Khayyam J. Math. 5 (2019), no. 1, 21-39
DOI: 10.22034/kjm.2018.73593

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

THE APPROXIMATE SOLUTIONS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS BY USING MODIFIED ADOMIAN DECOMPOSITION METHOD

AHMED A. HAMOUD[1*], KIRTIWANT P. GHADLE[2] AND SHAKIR M. ATSHAN[3]

Communicated by J. Brzdęk

Abstract. The main object of the present paper is to study the behavior of the approximated solutions of the Caputo fractional Volterra–Fredholm integro-differential equations by using modified Adomian decomposition method. Moreover, we discuss some new existence, uniqueness, and convergence results. Finally, an example is included to demonstrate the validity and applicability of the proposed technique.

1. Introduction

In this paper, we consider the Caputo fractional Volterra–Fredholm integro differential equation of the form:

$$^cD^{\alpha}u(x) = a(x)u(x) + g(x) + \int_0^x K_1(x,t)F_1(u(t))dt + \int_0^1 K_2(x,t)F_2(u(t))dt, \tag{1.1}$$

with the initial condition

$$u(0) = u_0, \tag{1.2}$$

where $^cD^{\alpha}$ is Caputo's fractional derivative, $0 < \alpha \leq 1$, and $u : J \longrightarrow \mathbb{R}$, where $J = [0, 1]$ is the continuous function which has to be determined, $g : J \longrightarrow \mathbb{R}$ and $K_i : J \times J \longrightarrow \mathbb{R}, i = 1, 2,$ are continuous functions. $F_i : \mathbb{R} \longrightarrow \mathbb{R}, i = 1, 2,$ are Lipschitz continuous functions.

Date: Received: 26 June 2018; Revised: 8 September 2018; Accepted: 29 September 2018.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 49M27; Secondary 45J05, 26A33.
Key words and phrases. Modified Adomian decomposition method, Caputo fractional derivative, fractional Volterra–Fredholm integro-differential equation, approximate solution.

An application of fractional derivatives was first given in 1823 by Abel [2] who applied the fractional calculus in the solution of an integral equation that arises in the formulation of the Tautochrone problem. The fractional integro-differential equations have attracted much more interest of mathematicians and physicists, which provides an efficiency for the description of many practical dynamical arising in engineering and scientific disciplines such as, physics, biology, electrochemistry, chemistry, economy, electromagnetic, control theory, and viscoelasticity [5, 7, 8, 9, 10, 13, 17, 18, 23].

The idea of ADM is originally emerged in a pioneering paper by Adomian [3]. Researchers who made most significant contributions in the applications and developments of ADM are Rach [19], Wazwaz [21], Abbaoui [1], among others. The modified decomposition method was introduced by Wazwaz [21]. In recent years, many authors focus on the development of numerical and analytical techniques for fractional integro-differential equations. For instance, we can remember the following works. Al-Samadi and Gumah [6] applied the homotopy analysis method for fractional SEIR epidemic model, Zurigat et al. [26] applied HAM for system of fractional integro-differential equations. Yang and Hou [23] applied the Laplace decomposition method to solve the fractional integro-differential equations, Mittal and Nigam [18] applied the Adomian decomposition method to approximate solutions for fractional integro-differential equations, and Ma and Huang [17] applied hybrid collocation method to study integro-differential equations of fractional order. Moreover, properties of the fractional integro-differential equations have been studied by several authors [4, 6, 12, 14, 22, 24, 26].

The main objective of the present paper is to study the behavior of the solution that can be formally determined by analytical approximated method as the modified Adomian decomposition method. Moreover, we proved the existence, uniqueness results, and convergence of the solution of the Caputo fractional Volterra–Fredholm integro-differential equation.

The rest of the paper is organized as follows: In Section 2, some preliminaries and basic definitions related to fractional calculus are recalled. In Section 3, modified Adomian decomposition method is constructed for solving the Caputo fractional Volterra–Fredholm integro-differential equations. In Section 4, the existence and uniqueness results and convergence of the solutions have been proved. In Section 5, the analytical example is presented to illustrate the accuracy of this method. Finally, we will give a report on our paper and a brief conclusion is given in Section 6.

2. Preliminaries

The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The most frequently used definitions of the fractional calculus involves the Riemann–Liouville fractional derivative, Caputo derivative [12, 15, 16, 20, 25].

Definition 2.1 (Riemann–Liouville fractional integral, [15]). The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function f is defined as

$$\begin{aligned} J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad x>0, \quad \alpha \in \mathbb{R}^+, \\ J^0 f(x) &= f(x), \end{aligned}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.2 (Caputo fractional derivative, [15]). The fractional derivative of $f(x)$ in the Caputo sense is defined by

$$\begin{aligned} {}^cD_x^\alpha f(x) &= J^{m-\alpha} D^m f(x) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \frac{d^m f(t)}{dt^m} dt, & m-1<\alpha<m, \\ \frac{d^m f(x)}{dx^m}, & \alpha = m, \quad m \in N, \end{cases} \end{aligned} \tag{2.1}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α will be considered.

Hence, we have the following properties:

(1) $J^\alpha J^v f = J^{\alpha+v} f, \quad \alpha, v > 0.$
(2) $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha},$
(3) $D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \quad \alpha > 0, \quad \beta > -1, \quad x > 0.$
(4) $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0, \quad m-1 < \alpha \le m.$

Definition 2.3 (Riemann–Liouville fractional derivative, [15]). The Riemann–Liouville fractional derivative of order $\alpha > 0$ is normally defined as

$$D^\alpha f(x) = D^m J^{m-\alpha} f(x), \qquad m-1 < \alpha \le m, \quad m \in \mathbb{N}.$$

Theorem 2.4 (Banach contraction principle, [25]). *Let (X, d) be a complete metric space; then each contraction mapping $T : X \longrightarrow X$ has a unique fixed point x of T in X; that is, $Tx = x$.*

Theorem 2.5 (Schauder's fixed point theorem, [15]). *Let X be a Banach space, and let A be a convex, closed subset of X. If $T : A \longrightarrow A$ is the map such that the set $\{Tu : u \in A\}$ is relatively compact in X (or T is continuous and completely continuous), then T has at least one fixed point $u^* \in A : Tu^* = u^*$.*

