (1.1).
- Choose $A = [a_{nk}]$ such that $a_{nk} = 1/n$ for $n \geq k$ and 0 otherwise, $\mathcal{E}_k = \mathbb{C}, f = (f_k \equiv f), \alpha = 0$. Now for $N = c_0$, the class $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$ reduces to $w_0(f,p)$ (1.4). Moreover, if we choose $p = (1)$ and take the space $N = c_0$ and l_∞, the class $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$ reduces to $w_0(f)$ (1.2) and $w_\infty(f)$ (1.3).
- If we choose $A = [a_{nk}]$ such that $a_{nk} = 1/k^s$ for all n, $f = (f_k \equiv f), q = (q_k \equiv q), \alpha = 1$ and $N = l_\infty$, the class $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$ reduces to $\mathcal{B}_g(p, f, q, s)$ (1.5). In this case if we take $\alpha = m$, then it reduces to $l(\Delta^m, f, p, q, s)$ (1.6).
- If we choose $A = [a_{nk}]$ such that $a_{nk} = |v_k|^{-(s/p_k)}$ for all $n = k$ and 0 otherwise, $f = (f_k \equiv f), q = (q_k \equiv q)$, and $\alpha = m$, the class $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$ reduces to $\mathcal{N}_p(\mathcal{E}_k, \Delta^m, f, s)$ (1.7).

3.2. **Results on the sequence space $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$.** We use Lemma 2.5 to find the following results. The constants C and H, which are used in these results, are same as in Lemma 2.5.

Lemma 3.1. (1) *Let $x = (x_k)$ and $y = (y_k)$ be two elements of $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$. Then,*

$$\begin{aligned} &\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} (x_k + y_k) \right) \right) \right]^{p_k} \\ &\leq C \left(\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} x_k \right) \right) \right]^{p_k} + \sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} y_k \right) \right) \right]^{p_k} \right) \end{aligned} \tag{3.3}$$

for all non-negative integer n.

(2) *Let a be an element of $\mathbb{C}$. Then,*

$$\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)}(a x_k) \right) \right) \right]^{p_k}$$
$$\leq \max \left\{ 1, (1 + \lfloor |a| \rfloor)^H \right\} \left(\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} x_k \right) \right) \right]^{p_k} \right) \tag{3.4}$$

for all non-negative integer n.

(3) *Let $f = (f_k)$ and $g = (g_k)$ be two sequences of modulus functions. Then,*

$$\sum_{k=0}^{\infty} a_{nk} \left[(f_k + g_k) \left(q_k \left(\Delta^{(\alpha)} x_k \right) \right) \right]^{p_k}$$
$$\leq C \left(\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} x_k \right) \right) \right]^{p_k} + \sum_{k=0}^{\infty} a_{nk} \left[g_k \left(q_k \left(\Delta^{(\alpha)} x_k \right) \right) \right]^{p_k} \right) \tag{3.5}$$

for all non-negative integer n.

(4) *Let $q = (q_k)$ and $q' = (q'_k)$ be two sequences of seminorms. Then,*

$$\sum_{k=0}^{\infty} a_{nk} \left[f_k \left((q_k + q'_k) \left(\Delta^{(\alpha)} x_k \right) \right) \right]^{p_k}$$
$$\leq C \left(\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} x_k \right) \right) \right]^{p_k} + \sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q'_k \left(\Delta^{(\alpha)} x_k \right) \right) \right]^{p_k} \right) \tag{3.6}$$

for all non-negative integer n.

Proof. Proof of this lemma is easy, so we omit it. □

Theorem 3.2. *The set $N \left[A, f, q, \Delta^{(\alpha)}, p \right]$ is a linear space over $\mathbb{C}$.*

Proof. Using part 1 of Lemma 3.1, we can easily prove this theorem. So we omit the proof. □

Theorem 3.3. *The sequence space $N \left[A, f, q, \Delta^{(\alpha)}, p \right]$ is a paranormed space with paranorm g defined by*

$$g(x) = \sup_n \left(\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} x_k \right) \right) \right]^{p_k} \right)^{\frac{1}{M}},$$

where $M = \max\{1, H\}$ and $H = \sup_k p_k < \infty$.

