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RUSCHEWEYH-TYPE HARMONIC FUNCTIONS DEFINED BY q- DIFFERENTIAL OPERATORS

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Abstract. A class of Ruscheweyh-type harmonic functions is defined, using q-differential operators and sufficient coefficient conditions for this class is determined. We then consider a subclass of the aforementioned class consisting of functions with real coefficients and obtain necessary and sufficient coefficient bounds, distortion theorem, extreme points, and convex combination conditions for such class. It is shown that the classes of functions considered in this paper contain various well-known as well as new classes of harmonic functions.

1. Introduction and preliminaries

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply-connected domain $\mathbb{D} \subset \Omega$, we can write $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ (see [2]). Let $\mathcal{H}$ denote the family of functions $f = h + \overline{g}$ which are harmonic, univalent, and orientation preserving in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ and are of the form

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}.$$

Obviously, $|b_1| < 1$ and the family $\mathcal{H}$ reduces to the well known class $\mathcal{S}$ of normalized analytic univalent functions if the co-analytic part of f is identically zero, that is, $g \equiv 0$.

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The Hadamard product or convolution of two power series $h_1(z)=\sum_{n=1}^{\infty} a_n z^n$ and $h_2(z)=\sum_{n=1}^{\infty} c_n z^n$ is given by $h_1(z)*h_2(z)=(h_1*h_2)(z)=\sum_{n=1}^{\infty} a_n c_n z^n$, and the convolution of two harmonic functions

$$f_1(z)=h_1(z)+\overline{g_1(z)} \text{ and } f_2(z)=h_2(z)+\overline{g_2(z)}$$

is given by

$$f_1(z)\widetilde{*}f_2(z)=(f_1\widetilde{*}f_2)(z)=h_1(z)*h_2(z)+\overline{g_1(z)*g_2(z)}.$$

Next we recall the notion of q-*operator* or q-*difference operator* that plays a vital role in the theory of hypergeometric series, quantum physics, and operator theory. In 1908, Jackson [3] initiated the application of q-*calculus* to analytic functions. For $0<q<1$, Jackson's q-*derivative* of the function $h(z)=z+\sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ is given by

$$\mathcal{D}_q h(z)=\begin{cases} \dfrac{h(z)-h(qz)}{(1-q)z} & for\ z\neq 0,\\ h'(0) & for\ z=0,\end{cases}$$

where $\mathcal{D}_q h(z)=1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1}$ and $[n]_q=\frac{1-q^n}{1-q}$.

In 2014, Kannas and Raducanu [7] introduced and investigated the Ruscheweyh-type q-*differential operator*

$$R_q^m h(z)=h(z)*F_{q,m+1}(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma_q(n+m)}{[n-1]!\Gamma_q(1+m)}a_n z^n,\quad m>-1,$$

where

$$F_{q,m+1}(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma_q(n+m)}{[n-1]!\Gamma_q(1+m)}z^n=z+\sum_{n=2}^{\infty}\frac{[m+1]_{n-1}}{[n-1]!}z^n.$$

Observe that

$$\begin{aligned} R_q^0 h(z) &= h(z),\\ R_q^1 h(z) &= zD_q h(z),\\ &\vdots\\ R_q^m h(z) &= \frac{zD_q^m(z^{m-1}h(z))}{[m]!},\end{aligned}$$

where $\mathcal{D}_q^2 h(z)=\mathcal{D}_q(\mathcal{D}_q h(z))$ and $\mathcal{D}_q^m h(z)=\mathcal{D}_q^{m-1}(\mathcal{D}_q h(z))$.

Obviously

$$R_q^m h(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma_q(n+m)}{[n-1]!\Gamma_q(1+m)}a_n z^n,$$

$$\lim_{q\to 1^-} R_q^m h(z)=R^m h(z)=h(z)*\frac{z}{(1-z)^{m+1}},$$

and

$$\lim_{q\to 1^-} F_{q,m+1}(z)=F_{m+1}(z)=\frac{z}{(1-z)^{m+1}}.$$

We remark that if $q \to 1^-$, then the Ruscheweyh q-differential operator reduces to the differential operator defined by Ruscheweyh [8],

$$\mathcal{D}_q(R_q^m h(z)) = 1 + \sum_{n=2}^{\infty} [n]_q \frac{\Gamma_q(n+m)}{[n-1]!\Gamma_q(1+m)} a_n z^{n-1}.$$