3. Modified Adomian decomposition method

Consider the equation (1.1) with the initial condition (1.2), where ${}^cD^\alpha$ is the operator defined as (2.1). Operating with J^α on both sides of the equation (1.1), we get

$$\begin{aligned} u(x) = u_0 + J^\alpha\Big(a(x)u(x) + g(x) &+ \int_0^x K_1(x,t) F_1(u(t))dt \\ &+ \int_0^1 K_2(x,t) F_2(u(t))dt \Big). \end{aligned}$$

Adomian's method defines the solution $u(x)$ by the series

$$u=\sum_{n=0}^{\infty}u_n, \tag{3.1}$$

and the nonlinear function F is decomposed as

$$F_1=\sum_{n=0}^{\infty}A_n, \qquad F_2=\sum_{n=0}^{\infty}B_n, \tag{3.2}$$

where A_n, B_n are the Adomian polynomials given by

$$A_n=\frac{1}{n!}\Big[\frac{d^n}{d\phi^n}(F_1\sum_{i=0}^{n}\phi^i u_i)\Big]_{\phi=0},$$
$$B_n=\frac{1}{n!}\Big[\frac{d^n}{d\phi^n}(F_2\sum_{i=0}^{n}\phi^i u_i)\Big]_{\phi=0}.$$

The Adomian polynomials were introduced in [1, 3, 7, 11] as:

$$\begin{aligned}
A_0 &= F_1(u_0),\\
A_1 &= u_1F_1^{'}(u_0),\\
A_2 &= u_2F_1^{'}(u_0)+\frac{1}{2}u_1^2F_1^{''}(u_0),\\
A_3 &= u_3F_1^{'}(u_0)+u_1u_2F_1^{''}(u_0)+\frac{1}{3}u_1^3F_1^{'''}(u_0),\\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
B_0 &= F_2(u_0),\\
B_1 &= u_1F_2^{'}(u_0),\\
B_2 &= u_2F_2^{'}(u_0)+\frac{1}{2}u_1^2F_2^{''}(u_0),\\
B_3 &= u_3F_2^{'}(u_0)+u_1u_2F_2^{''}(u_0)+\frac{1}{3}u_1^3F_2^{'''}(u_0),\\
&\vdots
\end{aligned}$$

The components $u_0, u_1, u_2, \ldots$ are determined recursively by

$$\begin{aligned}
u_0(x) &= u_0+J^{\alpha}\Big(g(x)\Big),\\
u_{k+1}(x) &= J^{\alpha}\Big(a(x)u_k(x)\Big)+J^{\alpha}\left(\int_0^x K_1(x,t)A_k dt+\int_0^1 K_2(x,t)B_k dt\right).
\end{aligned}$$

Having defined the components $u_0, u_1, u_2, \ldots,$ the solution u in a series form defined by (3.1) follows immediately. It is important to note that the decomposition method suggests that the 0th component u_0 is defined by the initial conditions and the function $g(x)$ is as described above. The other components namely $u_1, u_2, \ldots,$ are derived recurrently.

The modified decomposition method was introduced by Wazwaz [21]. This method is based on the assumption that the function $J^{\alpha}g(x) = R(x)$ can be divided into two parts, namely, $R_1(x)$ and $R_2(x)$. Under this assumption we set

$$R(x) = R_1(x) + R_2(x).$$

We apply this decomposition when the function $R(x)$ consists of several parts and can be decomposed into two different parts [1, 3, 10, 21]. In this case, $R(x)$ is usually a summation of a polynomial and trigonometric or transcendental functions. A proper choice for the part R_1 is important. For the method to be more efficient, we select R_1 as one term of $R(x)$ or at least a number of terms if possible and R_2 consists of the remaining terms of $R(x)$. In comparison with the standard decomposition method, the MADM minimizes the size of calculations and the cost of computational operations in the algorithm. Both standard and modified decomposition methods are reliable for solving linear or nonlinear problems such as Volterra–Fredholm integro-differential equations, but in order to decrease the complexity of the algorithm and simplify the calculations, we prefer to use the MADM. The MADM will accelerate the rapid convergence of the series solution in comparison with the standard Adomian decomposition method. The modified technique may give the exact solution for equations without the necessity to find the Adomian polynomials. We refer the reader to [21] for more details about the MADM. Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part R_1 is assigned to the component u_0, whereas the remaining part R_2 is combined with the other terms to define u_1. Consequently, the following modified recursive relation was developed:

$$\begin{aligned}
u_0(x) &= u_0 + R_1(x), \\
u_1(x) &= R_2(x) + J^{\alpha}\Big(a(x)u_0(x)\Big) + J^{\alpha}\Big(\int_0^x K_1(x,t)A_0dt + \int_0^1 K_2(x,t)B_0dt\Big). \\
&\vdots \\
u_{k+1}(x) &= J^{\alpha}\Big(a(x)u_k(x)\Big) + J^{\alpha}\left(\int_0^x K_1(x,t)A_kdt + \int_0^1 K_2(x,t)B_kdt\right), \; k \geq 1.
\end{aligned}$$

4. Main Results

In this section, we shall give an existence and uniqueness results of equation (1.1), with the initial condition (1.2) and prove it. Before starting and proving the main results, we introduce the following hypotheses:

(H1): There exist two constants $L_{F_1}, L_{F_2} > 0$ such that, for any $u_1, u_2 \in C(J,\mathbb{R})$

$$|F_1(u_1(x)) - F_1(u_2(x))| \leq L_{F_1} |u_1 - u_2|$$

and

$$|F_2(u_1(x)) - F_2(u_2(x))| \leq L_{F_2} |u_1 - u_2| .$$

(H2): There exist two functions $K_1^*, K_2^* \in C(D,\mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R}\times\mathbb{R} : 0 \leq t \leq x \leq 1\}$ such that

$$K_1^* = \sup_{x\in[0,1]} \int_0^x |K_1(x,t)|\, dt < \infty, \; K_2^* = \sup_{x\in[0,1]} \int_0^1 |K_2(x,t)|\, dt < \infty.$$

(H3): The two functions $a, g : J \to \mathbb{R}$ are continuous.