Proof. Let $x = (x_k)$ and $y = (y_k)$ be two elements of the space $N \left[A, f, q, \Delta^{(\alpha)}, p \right]$ and $\theta = (0, 0, 0, ...)$ be the zero of this space. Clearly, $g(x) \geq 0$, $g(\theta) = 0$ and $g(-x) = g(x)$. From Minkowski's inequality, we have

$$\left(\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)}(x_k + y_k) \right) \right) \right]^{p_k} \right)^{\frac{1}{M}}$$
$$\leq \left(\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} x_k \right) \right) \right]^{p_k} \right)^{\frac{1}{M}} + \left(\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} y_k \right) \right) \right]^{p_k} \right)^{\frac{1}{M}}. \tag{3.7}$$

Taking supremum of both sides of this inequality, we get $g(x+y) \leq g(x)+g(y)$. To prove g to be a jointly continuous function, it is enough to show that for a fixed $x=(x_k)$, $g(t_m x) \to 0$ as $t_m \to 0$. As $t_m \to 0$, there exists a natural number n_0 such that $|t_m| < 1$ for all $m \geq n_0$. Thus,

$$q_k\left(\Delta^{(\alpha)} t_m x_k\right) = |t_m| q_k\left(\Delta^{(\alpha)} x_k\right) < q_k\left(\Delta^{(\alpha)} x_k\right) \tag{3.8}$$

for all $m \geq n_0$. Consequently,

$$\sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)} t_m x_k\right)\right)\right]^{p_k} \leq \sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)} x_k\right)\right)\right]^{p_k} < \infty \tag{3.9}$$

for all $m \geq n_0$. Since the above sum is finite, there exists a natural number k_0 for each real number $\epsilon > 0$ such that

$$\sum_{k=k_0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)} t_m x_k\right)\right)\right]^{p_k} < \frac{\epsilon}{2}. \tag{3.10}$$

Now, we consider a function h such that

$$h(\mu) = \sum_{k=0}^{k_0} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)} \mu x_k\right)\right)\right]^{p_k}.$$

Clearly, h is a continuous function with $h(0)=0$. Then for each $\epsilon > 0$, there exists a real number $\delta > 0$ such that $|h(\mu)| < \frac{\epsilon}{2}$ whenever $|\mu| < \delta$. Since $t_m \to 0$, we can find a natural number n_1 such that $|t_m| < \delta$ for all $m \geq n_1$. Replacing μ by t_m, we get $h(t_m) < \frac{\epsilon}{2}$, i.e.

$$\sum_{k=0}^{k_0} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)} t_m x_k\right)\right)\right]^{p_k} < \frac{\epsilon}{2}. \tag{3.11}$$

Inequalities (3.10) and (3.11) imply that $g(t_m x) \to 0$ as $t_m \to 0$. Thus, g is a jointly continuous function. Hence, $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$ is a paranormed space with the paranorm g. □

Theorem 3.4. *Let $A=[a_{nk}]$ be a non-negative infinite matrix whose every column has at least one non-zero element. Let $(\mathcal{E}_k, q_k)$ be a sequence of complete seminormed spaces and $f=(f_k)$ be a sequence of modulus functions such that each f_k is strictly increasing. Then the sequence space $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$ is a complete paranormed space with respect to the paranorm g defined by*

$$g(x) = \sup_n \left(\sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)} x_k\right)\right)\right]^{p_k}\right)^{\frac{1}{M}},$$

where $M=\max\{1, H\}$ and $H=\sup_k p_k < \infty$.

Proof. We have already shown that the sequence space $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$ is a paranormed space with the paranorm g. It remains to prove that the space is complete with respect to g. For this let $(x^{(i)})_i$, where $x^{(i)}=(x_k^{(i)})_k$ be a Cauchy sequence in $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$. Then for every real number $\epsilon > 0$, there exists a natural number i_0 such that

$$g(x^{(i)} - x^{(j)}) < \epsilon \tag{3.12}$$

for all $i, j \geq i_0$. This implies that

$$\sup_n \left(\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} (x_k^{(i)} - x_k^{(j)}) \right) \right) \right]^{p_k} \right)^{\frac{1}{M}} < \epsilon \tag{3.13}$$