Recently, Jahangiri [5] applied q-difference operators to classes of harmonic functions and obtained coefficient bounds for such functions. Motivated by [7] and [5], we define a class of Ruscheweyh-type q-calculus harmonic functions $\mathcal{H}_q^m(\lambda, \gamma)$ consisting of functions $f \in \mathcal{H}$ satisfying

$$\Re \left(\frac{z\mathcal{D}_q(R_q^m f(z))}{(1-\lambda)z' + \lambda(R_q^m f(z))} \right) \geq \gamma, \tag{1.1}$$

where $z \in \mathbb{U}$, $0 \leq \lambda \leq 1$, $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$, and

$$z\mathcal{D}_q(R_q^m f(z)) = z\mathcal{D}_q(R_q^m h(z)) - \overline{z\mathcal{D}_q(R_q^m g(z))}.$$

We also define $\overline{\mathcal{H}}_q^m(\lambda, \gamma) \equiv \mathcal{H}_q^m(\lambda, \gamma) \cap \overline{\mathcal{H}}$, where $\overline{\mathcal{H}}$ is the subfamily of $\mathcal{H}$ consisting of harmonic functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n, \quad a_n \geq 0, \ b_n \geq 0. \tag{1.2}$$

The following special cases clearly demonstrate the significance of the class $\mathcal{H}_q^m(\lambda, \gamma)$.

(i) If $q \to 1^-$, then $\mathcal{H}_1^m(\lambda, \gamma) \equiv \mathcal{R}_{\mathcal{H}}(\lambda, \gamma)$ consists of functions $f \in \mathcal{H}$ satisfying

$$\Re \left(\frac{z\mathcal{D}(R^m f(z))}{(1-\lambda)z' + \lambda(R^m f(z))} \right) \geq \gamma, \quad (m > -1),$$

where $R^m f(z))$ is the differential operator defined by Ruscheweyh [8] and $\mathcal{H}_1^m(1, \gamma)$ is the class considered in [6].

(ii) If $q \to 1^-$ and $m = 0$, then $\mathcal{H}_1^0(\lambda, \gamma) \equiv \mathcal{H}(\lambda, \gamma)$ is the class defined in [9] that consists of functions $f \in \mathcal{H}$ satisfying

$$\Re \left(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} \right) \geq \gamma.$$

(iii) If $q \to 1^-$, $m = 0$ and $\lambda = 1$, then $\mathcal{H}_1^0(1, \gamma) \equiv \mathcal{SH}(\gamma)$ is the class defined in [4] that consists of functions $f \in \mathcal{H}$ satisfying

$$\Re \left(\frac{zf'(z)}{f(z)} \right) \geq \gamma.$$

(iv) If $q \to 1^-$, $m = 0$ and $\lambda = 0$, then $\mathcal{H}_1^0(0, \gamma) \equiv \mathcal{N}_{\mathcal{H}}(\gamma)$ is the class defined in [1] that consists of functions $f \in \mathcal{H}$ satisfying

$$\Re \ (f'(z)) \geq \gamma.$$

(v) If $\lambda = 0$, then $\mathcal{H}_q^m(0, \gamma) \equiv \mathcal{NH}_q^m(\gamma)$ consists of functions $f \in \mathcal{H}$ satisfying

$$\Re \left(\frac{z\mathcal{D}_q(R_q^m f(z))}{z'} \right) \geq \gamma \ .$$

(vi) If $\lambda = 1$, then $\mathcal{H}_q^m(1,\gamma) \equiv \mathcal{H}_q^m(\gamma)$ consists of functions $f \in \mathcal{H}$ satisfying

$$\Re \left(\frac{z\mathcal{D}_q(R_q^m f(z))}{R_q^m f(z)} \right) \geq \gamma \ .$$

It is the aim of this paper to obtain sufficient coefficient conditions for harmonic functions $f = h + \overline{g}$ to be in the class $\mathcal{H}_q^m(\lambda,\gamma)$. We also determine necessary and sufficient coefficient conditions for harmonic functions $f = h + \overline{g}$ to be in the class $\overline{\mathcal{H}}_q^m(\lambda,\gamma)$. Furthermore, distortion theorems and extreme points for functions in $\overline{\mathcal{H}}_q^m(\lambda,\gamma)$ are also obtained.

2. Main results

Throughout this section, unless otherwise stated, we shall use the notation

$$\Phi_q(n,m) = \frac{\Gamma_q(n+m)}{(n-1)!\Gamma_q(1+m)} \ .$$

First we obtain a sufficient coefficient condition for harmonic functions in $\mathcal{H}_q^m(\lambda,\gamma)$.