Lemma 4.1. *If $u_0(x) \in C(J, \mathbb{R})$, then $u(x) \in C(J, \mathbb{R}^+)$ is a solution of the problem (1.1) − (1.2) if and only if u satisfies*

$$\begin{aligned} u(x) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} a(s)u(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \Big(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau \\ &\qquad + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau \Big) ds \end{aligned}$$

for $x \in J$.

Our first result is based on Schauder's fixed point theorem for studying the existence of the solutions.

Theorem 4.2. *Assume that F_1, F_2 are continuous functions and* (H2), (H3) *hold. If*

$$\frac{\|a\|_\infty}{\Gamma(\alpha+1)} < 1, \tag{4.1}$$

then there exists at least a solution $u(x) \in C(J, \mathbb{R})$ to problem (1.1)–(1.2).

Proof. Let the operator $T : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ be defined by

$$\begin{aligned} (Tu)(x) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} a(s)u(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \Big(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau \\ &\qquad + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau \Big) ds, \end{aligned}$$

First, we prove that the operator T is completely continuous.

(**1**) We show that T is continuous.

Let u_n be a sequence such that $u_n \to u$ in $C(J,\mathbb{R})$. Then for each $u_n, u \in C(J,\mathbb{R})$ and for any $x \in J$, we have

$$\begin{aligned}
|(Tu_n)(x) & -(Tu)(x)| \\
& \leq \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|a(s)|\,|u_n(s)-u(s)|\,ds \\
& +\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\Big(\int_0^s |K_1(s,\tau)|\,|F_1(u_n(\tau))-F_1(u(\tau))|\,d\tau \\
& \qquad +\int_0^1 |K_2(s,\tau)|\,|F_2(u_n(\tau))-F_2(u(\tau))|\,d\tau\Big)ds \\
& \leq \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\sup_{s\in J}|a(s)|\sup_{s\in J}|u_n(s)-u(s)|\,ds \\
& +\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} \\
& \qquad \Big(\sup_{s,\tau\in J}\int_0^\tau |K_1(s,\tau)|\sup_{\tau\in J}|F_1(u_n(\tau))-F_1(u(\tau))|\,d\tau \\
& \qquad +\sup_{s,\tau\in J}\int_0^1 |K_2(s,\tau)|\sup_{\tau\in J}|F_2(u_n(\tau))-F_2(u(\tau))|\,d\tau\Big)ds \\
& \leq \|a\|_\infty\|u_n(.)-u(.)\|_\infty\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}ds \\
& +K_1^*\|F_1(u_n(.))-F_1(u(.))\|_\infty\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}ds \\
& +K_2^*\|F_2(u_n(.))-F_2(u(.))\|_\infty\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}ds.
\end{aligned}$$

Since $\int_0^x (x-s)^{\alpha-1}ds$ is bounded, $\lim_{n\to\infty} u_n(x) = u(x)$, and F_1, F_2 are continuous functions, we conclude that $\|Tu_n - Tu\|_\infty \to 0$ as $n \to \infty$, and thus, T is continuous on $C(J,\mathbb{R})$.

(**2**) We verify that T maps bounded sets into bounded sets in $C(J,\mathbb{R})$.

Indeed, we just show that, for any $\lambda > 0$, there exists a positive constant ℓ such that for each $u \in \mathbb{B}_\lambda = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \leq \lambda\}$, one has $\|Tu\|_\infty \leq \ell$.

Let $\mu_1 = \sup_{(u)\in J\times[0,\lambda]} F_1(u(x)) + 1$, and let $\mu_2 = \sup_{(u)\in J\times[0,\lambda]} F_2(u(x)) + 1$. Also for any $u \in \mathbb{B}_r$ and for each $x \in J$, we have

$$\begin{aligned}
|(Tu)(x)| &= |u_0| + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|a(s)|\,|u(s)|\,ds \\
& +\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|g(s)|\,ds \\
& +\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\Big(\int_0^s |K_1(s,\tau)|\,|F_1(u(\tau))|\,d\tau \\
& \qquad +\int_0^1 |K_2(s,\tau)|\,|F_2(u(\tau))|\,d\tau\Big)ds
\end{aligned}$$

$$\begin{aligned}
&\leq |u_0| + \|u\|_\infty \|a\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \|g\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{K_1^*\mu_1 x^\alpha}{\Gamma(\alpha+1)} + \frac{K_2^*\mu_2 x^\alpha}{\Gamma(\alpha+1)} \\
&\leq \left(|u_0| + \frac{\|a\|_\infty \lambda + \|g\|_\infty + K_1^*\mu_1 + K_2^*\mu_2}{\Gamma(\alpha+1)}\right) \\
&:= \ell.
\end{aligned}$$

Therefore, $\|Tu\| \leq \ell$ for every $u \in \mathbb{B}_r$, which implies that $T\mathbb{B}_r \subset \mathbb{B}_\ell$.

(**3**) We examine that T maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$.

Let $\mathbb{B}_\lambda$ be defined as in (**2**), and for each $u \in \mathbb{B}_\lambda$, $x_1, x_2 \in [0,1]$ with $x_1 < x_2$, we have