for all $i, j \geq i_0$. Then,

$$\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} (x_k^{(i)} - x_k^{(j)}) \right) \right) \right]^{p_k} < \epsilon^M \tag{3.14}$$

for all $i, j \geq i_0$ and for all n. Since each column of the matrix A has at least one non zero element, so we may suppose $a_{n_0 0}$ is a nonzero element in the first column. Then for $n = n_0$ and $k = 0$, we have

$$a_{n_0 0} \left[f_0 \left(q_0 \left(\Delta^{(\alpha)} (x_0^{(i)} - x_0^{(j)}) \right) \right) \right]^{p_0} < \epsilon^M \text{ for all } i, j \geq i_0.$$

Since $a_{n_0 0}$ and p_0 both are greater than zero, we can write

$$f_0 \left(q_0 \left(\Delta^{(\alpha)} (x_0^{(i)} - x_0^{(j)}) \right) \right) < \left(\frac{\epsilon^M}{a_{n_0 0}} \right)^{\frac{1}{p_0}} \text{ for all } i, j \geq i_0.$$

Since f_0 is a strictly increasing and continous function, we can find an interval $[0, c]$ where it is invertible. Then, we have

$$q_0 \left(\Delta^{(\alpha)} x_0^{(i)} - \Delta^{(\alpha)} x_0^{(j)} \right) < f_0^{-1} \left(\left(\frac{\epsilon^M}{a_{n_0 0}} \right)^{\frac{1}{p_0}} \right) = \epsilon' \text{ for all } i, j \geq i_0.$$

This shows that $(\Delta^{(\alpha)} x_0^{(i)}) = (x_0^{(i)})$ is a Cauchy sequence in $\mathcal{E}_0$ with respect to q_0. Since $\mathcal{E}_0$ is complete, $(x_0^{(i)})$ will converge to an element of $\mathcal{E}_0$, say x_0. If we repeat the above process for second column of the matrix A, then we find that the sequence $(\Delta^{(\alpha)} x_1^{(i)}) = (x_1^{(i)} - \alpha x_0^{(i)})$ is a Cauchy sequence. Also, $(x_1^{(i)}) = (\Delta^{(\alpha)} x_1^{(i)}) + \alpha (x_0^{(i)})$ is a Cauchy sequence, since it is a linear combination of two Cauchy sequences. In this way, we can prove that each sequence $(x_k^{(i)})$ is a Cauchy sequence and converges to x_k (say). Now, we claim that the sequence $(x^{(i)})$ converges to $x = (x_k)$ in $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$ with respect to g. Taking j tends to infinity in the Inequality (3.12), we get

$$g(x^{(i)} - x) < \epsilon \tag{3.15}$$

for all $i \geq i_0$. This shows that $(x^{(i)})$ converges to $x = (x_k)$. Now it remains to prove that $x = (x_k) \in N[A, f, q, \Delta^{(\alpha)}, p]$. We have,

$$\sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} x_k \right) \right) \right]^{p_k} = \sum_{k=0}^{\infty} a_{nk} \left[f_k \left(q_k \left(\Delta^{(\alpha)} \left(x_k - x_k^{(i_0)} + x_k^{(i_0)} \right) \right) \right) \right]^{p_k}.$$

From part 1. of Lemma (3.1), we can write

$$\begin{aligned}\sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k} \\ \leq C\left\{\sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}\left(x_k - x_k^{(i_0)}\right)\right)\right)\right]^{p_k} + \sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}(x_k^{(i_0)})\right)\right)\right]^{p_k}\right\}.\end{aligned} \tag{3.16}$$

Using inequality (3.15), we get

$$\sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k} < C\left\{\epsilon + \sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}(x_k^{(i_0)})\right)\right)\right]^{p_k}\right\}.$$

Since ϵ is arbitrary, we have

$$\sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k} < C\sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}(x_k^{(i_0)})\right)\right)\right]^{p_k}.$$

Since N is normal, the sequence $\left(\sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k}\right)_n \in N$. This implies that the sequence $x = (x_k) \in N\left[A, f, q, \Delta^{(\alpha)}, p\right]$. Thus the space $N\left[A, f, q, \Delta^{(\alpha)}, p\right]$ is a complete paranormed space. □

