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ALMOST CONFORMAL RICCI SOLITONS ON f-KENMOTSU MANIFOLDS

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Abstract. The object of the present paper is to study the ϕ-Ricci symmetric, locally ϕ-Ricci symmetric and cyclic Ricci parallel three-dimensional f-Kenmotsu manifold bearing the function f being constant. We have considered almost conformal Ricci soliton on the three-dimensional f-Kenmostu manifold and deduced the condition for an almost conformal Ricci soliton, which becomes conformal Ricci soliton. Finally, examples on three-dimensional f-Kenmotsu manifold depending on nature of f are constructed to illustrate the results.

1. Introduction

In 1982, Hamilton [11] introduced the concept of Ricci flow and proved its existence. This concept was developed to answer Thurston's geometric conjecture which says that each closed 3-manifold admits a geometric decomposition. Hamilton also [11] classified all compact manifolds with a positive curvature in dimension four. The Ricci flow equation is given by

$$\frac{\partial}{\partial t} g = -2S$$

on a compact Riemannian manifold M with the Riemannian metric g. A self-similar solution to the Ricci flow (see [26, 28]) is called a Ricci soliton [12] if it moves only by a one parameter family of diffeomorphism and scaling. The Ricci soliton equation is given by

$$\pounds_V g + 2S = 2\lambda g, \tag{1.1}$$

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where $\pounds$ is the Lie derivative, g is a Riemannian metric, S is the Ricci tensor, V is a complete vector field on M and λ is a scalar. The Ricci soliton is said to be shrinking, steady, and expanding according as λ is negative, zero, and positive, respectively.

A Ricci soliton with Killing V is reduced to the Einstein equation. During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians (see [1, 29, 31]). It has become more important after Perelman's application of Ricci solitons to solve the long standing Poincare conjecture posed in 1904. Thereafter Ricci solitons have been studied by various authors such as [7, 15, 16, 13, 14].

Recently Fischer developed the concept of conformal Ricci flow [9], which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on a smooth closed connected oriented n-manifold M is defined by the following equation [9]:

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg, \qquad r(g) = -1, \tag{1.2}$$

where a scalar p is a nondynamical field (time dependent scalar field), $r(g)$ is the scalar curvature of the manifold, and n is the dimension of manifold. The conformal Ricci flow equations are analogous to the Navier–Stokes equations of fluid mechanics.

In 2015, Basu and Bhattacharyya [2] introduced the notion of conformal Ricci soliton equation as

$$\pounds_V g + 2S + \left[2\lambda - (p + \frac{2}{n})\right] g = 0, \tag{1.3}$$

where λ is constant. This equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

The concept of Ricci almost soliton was first introduced by Pigola et al. [23]. Sharma has also done an excellent work in Ricci almost soliton [26]. A Riemannian manifold (M^n, g) admits a Ricci almost soliton [6], if there exist a complete vector field X and a smooth soliton function $\lambda : M^n \to \mathbb{R}$ satisfying,

$$R_{ij} + \frac{1}{2}(X_{ij} + X_{ji}) = \lambda g_{ij},$$

where R_{ij} and $X_{ij} + X_{ji}$ stand for the Ricci tensor and the Lie derivative $(\pounds_X g)$ in local coordinates, respectively.

A gradient Ricci soliton on a Riemannian manifold (M^n, g) is defined by [3]

$$S + \nabla\nabla h = \nu\, g,$$

for some constant ν and for a smooth function h on M, where h is called a potential function of the Ricci soliton and ∇ is the Levi–Civita connection on M. In particular a gradient shrinking Ricci soliton satisfies

$$S + \nabla\nabla h - \frac{1}{2\tau}\, g = 0,$$

where $\tau = T - t$ and T is the maximal time of the soliton. Again if the vector field of conformal Ricci soliton is the gradient of a function h, then we call it as a conformal gradient shrinking Ricci soliton [3] and such equation is

$$S + \nabla\nabla h = (\frac{1}{2\tau} - \frac{2}{n} - p)\, g,$$

where h is the Ricci potential function.