$$\begin{aligned}
|(Tu)(x_2) - (Tu)(x_1)| &\\
&\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{x_2} (x_2 - s)^{\alpha-1} a(s)u(s)ds - \int_0^{x_1} (x_1 - s)^{\alpha-1} a(s)u(s)ds \right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left| \int_0^{x_2} (x_2 - s)^{\alpha-1} g(s)ds - \int_0^{x_1} (x_1 - s)^{\alpha-1} g(s)ds \right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \Big| \int_0^{x_2} (x_2 - s)^{\alpha-1} \\
&\qquad \left(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau \right) ds \\
&\qquad - \int_0^{x_1} (x_1 - s)^{\alpha-1} \left(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau \right. \\
&\qquad \quad \left. + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau \right) ds \Big| \\
&= \frac{1}{\Gamma(\alpha)} \Big| \int_0^{x_2} (x_2 - s)^{\alpha-1} a(s)u(s)ds - \int_0^{x_1} (x_2 - s)^{\alpha-1} a(s)u(s)ds \\
&\quad + \int_0^{x_1} (x_2 - s)^{\alpha-1} a(s)u(s)ds - \int_0^{x_1} (x_1 - s)^{\alpha-1} a(s)u(s)ds \Big| \\
&\quad + \frac{1}{\Gamma(\alpha)} \Big| \int_0^{x_2} (x_2 - s)^{\alpha-1} g(s)ds - \int_0^{x_1} (x_2 - s)^{\alpha-1} g(s)ds \\
&\quad + \int_0^{x_1} (x_2 - s)^{\alpha-1} g(s)ds - \int_0^{x_1} (x_1 - s)^{\alpha-1} g(s)ds \Big| \\
&\quad + \frac{1}{\Gamma(\alpha)} \Big| \int_0^{x_2} (x_2 - s)^{\alpha-1} \\
&\qquad \left(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau \right) ds
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^{x_1} (x_2 - s)^{\alpha-1} \left(\int_0^s K_1(s,\tau) F_1(u(\tau)) d\tau \right.\\
&\left. + \int_0^1 K_2(s,\tau) F_2(u(\tau)) d\tau \right) ds\\
&+ \int_0^{x_1} (x_2 - s)^{\alpha-1} \left(\int_0^s K_1(s,\tau) F_1(u(\tau)) d\tau \right.\\
&\left. + \int_0^1 K_2(s,\tau) F_2(u(\tau)) d\tau \right) ds\\
&- \int_0^{x_1} (x_1 - s)^{\alpha-1} \left(\int_0^s K_1(s,\tau) F_1(u(\tau)) d\tau \right.\\
&\left. + \int_0^1 K_2(s,\tau) F_2(u(\tau)) d\tau \right) ds \Big| .
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
|(Tu)(x_2) - & (Tu)(x_1)|\\
& \leq \frac{1}{\Gamma(\alpha)} \Big(\int_{x_1}^{x_2} (x_2 - s)^{\alpha-1} |a(s)| \, |u(s)| \, ds\\
& \quad + \int_0^{x_1} (x_1 - s)^{\alpha-1} - (x_2 - s)^{\alpha-1} |a(s)| \, |u(s)| \, ds \Big)\\
& \quad + \frac{1}{\Gamma(\alpha)} \Big(\int_{x_1}^{x_2} (x_2 - s)^{\alpha-1} |g(s)| \, ds\\
& \quad + \int_0^{x_1} (x_1 - s)^{\alpha-1} - (x_2 - s)^{\alpha-1} |g(s)| \, ds \Big)\\
& \quad + \frac{1}{\Gamma(\alpha)} \Big(\int_{x_1}^{x_2} (x_2 - s)^{\alpha-1} \Big(\int_0^s |K_1(s,\tau)| \, |F_1(u(\tau))| \, d\tau\\
& \quad + \int_0^1 |K_2(s,\tau)| \, |F_2(u(\tau))| \, d\tau \Big) ds\\
& \quad + \int_0^{x_1} (x_1 - s)^{\alpha-1}\\
& \qquad - (x_2 - s)^{\alpha-1} \Big(\int_0^s |K_1(s,\tau)| \, |F_1(u(\tau))| \, d\tau\\
& \qquad + \int_0^1 |K_2(s,\tau)| \, |F_2(u(\tau))| \, d\tau \Big) ds \Big)\\
& = I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$\begin{aligned}
I_1 &= \frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left|a(s)\right|\left|u(s)\right|ds\right.\\
&\quad\left.+\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\left|a(s)\right|\left|u(s)\right|ds\right)\\
&\leq \frac{(x_2-x_1)^{\alpha}}{\Gamma(\alpha+1)}\left\|a\right\|_{\infty}\lambda+\frac{x_1^{\alpha}}{\Gamma(\alpha+1)}\left\|a\right\|_{\infty}\lambda+\frac{(x_2-x_1)^{\alpha}}{\Gamma(\alpha+1)}\left\|a\right\|_{\infty}\lambda\\
&\quad-\frac{x_2^{\alpha}}{\Gamma(\alpha+1)}\left\|a\right\|_{\infty}\lambda\\
&= \frac{\left\|a\right\|_{\infty}\lambda}{\Gamma(\alpha+1)}\left(2\left(x_2-x_1\right)^{\alpha}+\left(x_1^{\alpha}-x_2^{\alpha}\right)\right)\\
&\leq \frac{\left\|a\right\|_{\infty}\lambda}{\Gamma(\alpha+1)}2\left(x_2-x_1\right)^{\alpha}, \qquad (4.2)
\end{aligned}$$

$$\begin{aligned}
I_2 &= \frac{1}{\Gamma(\alpha)}\Big(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left|g(s)\right|ds\\
&\quad+\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\left|g(s)\right|ds\Big)\\
&\leq \frac{(x_2-x_1)^{\alpha}}{\Gamma(\alpha+1)}\left\|g\right\|_{\infty}+\frac{x_1^{\alpha}}{\Gamma(\alpha+1)}\left\|g\right\|_{\infty}+\frac{(x_2-x_1)^{\alpha}}{\Gamma(\alpha+1)}\left\|g\right\|_{\infty}-\frac{x_2^{\alpha}}{\Gamma(\alpha+1)}\left\|g\right\|_{\infty}\\
&= \frac{\left\|g\right\|_{\infty}}{\Gamma(\alpha+1)}\left(2\left(x_2-x_1\right)^{\alpha}+\left(x_1^{\alpha}-x_2^{\alpha}\right)\right)\\
&\leq \frac{\left\|g\right\|_{\infty}}{\Gamma(\alpha+1)}2\left(x_2-x_1\right)^{\alpha}, \qquad (4.3)
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\Big(\int_0^{s}\left|K_1(s,\tau)\right|\left|F_1(u(\tau))\right|d\tau\\
&\quad+\int_0^{1}\left|K_2(s,\tau)\right|\left|F_2(u(\tau))\right|d\tau\Big)ds\\
&\quad+\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\Big(\int_0^{s}\left|K_1(s,\tau)\right|\left|F_1(u(\tau))\right|d\tau\\
&\quad+\int_0^{1}\left|K_2(s,\tau)\right|\left|F_2(u(\tau))\right|d\tau\Big)ds\\
&\leq \frac{\Big(K_1^{*}\mu_1+K_2^{*}\mu_2\Big)}{\Gamma(\alpha+1)}\left(2\left(x_2-x_1\right)^{\alpha}+\left(x_1^{\alpha}-x_2^{\alpha}\right)\right)\\
&\leq \frac{(K_1^{*}\mu_1+K_2^{*}\mu_2)}{\Gamma(\alpha+1)}2\left(x_2-x_1\right)^{\alpha}. \qquad (4.4)
\end{aligned}$$