Theorem 2.1. *Let $f = h + \overline{g} \in \mathcal{H}$. If*

$$\sum_{n=1}^{\infty} \left(\frac{[n]_q - \gamma\lambda}{1-\gamma} |a_n| + \frac{[n]_q + \gamma\lambda}{1-\gamma} |b_n| \right) \Phi_q(n,m) \leq 2, \tag{2.1}$$

where $a_1 = 1$ and $0 \leq \gamma < 1$, then $f \in \mathcal{H}_q^m(\lambda,\gamma)$.

Proof. We will show that if the coefficients of the harmonic function $f = h + \overline{g} \in \mathcal{H}$ satisfy the inequality (2.1), then $f = h + \overline{g}$ satisfies the condition (1.1). In other words, we need to show that

$$\Re \left(\frac{z\mathcal{D}_q(R_q^m h(z)) - \overline{z\mathcal{D}_q(R_q^m g(z))}}{(1-\lambda)z' + \lambda(R_q^m f(z))} \right) = \Re \left(\frac{A(z)}{B(z)} \right) \geq \gamma, \tag{2.2}$$

where

$$\begin{aligned} A(z) &= z\mathcal{D}_q(R_q^m h(z)) - \overline{z\mathcal{D}_q(R_q^m g(z))} \\ &= z + \sum_{n=2}^{\infty} [n]_q \Phi_q(n,m) a_n z^n - \sum_{n=1}^{\infty} [n]_q \Phi_q(n,m) \overline{b}_n \overline{z}^n \ , \end{aligned}$$

and

$$\begin{aligned} B(z) &= (1-\lambda)z' + \lambda(R_q^m f(z)) \\ &= z + \sum_{n=2}^{\infty} \lambda \Phi_q(n,m) a_n z^n + \sum_{n=1}^{\infty} \lambda \Phi_q(n,m) \overline{b}_n \overline{z}^n \ . \end{aligned}$$

Using the fact that $\Re\, \{w\} \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it suffices to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0. \tag{2.3}$$

Substituting for $A(z)$ and $B(z)$ in (2.3), we get

$$
\begin{aligned}
&|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)| \\
&= \Big|\ (2-\gamma)z+\sum_{n=2}^{\infty}[[n]_q+(1-\gamma)\lambda]\Phi_q(n,m)a_n z^n \\
&\qquad -\sum_{n=1}^{\infty}[[n]_q-(1-\gamma)\lambda]\Phi_q(n,m)\bar{b}_n\ \overline{z}^n\ \Big| \\
&\quad -\Big|-\gamma z+\sum_{n=2}^{\infty}[[n]_q-(1+\gamma)\lambda]\Phi_q(n,m)a_n z^n \\
&\qquad -\sum_{n=1}^{\infty}[[n]_q+(1+\gamma)\lambda]\Phi_q(n,m)\bar{b}_n\overline{z}^n\Big| \\
&\geq (2-\gamma)|z|-\sum_{n=2}^{\infty}[[n]_q+(1-\gamma)\lambda]\Phi_q(n,m)|a_n||z|^n \\
&\quad -\sum_{n=1}^{\infty}[[n]_q-(1-\gamma)\lambda]\Phi_q(n,m)|b_n|\ |z|^n \\
&\quad -\gamma|z|-\sum_{n=2}^{\infty}[[n]_q-(1+\gamma)\lambda]\Phi_q(n,m)|a_n|\ |z|^n \\
&\quad -\sum_{n=1}^{\infty}[[n]_q+(1+\gamma)\lambda]\Phi_q(n,m)|b_n|\ |z|^n \\
&\geq 2(1-\gamma)|z|\left(2-\sum_{n=1}^{\infty}\left[\frac{[n]_q-\gamma\lambda}{1-\gamma}|a_n|+\frac{[n]_q+\gamma\lambda}{1-\gamma}|b_n|\right]\Phi_q(n,m)|z|^{n-1}\right) \\
&\geq 2(1-\gamma)\left(2-\sum_{n=1}^{\infty}\left[\frac{[n]_q-\gamma\lambda}{1-\gamma}|a_n|+\frac{[n]_q+\gamma\lambda}{1-\gamma}|b_n|\right]\Phi_q(n,m)\right).
\end{aligned}
$$

The above expression is nonnegative by (2.1), and so $f\in\mathcal{H}_q^m(\lambda,\gamma)$. □