Theorem 3.5. *Let $f = (f_k)$ and $g = (g_k)$ be two sequences of modulus functions. Then*

$$N\left[A, f, q, \Delta^{(\alpha)}, p\right] \cap N\left[A, g, q, \Delta^{(\alpha)}, p\right] \subseteq N\left[A, f+g, q, \Delta^{(\alpha)}, p\right].$$

Proof. Let $x = (x_k) \in N\left[A, f, q, \Delta^{(\alpha)}, p\right] \cap N\left[A, g, q, \Delta^{(\alpha)}, p\right]$. Then the sequences $\left(\sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k}\right)_n$ and $\left(\sum_{k=0}^{\infty} a_{nk}\left[g_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k}\right)_n$ both are in N. Now, from Part 3. of Lemma 3.1, we can write

$$\begin{aligned}\sum_{k=0}^{\infty} a_{nk}\left[(f_k+g_k)\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k} \\ \leq C\left\{\sum a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k} + \sum a_{nk}\left[g_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k}\right\}.\end{aligned} \tag{3.17}$$

Since N is normal, $\left(\sum_{k=0}^{\infty} a_{nk}\left[(f_k+g_k)\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k}\right)_n \in N$. Then, $x = (x_k) \in N\left[A, f+g, q, \Delta^{(\alpha)}, p\right]$. Hence, the above inclusion relation holds. □

Theorem 3.6. *Let $q = (q_k)$ and $q' = (q'_k)$ be two sequences of seminorms. Suppose addition of two sequences of seminorms is defined as $q + q' = (q_k + q'_k)$. Then,*

$$N\left[A, f, q, \Delta^{(\alpha)}, p\right] \cap N\left[A, f, q', \Delta^{(\alpha)}, p\right] \subseteq N\left[A, f, q+q', \Delta^{(\alpha)}, p\right].$$

Proof. Proof of this theorem runs on the similar lines as that of the Theorem 3.5. So we omit the proof. □

Theorem 3.7. *If $q = (q_k)$ and $q' = (q'_k)$ are two sequences of seminorms such that q_k is stronger than q'_k for each k, then*

$$N\left[A, f, q, \Delta^{(\alpha)}, p\right] \subseteq N\left[A, f, q', \Delta^{(\alpha)}, p\right].$$

Proof. Let $x = (x_k) \in N\left[A, f, q, \Delta^{(\alpha)}, p\right]$. Then, $\left(\sum\limits_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k}\right)_n \in N$. Since each q_k is stronger than corresponding q'_k, we have a natural number $\mathcal{M}_k$ corresponding to each non-negative integer k such that $q'_k(t) \leq \mathcal{M}_k q_k(t)$. Let $\mathcal{M} = \max\{\mathcal{M}_k\}$. Then, $q'_k(t) \leq \mathcal{M} q_k(t)$ for all non-negative integers k. Consequently, $q'_k\left(\Delta^{(\alpha)}x_k\right) \leq \mathcal{M} q_k\left(\Delta^{(\alpha)}x_k\right)$. Using Remark 2.6 and Lemma 2.5, we get

$$\sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q'_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k} \leq \max\left\{1, \mathcal{M}^H\right\} \sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k}. \tag{3.18}$$

As the sequence space N is normal, $\left(\sum\limits_{k=0}^{\infty} a_{nk}\left[f_k\left(q'_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k}\right)_n \in N$, i.e. $x = (x_k) \in N\left[A, f, q', \Delta^{(\alpha)}, p\right]$. Hence, the inclusion holds. □

Corollary 3.8. *If $q = (q_k)$ and $q' = (q'_k)$ are two sequences of seminorms such that q_k is equivalent to q'_k for each k, then*

$$N\left[A, f, q, \Delta^{(\alpha)}, p\right] = N\left[A, f, q', \Delta^{(\alpha)}, p\right].$$

Theorem 3.9. *Let $f = (f_k)$ and $f' = (f'_k)$ be two sequences of modulus functions such that $f_k(1)$ is finite for each k. Let $A = [a_{nk}]$ be a non-negative infinite matrix such that $\left(\sum\limits_{k=0}^{\infty} a_{nk}\right)_n \in N$. Then,*

$$N\left[A, f', q, \Delta^{(\alpha)}, p\right] \subseteq N\left[A, f \circ f', q, \Delta^{(\alpha)}, p\right],$$

where the composition of two sequence of functions is defined as $f \circ f' = (f_k \circ f'_k)$.