We can conclude the right-hand side of (4.2), (4.3), and (4.4) is independent of $u \in \mathbb{B}_\lambda$ and tends to zero as $x_2 - x_1 \to 0$. This leads to $|(Tu)(x_2) - (Tu)(x_1)| \to 0$ as $x_2 \to x_1$, that is, the set $\{T\mathbb{B}_\lambda\}$ is equicontinuous.

From I_1 to I_3 together with the Arzela–Ascoli theorem, we can conclude that $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is completely continuous.

Finally, we need to investigate that there exists a closed convex bounded subset $\mathbb{B}_{\widetilde{\lambda}} = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \le \widetilde{\lambda}\}$ such that $T\mathbb{B}_{\widetilde{\lambda}} \subseteq \mathbb{B}_{\widetilde{\lambda}}$. For each positive integer $\widetilde{\lambda}$, $\mathbb{B}_{\widetilde{\lambda}}$ is clearly a closed, convex, and bounded subset of $C(J,\mathbb{R})$. We claim that there exists a positive integer ϵ such that $T\mathbb{B}_\epsilon \subseteq \mathbb{B}_\epsilon$. If this property is false, then for every positive integer $\widetilde{\lambda}$, there exists $u_{\widetilde{\lambda}} \in \mathbb{B}_{\widetilde{\lambda}}$ such that $(Tu_{\widetilde{\lambda}}) \notin T\mathbb{B}_{\widetilde{\lambda}}$, that is, $\left\|Tu_{\widetilde{\lambda}}(t)\right\|_\infty > \widetilde{\lambda}$ for some $x_{\widetilde{\lambda}} \in J$, where $x_{\widetilde{\lambda}}$ denotes x depending on $\widetilde{\lambda}$. But by using the previous hypothesis, we have

$$\begin{aligned}
&|u_0| + \|u\|_\infty \|a\|_\infty \frac{t^\alpha}{\Gamma(\alpha+1)} + \|g\|_\infty \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{K_1^*\mu_1 x^\alpha}{\Gamma(\alpha+1)} + \frac{K_2^*\mu_2 x^\alpha}{\Gamma(\alpha+1)} \\
&\quad \le \left(|u_0| + \frac{\|a\|_\infty \lambda + \|g\|_\infty + K_1^*\mu_1 + K_2^*\mu_2}{\Gamma(\alpha+1)}\right),
\end{aligned}$$

and hence

$$\begin{aligned}
\widetilde{\lambda} &< \left\|Tu_{\widetilde{\lambda}}\right\|_\infty \\
&= \sup_{x\in J}\left|(Tu_{\widetilde{\lambda}})(x)\right| \\
&\le \sup_{x\in J}\Big\{ |u_0| + \left|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} a(s)\,|u(s)|\,ds\right| \\
&\quad + \left|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} g(s) ds\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\Big(\int_0^s |K_1(s,\tau)|\,|F_1(u(\tau))|\,d\tau \\
&\quad + \int_0^1 |K_2(s,\tau)|\,|F_2(u(\tau))|\,d\tau\Big) ds\Big\} ds \\
&\le \sup_{x\in J}\Big\{ |u_0| + \|u\|_\infty \|a\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \|g\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{K_1^*\mu_1 x^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \frac{K_2^*\mu_2 x^\alpha}{\Gamma(\alpha+1)}\Big\} \\
&\le \sup_{x\in J}\left(|u_0| + \frac{\|a\|_\infty \widetilde{\lambda} + \|g\|_\infty + K_1^*\mu_1 + K_2^*\mu_2}{\Gamma(\alpha+1)}\right).
\end{aligned}$$

Dividing both sides by $\widetilde{\lambda}$ and taking the limit as $\widetilde{\lambda} \to +\infty$, we obtain

$$1 < \frac{\|a\|_\infty}{\Gamma(\alpha+1)},$$

which contradicts assumption (4.1). Hence, for some positive integer $\widetilde{\lambda}$, we must have $T\mathbb{B}_{\widetilde{\lambda}} \subseteq \mathbb{B}_{\widetilde{\lambda}}$.

An application of Schauder's fixed point theorem shows that there exists at least one fixed point u for T in $C(J,\mathbb{R})$. Then u is the solution to (1.1)–(1.2) on J, and the proof is completed. □

Now, we will study the uniqueness result of the solution based on the Banach contraction principle.

Theorem 4.3. *Assume that* (H1)–(H3) *hold. If*

$$\left(\frac{\|a\|_\infty + K_1^* L_{F_1} + K_2^* L_{F_2}}{\Gamma(\alpha+1)}\right) < 1, \tag{4.5}$$

then there exists a unique solution $u(x) \in C(J)$ *to* (1.1)–(1.2).

Proof. By Lemma 4.1, we know that a function u is a solution to (1.1)–(1.2) if and only if u satisfies

$$\begin{aligned} u(x) &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} a(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} g(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s K_1(s,\tau)F_1(u(\tau))d\tau \right.\\ &\left. + \int_0^1 K_2(s,\tau)F_2(u(\tau))d\tau\right) ds. \end{aligned}$$

Let the operator $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ be defined as in Theorem 4.2. We can see that, if $u \in C(J,\mathbb{R})$ is a fixed point of T, then u is a solution of (1.1)–(1.2).