The present paper is to study of the three-dimensional f-Kenmotsu manifold. After introduction, section 2 is concerned with some preliminaries. In section 3, we have proved that ϕ-symmetry and ϕ-Ricci symmetry are equivalent in the said manifold with a certain condition. Section 4 deals with the study of cyclic Ricci parallel three-dimensional f-Kenmotsu manifold. In section 5, conformal Ricci soliton on the said manifold have been studied. Also in section 6, we have considered the application of torqued vector field to conformal Ricci soliton. Finally, in the last section 7, some interesting examples of three-dimensional regular f-Kenmotsu manifold are constructed to illustrate the results.

2. Preliminaries

An odd dimensional smooth manifold M is said to be an *almost contact metric manifold*, if there exist a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η, and a Riemannian metric g on M such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \eta(X) = g(X,\xi), \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any vector fields $X, Y \in \chi(M)$. Such a manifold of dimension $2n+1$ is denoted by $M^{2n+1}(\phi,\xi,\eta,g)$; see [4]. Also $M^{2n+1}(\phi,\xi,\eta,g)$ is called an f-Kenmotsu manifold if the covariant differentiation of ϕ satisfies

$$(\nabla_X \phi)Y = f\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.3}$$

where $f \in C^\infty(M)$ is such that $df \wedge \eta = 0$ (see [18, 21]). If $f = \beta$ is nonzero constant, then the manifold is a β-Kenmotsu manifold [17]. 1-Kenmotsu manifold is a Kenmotsu manifold (see [18, 24]). If $f = 0$, then the manifold is cosymplectic [17]. An f-Kenmotsu manifold is said to be regular if $f^2 + f' \neq 0$, where $f' = \xi f$. For an f-Kenmotsu manifold, it follows from (2.3) that

$$\nabla_X \xi = f\{X - \eta(X)\,\xi\}. \tag{2.4}$$

The condition $df \wedge \eta = 0$ holds if dim. $M \geq 5$, in general and does not hold if dim. $M = 3$ [22]. In a three-dimensional Riemannian manifold, we have

$$\begin{aligned} R(X,Y)Z \;=\;& g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ & - S(X,Z)Y - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{2.5}$$

In a three-dimensional f-Kenmotsu manifold, we have

$$\begin{aligned} R(X,Y)Z \;=\;& \left(\frac{r}{2} + 2f^2 + 2f'\right)(X \wedge Y)Z \\ & - \left(\frac{r}{2} + 3f^2 + 3f'\right)\{\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z\} \end{aligned} \tag{2.6}$$

$$S(X,Y) = \left(\frac{r}{2} + f^2 + f'\right) g(X,Y) - \left(\frac{r}{2} + 3f^2 + 3f'\right) \eta(X)\eta(Y), \tag{2.7}$$

$$QX = \left(\frac{r}{2} + f^2 + f'\right) X - \left(\frac{r}{2} + 3f^2 + 3f'\right) \eta(X)\xi, \tag{2.8}$$

where r is the scalar curvature of M; see [21]. From (2.6) and (2.7), we obtain

$$R(X,Y)\xi = -\left(f^2 + f'\right)\left[\eta(Y)X - \eta(X)Y\right], \tag{2.9}$$

$$S(X,\xi) = -2\left(f^2 + f'\right)\eta(X), \tag{2.10}$$

$$S(\xi,\xi) = -2\left(f^2 + f'\right), \tag{2.11}$$

$$Q\xi = -2\left(f^2 + f'\right)\xi. \tag{2.12}$$

As a consequence of (2.4), we also have

$$(\nabla_X \eta)(Y) = fg(\phi X, \phi Y). \tag{2.13}$$

Also from (2.10) it follows that

$$S(\phi X, \phi Y) = S(X,Y) + 2\left(f^2 + f'\right)\eta(X)\eta(Y) \tag{2.14}$$

for all vector fields X and Y.