Proof. Let $x = (x_k) \in N[A, f', q, \Delta^{(\alpha)}, p]$. Then $(\sum\limits_{k=0}^{\infty} a_{nk}[f'_k(q_k(\Delta^{(\alpha)}x_k))]^{p_k})_n \in N$. Since each f_k is continuous and $f_k(0) = 0$ for each k, we can choose $\delta \in (0, 1)$ corresponding to an arbitrary $\epsilon > 0$ such that $f_k(t) < \epsilon$ for $0 \leq t \leq \delta$. Now, we take $t_k = f'_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)$ and consider

$$\sum_{k=1}^{\infty} a_{nk}[f_k(t_k)]^{p_k} = \sum_{1} a_{nk}[f_k(t_k)]^{p_k} + \sum_{2} a_{nk}[f_k(t_k)]^{p_k}, \tag{3.19}$$

where the first summation is over $t_k \leq \delta$ and the second is over $t_k > \delta$. For $t_k \leq \delta$, we have $f_k(t_k) < \epsilon$ and so $\sum\limits_{1} a_{nk}[f_k(t_k)]^{p_k} < \sum\limits_{1} a_{nk}[\epsilon]^{p_k}$. Then from Lemma 2.5, we get

$$\sum_{1} a_{nk}[f_k(t_k)]^{p_k} < \max\left(1, \epsilon^H\right) \sum_{1} a_{nk} \leq \max\left(1, \epsilon^H\right) \sum_{k=0}^{\infty} a_{nk}. \tag{3.20}$$

For $t_k > \delta$, we have $t_k < \frac{t_k}{\delta}$. So, from Remark 2.6, we get

$$f_k(t_k) < f_k\left(\frac{t_k}{\delta}\right) \le \left(1 + \left\lfloor \frac{t_k}{\delta} \right\rfloor\right) f_k(1) \le 2f_k(1)\frac{t_k}{\delta}. \tag{3.21}$$

Let $\eta = \max_k(f_k(1))$, then $f_k(t_k) < 2\eta\frac{t_k}{\delta}$. Using Lemma 2.5, we find that

$$\sum_2 a_{nk}[f_k(t_k)]^{p_k} \le \max\left(1, \left(\frac{2\eta}{\delta}\right)^H\right)\sum_{k=0}^{\infty} a_{nk}[t_k]^{p_k}. \tag{3.22}$$

By Inequalities (3.19), (3.20) and (3.22), we have

$$\sum_{k=1}^{\infty} a_{nk}[f_k(t_k)]^{p_k} \le \max\left(1, \epsilon^H\right)\sum_{k=0}^{\infty} a_{nk} + \max\left(1, \left(\frac{2\eta}{\delta}\right)^H\right)\sum_{k=0}^{\infty} a_{nk}[t_k]^{p_k}.$$

Since N is normal,

$$\left(\sum_{k=1}^{\infty} a_{nk}[f_k(t_k)]^{p_k}\right)_n = \left(\sum_{k=1}^{\infty} a_{nk}\left[f_k \circ f_k'\left(q_k\left(\Delta^{(\alpha)} x_k\right)\right)\right]^{p_k}\right)_n \in N.$$

Then, $x = (x_k)$ belongs to $N\left[A, f \circ f', q, \Delta^{(\alpha)}, p\right]$. Hence, the inclusion holds. $\square$

Theorem 3.10. *Let $f = (f_k)$ be a sequence of modulus functions such that $f_k(t) \le f_{k-1}(t)$ for all $t \in [0, \infty)$ and $q = (q_k)$ be a sequence of seminorms such that $q_k(t) \le q_{k-1}(t)$ for all t. Suppose $A = [a_{nk}]$ is a non-negative infinite matrix such that $a_{nk} \le a_{n(k-1)}$ for all non-negative integers n and k and suppose $p = (p_k \equiv p)$ is a constant sequence of positive real number. Then $N\left[A, f, q, \Delta^{(\alpha-1)}, p\right] \subset N\left[A, f, q, \Delta^{(\alpha)}, p\right]$ and the inclusion is strict, in general.*