Now we prove T has a fixed point u in $C(J,\mathbb{R})$. For that, let $u_1, u_2 \in C(J,\mathbb{R})$ such that, for any $x \in [0,1]$,

$$\begin{aligned} u_1(x) &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} a(s)u_1(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} g(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\Big(\int_0^s K_1(s,\tau)F_1(u_1(\tau))d\tau \\ &+ \int_0^1 K_2(s,\tau)F_2(u_1(\tau))d\tau\Big) ds, \end{aligned}$$

and

$$\begin{aligned} u_2(x) &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} a(s)u_2(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} g(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\Big(\int_0^s K_1(s,\tau)F_1(u_2(\tau))d\tau \\ &+ \int_0^1 K_2(s,\tau)F_2(u_2(\tau))d\tau\Big) ds. \end{aligned}$$

Consequently, we get

$$\begin{aligned}
|(Tu_1)(x) \quad - \quad & (Tu_2)(x)| \\
\leq \quad & \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} |a(s)| \, |u_1(s) - u_2(s)| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \Big(\int_0^s |K_1(s,\tau)| \, |F_1(u_1(\tau)) - F_1(u_2(\tau))| \, d\tau \\
& \quad + \int_0^1 |K_2(s,\tau)| \, |F_2(u_1(\tau)) - F_2(u_2(\tau))| \, d\tau \Big) ds \\
\leq \quad & \frac{\|a\|_\infty}{\Gamma(\alpha+1)} |u_1(x) - u_2(x)| + \frac{K_1^* L_{F_1}}{\Gamma(\alpha+1)} |u_1(x) - u_2(x)| \\
& + \frac{K_2^* L_{F_2}}{\Gamma(\alpha+1)} |u_1(x) - u_2(x)| \\
= \quad & \left(\frac{\|a\|_\infty + K_1^* L_{F_1} + K_2^* L_{F_2}}{\Gamma(\alpha+1)} \right) |u_1(x) - u_2(x)| \, .
\end{aligned}$$

From the inequality (4.5) we have

$$\|Tu_1 - Tu_2\|_\infty \leq \|u_1 - u_2\|_\infty \, .$$

This means that T is a contraction map. By the Banach contraction principle, we can conclude that T has a unique fixed point u in $C(J, \mathbb{R})$. □

Now, we will study the convergence theorem of the solutions based on the MADM.

Theorem 4.4. *Suppose that* (H1)–(H3) *and* (4.5) *hold, if the series solution* $u(x) = \sum_{i=0}^{\infty} u_i(x)$ *and* $\|u_1\|_\infty < \infty$ *obtained by the* m*-order deformation is convergent, then it converges to the exact solution of the fractional Volterra–Fredholm integro-differential equation* (1.1)–(1.2).

Proof. The notation $(C[0,1], \|.\|)$ denotes the Banach space of all continuous functions on J with $|u_1(x)| \leq \infty$ for all x in J.

First we define the sequence of partial sums s_n, and let s_n and s_m be arbitrary partial sums with $n \geq m$. We are going to prove that $s_n = \sum_{i=0}^{n} u_i(x)$ is a Cauchy sequence in this Banach space:

$$\begin{aligned}
\|s_n - s_m\|_\infty &= \max_{\forall x \in J} |s_n - s_m| \\
&= \max_{\forall x \in J} \Big| \sum_{i=0}^{n} u_i(x) - \sum_{i=0}^{m} u_i(x) \Big| \\
&= \max_{\forall x \in J} \Big| \sum_{i=m+1}^{n} u_i(x) \Big| \\
&= \max_{\forall x \in J} \Big| \sum_{i=m+1}^{n} (\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} [a(t)u_i(t) \\
&\quad + \int_0^t K_1(t,s) A_i(s) ds + \int_0^1 K_2(t,s) B_i(s) ds] dt) \Big| \\
&= \max_{\forall x \in J} \Big| \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} [a(t) \sum_{i=m}^{n-1} u_i(t) \\
&\quad + \int_0^t K_1(t,s) \sum_{i=m}^{n-1} A_i(s) ds + \int_0^1 K_2(t,s) \sum_{i=m}^{n-1} B_i(s) ds)] dt \Big|.
\end{aligned}$$

From (3.1) and (3.2), we have

$$\sum_{i=m}^{n-1} A_i = F_1(s_{n-1}) - F_1(s_{m-1}),$$

$$\sum_{i=m}^{n-1} B_i = F_2(s_{n-1}) - F_2(s_{m-1}),$$

$$\sum_{i=m}^{n-1} u_i = u(s_{n-1}) - u(s_{m-1}).$$

So,

$$\begin{aligned}
\|s_n - s_m\|_\infty &= \max_{\forall x \in J} \Big(\Big| \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} [a(t)(u(s_{n-1}) - u(s_{m-1})) \\
&\quad + \int_0^t K_1(t,s)(F_1(s_{n-1}) - F_1(s_{m-1})) ds \\
&\quad + \int_0^1 K_2(t,s)(F_2(s_{n-1}) - F_2(s_{m-1})) ds] dt \Big| \Big), \\
&\leq \max_{\forall x \in J} \Big(\frac{1}{\Gamma(\alpha)} \int_0^x \Big| x-t \Big|^{\alpha-1} \Big[\Big| a(t) \Big| \Big| u(s_{n-1}) - u(s_{m-1}) \Big| \\
&\quad + \int_0^t \Big| K_1(t,s) \Big| \Big| (F_1(s_{n-1}) - F_1(s_{m-1})) \Big| ds \\
&\quad + \int_0^1 \Big| K_2(t,s) \Big| \Big| (F_2(s_{n-1}) - F_2(s_{m-1})) \Big| ds \Big] dt \Big),
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha+1)}\Big[\|a(t)\|_\infty\|s_{n-1}-s_{m-1}\|_\infty + K_1^* L_{F_1}\|s_{n-1}-s_{m-1}\|_\infty \\
&\qquad + K_2^* L_{F_2}\|s_{n-1}-s_{m-1}\|_\infty\Big], \\
&= \left(\frac{\|a\|_\infty + K_1^* L_{F_1} + K_2^* L_{F_2}}{\Gamma(\alpha+1)}\right)\|s_{n-1}-s_{m-1}\|_\infty, \\
&= \delta\|s_{n-1}-s_{m-1}\|_\infty,
\end{aligned}$$

where

$$\delta = \left(\frac{\|a\|_\infty + K_1^* L_{F_1} + K_2^* L_{F_2}}{\Gamma(\alpha+1)}\right).$$