The harmonic function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{1-\gamma}{([n]_q-\gamma\lambda)\Phi_q(n,m)}x_n z^n+\sum_{n=1}^{\infty}\frac{1-\gamma}{([n]_q+\gamma\lambda)\Phi_q(n,m)}\overline{y}_n(\overline{z})^n, \quad (2.4)$$

where $\sum\limits_{n=2}^{\infty}|x_n|+\sum\limits_{n=1}^{\infty}|y_n|=1$, shows that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.4) are in $\mathcal{H}_q^m(\lambda,\gamma)$, because

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(\frac{([n]_q-\gamma\lambda)\Phi_q(n,m)}{1-\gamma}|a_n|+\frac{([n]_q+\gamma\lambda)\Phi_q(n,m)}{1-\gamma}|b_n|\right) \\
&\qquad\qquad = 1+\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=2.
\end{aligned}
$$

The necessary and sufficient coefficient conditions for the harmonic functions $f = h + \overline{g}$ to be in $\overline{\mathcal{H}}_q^m(\lambda, \gamma)$ are given by the following theorem.

Theorem 2.2. *For $a_1 = 1$ and $0 \leq \gamma < 1$, we have $f = h + \overline{g} \in \overline{\mathcal{H}}_q^m(\lambda, \gamma)$ if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{[n]_q - \gamma\lambda}{1-\gamma} \, a_n + \frac{[n]_q + \gamma\lambda}{1-\gamma} \, b_n \right) \Phi_q(n, m) \leq 2. \tag{2.5}$$

Proof. Since $\overline{\mathcal{H}}_q^m(\lambda, \gamma) \subset \mathcal{H}_q^m(\lambda, \gamma)$, we need to prove the "only if" part of the theorem. In other words, for functions f of the form (1.2), we will show that if the condition (2.2) holds, then the coefficients of the function $f = h + \overline{g}$ satisfy the inequality (2.5). We note that the condition (2.2) is equivalent to

$$\Re \left(\frac{(1-\gamma)z - \sum_{n=2}^{\infty} ([n]_q - \gamma\lambda)\Phi_q(n, m) a_n z^n - \sum_{n=1}^{\infty} [[n]_q + \gamma\lambda]\Phi_q(n, m) \overline{b}_n \overline{z}^n}{z - \sum_{n=2}^{\infty} \lambda\Phi_q(n, m) a_n z^n + \sum_{n=1}^{\infty} \lambda\Phi_q(n, m) \overline{b}_n \overline{z}^n} \right) \geq 0.$$

The above required condition must hold for all values of z in $\mathbb{U}$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1,$, we must have

$$\frac{(1-\gamma) - \sum_{n=2}^{\infty} ([n]_q - \gamma\lambda)\Phi_q(n, m) a_n r^{n-1} - \sum_{n=1}^{\infty} ([n]_q + \gamma\lambda)\Phi_q(n, m) b_n r^{n-1}}{1 - \sum_{n=2}^{\infty} \lambda\Phi_q(n, m) a_n r^{n-1} + \sum_{n=1}^{\infty} \lambda\Phi_q(n, m) b_n r^{n-1}} \geq 0. \tag{2.6}$$

If the condition (2.5) does not hold, then the numerator in (2.6) is negative for r sufficiently close to 1. Hence, there exist $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.6) is negative. This contradicts the required condition for $f \in \overline{\mathcal{H}}_q^m(\lambda, \gamma)$ and so the proof is complete. □

The following theorem gives the distortion bounds for functions in $\overline{\mathcal{H}}_q^m(\lambda, \gamma)$ which yields a covering result for the class $\overline{\mathcal{H}}_q^m(\lambda, \gamma)$.

Theorem 2.3. *Let $f \in \overline{\mathcal{H}}_q^m(\lambda, \gamma)$. Then for $|z| = r < 1$, we have*

$$\begin{aligned} (1-b_1)r - &\frac{1}{\Phi_q(2, m)} \left(\frac{1-\gamma}{[2]_q - \gamma\lambda} - \frac{1+\gamma}{[2]_q - \gamma\lambda} b_1 \right) r^2 \\ &\leq |f(z)| \\ &\leq (1+b_1)r + \frac{1}{\Phi_q(2, m)} \left(\frac{1-\gamma}{[2]_q - \gamma\lambda} - \frac{1+\gamma}{[2]_q - \gamma\lambda} b_1 \right) r^2. \end{aligned}$$