Proof. Let $x = (x_k) \in N[A, f, q, \Delta^{(\alpha-1)}, p]$. Then, $(\sum_{k=0}^{\infty} a_{nk}\ [f_k\ (q_k\ (\Delta^{(\alpha-1)}\ x_k))\ ]^p\ )_n \in N$. As $f_k(t) \le f_{k-1}(t)$, $q_k(t) \le q_{k-1}(t)$ and $a_{nk} \le a_{n(k-1)}$, we have

$$\begin{aligned}\sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha-1)} x_{k-1}\right)\right)\right]^p &\le \sum_{k=0}^{\infty} a_{n(k-1)}\left[f_{k-1}\left(q_{k-1}\left(\Delta^{(\alpha-1)} x_{k-1}\right)\right)\right]^p \\ &= \sum_{k=0}^{\infty} a_{n(k)}\left[f_k\left(q_k\left(\Delta^{(\alpha-1)} x_k\right)\right)\right]^p.\end{aligned}$$

Since N is normal, $\left(\sum\limits_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha-1)} x_{k-1}\right)\right)\right]^p\right)_n \in N$. Now,

$$\begin{aligned}&\sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)} x_k\right)\right)\right]^p \\ &= \sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha-1)}(x_k - x_{k-1})\right)\right)\right]^p \\ &\le C\left\{\sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha-1)} x_k\right)\right)\right]^p + \sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha-1)} x_{k-1}\right)\right)\right]^p\right\}.\end{aligned}$$

Again N is normal implies $\left(\sum\limits_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k}\right)_n \in N$. This implies that $x=(x_k)\in N\left[A,f,q,\Delta^{(\alpha)},p\right]$. Thus the inclusion holds. $\square$

To show strictness of the above inclusion relation, we consider the following example in which we show that there exists a sequence $x=(x_k)$ in the sequence space $l_\infty\left[A,f,q,\Delta^{(\alpha)},p\right]$, but not in the sequence space $l_\infty\left[A,f,q,\Delta^{(\alpha-1)},p\right]$.

Example 3.11. Let $A=I$ be the identity matrix of infinite order. Consider the terms of the sequences $f=(f_k)$, $q=(q_k)$, $p=(p_k)$ and $x=(x_k)$ as $f_k(x)=x$, $q_k(x)=|x|$, $p_k=1$ and $x_k=1$ for all non-negative integer k respectively. Then,

$$\begin{aligned}\sup_n \sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k} &= \sup_n |1-\frac{(\alpha)}{1!}+\frac{(\alpha)(\alpha-1)}{2!}+\dots\\ &\quad +(-1)^n\frac{(\alpha)(\alpha-1)...(\alpha-(n-1))}{n!}|\\ &= \sup_n |T_n(\alpha)|(say).\end{aligned}$$

If we take $\alpha=\frac{1}{2}$, then $\sup\limits_n |T_n(\alpha)|<\infty$, whereas $\sup\limits_n |T_n(\alpha-1)|=\infty$. So, $x=(x_k)\in l_\infty\left[I,f,q,\Delta^{(\frac{1}{2})},p\right]$, but $x=(x_k)\notin l_\infty\left[I,f,q,\Delta^{(\frac{-1}{2})},p\right]$. Hence the inclusion relation is strict in general.

Theorem 3.12. *The sequence space $N\left[A,f,q,\Delta^{(\alpha)},p\right]$ is not solid in general.*

We consider the following example to show this result.