Let $n = m+1$; then

$$\|s_n - s_m\|_\infty \leq \delta\|s_m - s_{m-1}\|_\infty \leq \delta^2\|s_{m-1}-s_{m-2}\|_\infty \leq \cdots \leq \delta^m\|s_1 - s_0\|_\infty;$$

so,

$$\begin{aligned}
\|s_n - s_m\|_\infty &\leq \|s_{m+1}-s_m\|_\infty + \|s_{m+2}-s_{m+1}\|_\infty + \cdots + \|s_n - s_{n-1}\|_\infty \\
&\leq [\delta^m + \delta^{m+1} + \cdots + \delta^{n-1}]\|s_1 - s_0\|_\infty \\
&\leq \delta^m[1 + \delta + \delta^2 + \cdots + \delta^{n-m-1}]\|s_1 - s_0\|_\infty \\
&\leq \delta^m\left(\frac{1-\delta^{n-m}}{1-\delta}\right)\|u_1\|_\infty.
\end{aligned}$$

Since $0 < \delta < 1$, we have $(1-\delta^{n-m}) < 1$, and then

$$\|s_n - s_m\|_\infty \leq \frac{\delta^m}{1-\delta}\|u_1\|_\infty.$$

But $|u_1(x)| < \infty$, so $\|s_n - s_m\|_\infty \longrightarrow 0$ as $m \longrightarrow \infty$.

We conclude that s_n is a Cauchy sequence in $C[0,1]$; therefore $u = \lim_{n\to\infty} u_n$. Thus the series is convergent and the proof is complete. □

5. Illustrative example

In this section, we present the analytical technique based on MADM to solve Caputo fractional Volterra–Fredholm integro-differential equations.

Example 5.1. Consider the following Caputo fractional Volterra–Fredholm integro-differential equation.

$$^{c}D^{0.5}[u(x)] = \frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2} - \frac{x^2 e^x}{3}u(x) + \int_0^x e^x s u(s)ds + \int_0^1 x^2 u(s)ds, \quad (5.1)$$

with the initial condition

$$u(0) = 0,$$

and the the exact solution is $u(x) = x$. Applying the operator $J^{0.5}$ to both sides of equation (5.1)

$$\begin{aligned} u(x) &= 0 + J^{0.5}\Big[\frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}\Big] + J^{0.5}\Big[-\frac{x^2e^x}{3}u(x)\Big] \\ &\quad + J^{0.5}\Big[\int_0^x e^x s u(s)ds + \int_0^1 x^2 u(s)ds\Big]. \end{aligned}$$

Then,

$$\begin{aligned} u(x) &= J^{0.5}\Big[\frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}\Big] + J^{0.5}\Big[-\frac{x^2e^x}{3}u(x)\Big] \\ &\quad + J^{0.5}\Big[\int_0^x e^x s u(s)ds + \int_0^1 x^2 u(s)ds\Big]. \end{aligned} \tag{5.2}$$

From equation (5.1) we see $g(x) = \frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}$. Suppose $R(x) = J^{0.5}g(x)$, from equation (5.2) we have

$$\begin{aligned} R(x) &= J^{0.5}g(x) = J^{0.5}\Big[\frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}\Big], \\ &= \frac{1}{\Gamma(1.5)\Gamma(0.5)}\int_0^x (x-s)^{-0.5}s^{0.5}ds - \frac{1}{2\Gamma(0.5)}\int_0^x (x-s)^{-0.5}s^2 ds, \\ &= \frac{1}{\Gamma(1.5)\Gamma(0.5)}\int_0^x (1-\frac{s}{x})^{-0.5}x^{-0.5}s^{0.5}ds \\ &\quad - \frac{1}{2\Gamma(0.5)}\int_0^x (1-\frac{s}{x})^{-0.5}x^{-0.5}s^2 ds, \\ &= \frac{1}{\Gamma(1.5)\Gamma(0.5)}\int_0^1 (1-\tau)^{-0.5}\tau^{0.5}x d\tau - \frac{1}{2\Gamma(0.5)}\int_0^1 (1-\tau)^{-0.5}x^{2.5}\tau^2 d\tau, \\ &= \frac{x}{\Gamma(1.5)\Gamma(0.5)}\beta(0.5,1.5) - \frac{x^{2.5}}{2\Gamma(0.5)}\beta(0.5,3), \\ &= x - \frac{x^{2.5}}{\Gamma(3.5)}. \end{aligned}$$

Now, we apply the modified Adomian decomposition method,

$$R(x) = R_1(x) + R_2(x) = J^{0.5}[\frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}] = x - \frac{x^{2.5}}{\Gamma(3.5)}.$$

The modified recursive relation

$$\begin{aligned}
u_0(x) &= R_1(x) = x. \\
u_1(x) &= R_2(x) + J^{0.5}\left(f(x)u_0(x)\right) \\
&\quad + J^{0.5}\left(\int_0^x K_1(x,s)A_0 ds + \int_0^1 K_2(x,s)B_0 ds\right) \\
&= -\frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^2e^x}{3}u_0(x)\right) \\
&\quad + J^{0.5}\left(\int_0^x e^x s A_0(s) ds + \int_0^1 x^2 B_0(s) ds\right) \\
&= -\frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^2e^x}{3}x\right) + J^{0.5}\left(\int_0^x e^x s^2 ds + \int_0^1 x^2 s ds\right) \\
&= -\frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^3e^x}{3}\right) + J^{0.5}\left(\frac{e^x x^3}{3} + \frac{x^2}{2}\right) \\
&= -\frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^3e^x}{3}\right) + J^{0.5}\left(\frac{e^x x^3}{3}\right) + J^{0.5}\left(\frac{x^2}{2}\right) \\
&= 0, \\
u_2(x) &= 0, \\
&\vdots \\
u_n(x) &= 0.
\end{aligned}$$

Therefore, the obtained solution is

$$u(x) = \sum_{i=0}^{\infty} u_i(x) = x.$$

6. Conclusions

The modified Adomian decomposition method is successfully applied to find the approximate solution of Caputo fractional Volterra–Fredholm integro-differential equation. The reliability of the method and reduction in the size of the computational work give this method a wider applicability. The method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and nonlinear fractional Volterra–Fredholm integro-differential equations. Moreover, we proved the existence and uniqueness of the solution. The convergence theorem and the illustrative example establish the precision and efficiency of the proposed technique.