Proof. We shall only prove the right hand inequality, since the proof for the left hand inequality is similar to that given for the right hand side inequality. Taking the absolute value of $f(z)$, we obtain

$$\begin{aligned}
|f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n \right| \\
&\leq (1+b_1)|z| + \sum_{n=2}^{\infty} (a_n + b_n)|z|^n \\
&\leq (1+b_1)r + \sum_{n=2}^{\infty} (a_n + b_n) r^2 \\
&\leq (1+b_1)r + \frac{(1-\gamma)}{([2]_q - \gamma\lambda)\Phi_q(2,m)} \\
&\quad \times \sum_{n=2}^{\infty} \left(\frac{([2]_q - \gamma\lambda)\Phi_q(2,m)}{(1-\gamma)} a_n + \frac{([2]_q - \gamma\lambda)\Phi_q(2,m)}{(1-\gamma)} b_n \right) r^2 \\
&\leq (1+b_1)r + \frac{(1-\gamma)1}{([2]_q - \gamma\lambda)\Phi_q(2,m)} \left(1 - \frac{1+\gamma}{1-\gamma} b_1 \right) r^2 \\
&\leq (1+b_1)r + \frac{1}{\Phi_q(2,m)} \left(\frac{1-\gamma}{[2]_q - \gamma\lambda} - \frac{1+\gamma}{[2]_q - \gamma\lambda} b_1 \right) r^2.
\end{aligned}$$

□

As a consequence of Theorem 2.3, we obtain the following covering result.

Corollary 2.4. *If $f(z) \in \overline{\mathcal{H}}_q^m(\lambda, \gamma)$, then*

$$\left\{ w : |w| < \frac{([2]_q - \gamma\lambda)\Phi_q(2,m) - (1-\gamma)}{([2]_q - \gamma\lambda)\Phi_q(2,m)} - \frac{([2]_q - \gamma\lambda)\Phi_q(2,m) - (1+\gamma)}{([2]_q - \gamma\lambda)\Phi_q(2,m)} b_1 \right\} \subset f(\mathbb{U}).$$

Proof. Using the left hand inequality of Theorem 2.3 and letting $r \to 1$, it follows that

$$\begin{aligned}
&(1-b_1) - \frac{1}{\Phi_q(2,m)} \left(\frac{1-\gamma}{[2]_q - \gamma\lambda} - \frac{1+\gamma}{[2]_q - \gamma\lambda} b_1 \right) \\
&\quad = (1-b_1) - \frac{1}{\Phi_q(2,m)([2]_q - \gamma\lambda)} [1 - \gamma - (1+\gamma)b_1] \\
&\quad = \frac{(1-b_1)\Phi_q(2,m)([2]_q - \gamma\lambda) - (1-\gamma) + (1+\gamma)b_1}{\Phi_q(2,m)([2]_q - \gamma\lambda)} \\
&\quad = \left(\frac{([2]_q - \gamma\lambda)\Phi_q(2,m) - (1-\gamma)}{([2]_q - \gamma\lambda)\Phi_q(2,m)} - \frac{([2]_q - \gamma\lambda)\Phi_q(2,m) - (1+\gamma)}{([2]_q - \gamma\lambda)\Phi_q(2,m)} |b_1| \right) \\
&\quad \subset f(\mathbb{U}).
\end{aligned}$$

□

Next we determine the extreme points of closed convex hulls of $\overline{\mathcal{H}}_q^m(\lambda, \gamma)$ denoted by $clco\overline{\mathcal{H}}_q^m(\lambda, \gamma)$.

Theorem 2.5. *A function* $f(z) \in \overline{\mathcal{H}}_q^m(\lambda, \gamma)$ *if and only if*

$$f(z) = \sum_{n=1}^{\infty} \left(X_n h_n(z) + Y_n g_n(z) \right),$$

where $h_1(z) = z,\ h_n(z) = z - \frac{1-\gamma}{([n]_q - \gamma\lambda)\Phi_q(n,m)} z^n\ (n \geq 2),\ g_n(z) = z + \frac{1-\gamma}{([n]_q + \gamma\lambda)\Phi_q(n,m)} \overline{z}^n$ $(n \geq 2),\quad \sum_{n=1}^{\infty} (X_n + Y_n) = 1,\ \ X_n \geq 0$ *and* $Y_n \geq 0$. *In particular, the extreme points of* $\overline{\mathcal{H}}_q^m(\lambda, \gamma)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. First, we note that for f as in the theorem above, we may write