Example 3.13. As in the previous example, suppose $A=I$, $f_k(x)=x$, $q_k(x)=|x|$ and $p_k=1$ for all non-negative integer k. Let $\alpha=-\frac{1}{2}$. Consider the sequences $x=(x_k)=((-1)^k)$ and $y=(y_k)=(1)$. Clearly, $|y_k|\le|x_k|$ for each k. Now,

$$\begin{aligned}\sup_n \sum_{k=0}^{\infty} a_{nk}\left[f_k\left(q_k\left(\Delta^{(\alpha)}x_k\right)\right)\right]^{p_k} &= \sup_n |1+\frac{(\alpha)}{1!}+\frac{(\alpha)(\alpha-1)}{2!}+\dots\\ &\quad +\frac{(\alpha)(\alpha-1)\dots(\alpha-(n-1))}{n!}|\\ &= \sup_n |T'_n|\ (say).\end{aligned}$$

The sequence (T'_n) converges to $(1+1)^\alpha=(1+1)^{-\frac{1}{2}}=\frac{1}{\sqrt{2}}$. Therefore, $\sup\limits_n |T'_n|<\infty$. Now,

$$\begin{aligned}\sup_n \sum_{k=0}^{\infty} a_{nk}[f_k(q_k(\Delta^{(\alpha)}y_k))]^{p_k} &= \sup_n |1-\frac{(\alpha)}{1!}+\frac{(\alpha)(\alpha-1)}{2!}+\dots\\ &\quad +(-1)^n\frac{(\alpha)(\alpha-1)...(\alpha-(n-1))}{n!}|\\ &= \sup_n |(T''_n)|,\end{aligned}$$

where $T''_n \to (1+(-1))^{\alpha} = (1+(-1))^{-\frac{1}{2}} = \infty$. Therefore, $\sup\limits_n |T''_n| = \infty$. So, $x = (x_k) \in l_\infty\left[A, f, q, \Delta^{(\alpha)}, p\right]$ and $y = (y_k) \notin l_\infty\left[A, f, q, \Delta^{(\alpha)}, p\right]$, whereas $|y_k| \le |x_k|$ for each k, i.e. the sequence space $l_\infty\left[A, f, q, \Delta^{(\alpha)}, p\right]$ is not solid.

Theorem 3.14. *Let us denote the infinite matrix of ones by J, i.e. all entries of J are 1. If $0 < \inf\limits_k t_k < t_k \le r_k < \sup\limits_k r_k < \infty$ for all non-negative integer k, then the inclusion $l_\infty\left[J, f, q, \Delta^{(\alpha)}, t\right] \subseteq l_\infty\left[J, f, q, \Delta^{(\alpha)}, r\right]$ holds.*

Proof. Let $x = (x_k) \in l_\infty\left[J, f, q, \Delta^{(\alpha)}, t\right]$. This implies that

$$\sup_n \sum_{k=0}^{\infty} \left[f_k\left(q_k\left(\Delta^{(\alpha)} x_k\right)\right)\right]^{t_k} = \sum_{k=0}^{\infty} \left[f_k\left(q_k\left(\Delta^{(\alpha)} x_k\right)\right)\right]^{t_k} < \infty.$$

Then for sufficiently large k, say k_0, we have

$$\left[f_k\left(q_k\left(\Delta^{(\alpha)} x_k\right)\right)\right]^{t_k} \le 1 \tag{3.23}$$

for all $k \ge k_0$. Consequently,

$$\left\{\left[f_k\left(q_k\left(\Delta^{(\alpha)} x_k\right)\right)\right]^{t_k}\right\}^{\frac{r_k}{t_k}} \le \left[f_k\left(q_k\left(\Delta^{(\alpha)} x_k\right)\right)\right]^{t_k}.$$

for all $k \ge k_0$. Taking summation from k_0 to ∞ both side, we get

$$\sum_{k=k_0}^{\infty} \left[f_k\left(q_k\left(\Delta^{(\alpha)} x_k\right)\right)\right]^{r_k} \le \sum_{k=k_0}^{\infty} \left[f_k\left(q_k\left(\Delta^{(\alpha)} x_k\right)\right)\right]^{t_k} < \infty,$$

This implies that $\sup\limits_n \sum\limits_{k=0}^{\infty} \left[f_k\left(q_k\left(\Delta^{(\alpha)} x_k\right)\right)\right]^{r_k} < \infty$. So, $x = (x_k)$ belongs to $l_\infty\left[J, f, q, \Delta^{(\alpha)}, r\right]$. Thus, the inclusion relation holds. $\square$

4. Conclusions

The purpose of this article was to give an insight to the concepts of fractional difference operator and modulus function by defining a new class of sequence spaces and studying properties of the spaces of this class.

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