References

1. K. Abbaoui, Y. Cherruault, *Convergence of Adomian's method applied to nonlinear equations*, Math. Comput. Modelling **20** (1994), no. 9, 69–73.

2. N. Abel, *Solution de quelques problemes a laide dintegrales definites*, Christiania Grondahl, Norway (1881) 16–18.
3. G. Adomian, *A review of the decomposition method in applied mathematics*, J. Math. Anal. Appl. **135** (1988), no. 2, 501–544.
4. B. Ahmad, S. Sivasundaram, *Some existence results for fractional integro-differential equations with nonlinear conditions*, Commun. Appl. Anal. **12** (2008) 107–112.
5. S. Alkan, V. Hatipoglu, *Approximate solutions of Volterra–Fredholm integro-differential equations of fractional order*, Tbilisi Math. J. **10** (2017), no. 2, 1–13.
6. M. AL-Smadi, G. Gumah, *On the homotopy analysis method for fractional SEIR epidemic model*, Research J. Appl. Sci. Engrg. Technol. **7** (2014), no. 18, 3809–3820.
7. A.A. Hamoud, A.D. Azeez, K.P. Ghadle, *A study of some iterative methods for solving fuzzy Volterra–Fredholm integral equations*, Indonesian J. Elec. Eng. & Comp. Sci., **11** (2018), no. 3, 1228–1235.
8. A.A Hamoud, K.P. Ghadle, *The reliable modified of Laplace Adomian decomposition method to solve nonlinear interval Volterra–Fredholm integral equations*, Korean J. Math. **25** (2017), no. 3, 323–334.
9. A.A. Hamoud, K.P. Ghadle, *The combined modified Laplace with Adomian decomposition method for solving the nonlinear Volterra–Fredholm integro-differential equations*, J. Korean Soc. Ind. Appl. Math. **21** (2017) 17–28.
10. A.A. Hamoud, K.P. Ghadle, *Modified Adomian decomposition method for solving fuzzy Volterra–Fredholm integral equations*, J. Indian Math. Soc. **85** (2018), no. 1-2, 52–69.
11. A.A. Hamoud, K.P. Ghadle, *The approximate solutions of fractional Volterra–Fredholm integro-differential equations by using analytical techniques*, Probl. Anal. Issues Anal. **7(25)** (2018), no. 1, 41–58.
12. A.A. Hamoud, K.P. Ghadle, M.SH. Bani Issa, Giniswamy, *Existence and uniqueness theorems for fractional Volterra–Fredholm integro-differential equations*, Int. J. Appl. Math. **31** (2018), no. 3, 333–348.
13. A.A. Hamoud, K.P. Ghadle, *Existence and uniqueness of solutions for fractional mixed Volterra–Fredholm integro-differential equations*, Indian J. Math. **60** (2018), no. 3, to appear.
14. K. Karthikeyan, J. Trujillo, *Existence and uniqueness results for fractional integro-differential equations with boundary value conditions*, Commun. Nonlinear Sci. Numer. Simul. **17** (2012) 4037–4043.
15. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Math. Stud. Elsevier, Amsterdam, 2006.
16. V. Lakshmikantham, *Theory of fractional functional differential equations*, Nonlinear Anal. **69** (2008), no. 10, 3337–3343.
17. X. Ma, C. Huang, *Numerical solution of fractional integro-differential equations by a hybrid collocation method*, Appl. Math. Comput. **219** (2013), no. 12, 6750–6760.
18. R. Mittal, R. Nigam, *Solution of fractional integro-differential equations by Adomian decomposition method*, Int. J. Appl. Math. Mech. **4** (2008), no. 2, 87–94.
19. R. Rach, *On the Adomian decomposition method and comparisons with Picard's method.* J. Math. Anal. Appl. **128** (1987), no. 2, 480–483.
20. S. Samko, A. Kilbas, O. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.
21. A.M. Wazwaz, A reliable modification of Adomian decomposition method, *Appl. Math. Comput.* **102** (1999) 77–86.
22. J. Wu, Y. Liu, *Existence and uniqueness of solutions for the fractional integro-differential equations in Banach spaces*, Electron. J. Differential Equations, **2009** (2009), No. 129, 8 pp.
23. C. Yang, J. Hou, *Numerical solution of integro-differential equations of fractional order by Laplace decomposition method*, Wseas Trans. Math. **12** (2013), no. 12, 1173–1183.

24. X. Zhang, B. Tang, Y. He, *Homotopy analysis method for higher-order fractional integro-differential equations*, Comput. Math. Appl. **62** (2011), no. 8, 3194–3203.
25. Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.
26. M. Zurigat, S. Momani, A. Alawneh, *Homotopy analysis method for systems of fractional integro-differential equations*, Neur. Parallel Sci. Comput. **17** (2009) 169–186.

[1]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University Aurangabad 431-004, India.
[1]Department of Mathematics, Taiz University, Taiz, Yemen.
E-mail address: drahmed985@yahoo.com

[2]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University Aurangabad 431-004, India.
E-mail address: drkp.ghadle@gmail.com

[3] Department of Mathematics, Thi Qar Directorates of Education, Ministry of Education, Iraq.
E-mail address: s11h32@yahoo.com