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \left(X_n h_n(z) + Y_n g_n(z) \right) \\ &= \sum_{n=1}^{\infty} (X_n + Y_n) z - \sum_{n=2}^{\infty} \frac{1-\gamma}{([n]_q - \gamma\lambda)\Phi_q(n,m)} X_n z^n \\ &+ \sum_{n=1}^{\infty} \frac{1-\gamma}{([n]_q + \gamma\lambda)\Phi_q(n,m)} Y_n \overline{z}^n. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{([n]_q - \gamma\lambda)\Phi_q(n,m)}{1-\gamma} |a_n| &+ \sum_{n=1}^{\infty} \frac{([n]_q + \gamma\lambda)\Phi_q(n,m)}{1-\gamma} |b_n| \\ &= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1, \end{aligned}$$

and so $f(z) \in clco\overline{\mathcal{H}}_q^m(\lambda, \gamma)$.

Conversely, suppose that $f(z) \in clco\overline{\mathcal{H}}_q^m(\lambda, \gamma)$. Set

$$X_n = \frac{([n]_q - \gamma\lambda)\Phi_q(n,m)}{1-\gamma} |a_n| \quad (0 \leq X_n \leq 1, n \geq 2)$$

$$Y_n = \frac{([n]_q + \gamma\lambda)\Phi_q(n,m)}{1-\gamma} |b_n| \quad (0 \leq Y_n \leq 1, n \geq 1)$$

and $X_1 = 1 - \sum\limits_{n=2}^{\infty} X_n - \sum\limits_{n=1}^{\infty} Y_n$. Therefore $f(z)$ can be rewritten as

$$\begin{aligned} f(z) &= z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n \\ &= z - \sum_{n=2}^{\infty} \frac{1-\gamma}{([n]_q - \gamma\lambda)\Phi_q(n,m)} X_n z^n + \sum_{n=1}^{\infty} \frac{1-\gamma}{([n]_q - \gamma\lambda)\Phi_q(n,m)} Y_n \overline{z}^n \\ &= z + \sum_{n=2}^{\infty} (h_n(z) - z) X_n + \sum_{n=1}^{\infty} (g_n(z) - z) Y_n \\ &= z\{1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n\} + \sum_{n=2}^{\infty} h_n(z) X_n + \sum_{n=1}^{\infty} g_n(z) Y_n \\ &= \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_n(z)), \end{aligned}$$

as required. □

Finally, we show that $\overline{\mathcal{H}}_q^m(\lambda,\gamma)$ is closed under convex combinations of its members.

Theorem 2.6. *The family $\overline{\mathcal{H}}_q^m(\lambda,\gamma)$ is closed under convex combinations.*

Proof. For $i = 1, 2, \ldots$, suppose that $f_i \in \overline{\mathcal{H}}_q^m(\lambda,\gamma)$, where

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{i,n} z^n + \sum_{n=2}^{\infty} \overline{b}_{i,n} \overline{z}^n.$$

Then, by Theorem 2.2

$$\sum_{n=2}^{\infty} \frac{([n]_q - \gamma\lambda)\Phi_q(n,m)}{(1-\gamma)} a_{i,n} + \sum_{n=1}^{\infty} \frac{([n]_q + \gamma\lambda)\Phi_q(n,m)}{(1-\gamma)} b_{i,n} \le 1.$$

For $\sum\limits_{i=1}^{\infty} t_i = 1$, $0 \le t_i \le 1$, the convex combinations of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{i,n} \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{i,n} \right) \overline{z}^n.$$

Using the inequality (2.5), we obtain

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{([n]_q - \gamma\lambda)\Phi_q(n,m)}{1-\gamma} \left(\sum_{i=1}^{\infty} t_i a_{i,n} \right) + \sum_{n=1}^{\infty} \frac{([n]_q + \gamma\lambda)\Phi_q(n,m)}{1-\gamma} \left(\sum_{i=1}^{\infty} t_i b_{i,n} \right) \\ &\quad = \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{([n]_q - \gamma\lambda)\Phi_q(n,m)}{1-\gamma} a_{i,n} + \sum_{n=1}^{\infty} \frac{([n]_q + \gamma\lambda)\Phi_q(n,m)}{1-\gamma} b_{i,n} \right) \\ &\quad \le \sum_{i=1}^{\infty} t_i = 1, \end{aligned}$$

and therefore $\sum_{i=1}^{\infty} t_i f_i \in \overline{\mathcal{H}}_q^m(\lambda, \gamma)$. $\square$

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