A nowhere vanishing vector field τ on a Riemannian or pseudo Riemannian manifold M is called torse-forming if

$$\nabla_X \tau = \varphi X + \alpha(X)\tau, \tag{2.15}$$

where φ is a function, α is a 1-form. The vector field τ is called concircular (see [8, 20, 30, 32, 33]) if the 1-form α vanishes identically. The vector field τ is called concurrent if the 1-form α vanishes identically and the function $\varphi = 1$ (see [25, 34]). The vector field τ is called recurrent if the function $\varphi = 0$. Finally if $\varphi = \alpha = 0$, then the vector field τ is called a parallel vector field. The nowhere zero vector field τ is called a torqued vector field if it satisfies

$$\nabla_X \tau = \varphi X + \alpha(X)\tau, \qquad \alpha(\tau) = 0, \tag{2.16}$$

where the function φ is called the torqued function and 1-form α is called the torqued form of τ; see [8].

A pseudo Riemannian manifold (M,g) is called a quasi-Einstein manifold if

$$S = pg + q\alpha \otimes \alpha, \tag{2.17}$$

where p and q are functions and α is a 1-form. A pseudo Riemannian manifold (M,g) is called almost quasi-Einstein manifold if

$$S = pg + q(\beta \otimes \mu + \mu \otimes \beta), \tag{2.18}$$

where p and q are functions and β and μ are 1-forms.

An f-Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be ϕ-symmetric if its curvature tensor R bears the condition

$$\phi^2((\nabla_X R)(Y,Z)W) = 0 \tag{2.19}$$

for all vector fields X, Y, Z, W; see [27]. In particular, if X, Y, Z, W are orthogonal to ξ, then $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be locally ϕ-symmetric.

An f-Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be ϕ-Ricci symmetric if its Ricci-operator Q bears the condition

$$\phi^2((\nabla_X Q)(Y)) = 0 \tag{2.20}$$

for all vector fields X and Y. If X and Y are orthogonal to ξ, then $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be locally ϕ-Ricci symmetric.

It may be noted that ϕ-symmetry implies ϕ-Ricci symmetry, but the converse is not valid in general.

3. ϕ-Ricci Symmetric three-dimensional f-Kenmotsu manifold

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be ϕ-Ricci symmetric f-Kenmotsu manifold. As a consequence of (2.1), we get from (2.20) that

$$-g((\nabla_X Q)(Y),Z) + S(\nabla_X Y,Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0. \tag{3.1}$$

Setting $Y=\xi$ in (3.1), we get

$$-g((\nabla_X Q)(\xi),Z) + S(\nabla_X \xi,Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{3.2}$$

Making use of (2.4) and (2.8) in (3.2), we get

$$\begin{aligned} -g(-2(2f(Xf) + (Xf^{'})\xi - 2(f^2+f^{'})(\nabla_X\xi),Z)) \\ +S(\nabla_X\xi,Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \end{aligned} \tag{3.3}$$

Substituting $X=\phi X, Z=\phi Z$ in (3.3), we obtain

$$S(\phi X,\phi Z) = -2\left(f^2+f^{'}\right)\ g(\phi X,\phi Z), \quad f\neq 0. \tag{3.4}$$

In view of (2.14) and (3.4), we have

$$S(X,Z) = -2\left(f^2+f^{'}\right)g(X,Z). \tag{3.5}$$

From (2.5) and (3.5), we get

$$R(X,Y)Z = -\left(f^2+f^{'}\right)\left[g(Y,Z)X - g(X,Z)Y\right]. \tag{3.6}$$

Thus for f being constant, it follows that

$$\phi^2((\nabla_X R)(Y,Z)W) = 0.$$

This leads to the following.

Theorem 3.1. *On a three-dimensional noncosymplectic f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$, ϕ-symmetric and ϕ-Ricci symmetric are equivalent provided f is constant.*

Again differentiating (2.8) covariantly along X, we obtain

$$\begin{aligned} (\nabla_X Q)(Y) &= \frac{1}{2}\{dr(X)Y - dr(X)\eta(Y)\xi\} + (2f(Xf)) \\ &\quad +(Xf^{'})Y - (6f(Xf) + 3(Xf^{'})\eta(Y)\xi \\ &\quad -(3f^2+3f^{'})\{(\nabla_X\eta)(Y)\xi + \eta(Y)(\nabla_X\xi)\}. \end{aligned} \tag{3.7}$$

Taking ϕ^2 on both side of (3.7) and keeping in mind (2.1), we get

$$\begin{aligned}\phi^2((\nabla_X Q)(Y)) &= \frac{1}{2}\{(dr(X)+2(2f(Xf))+(Xf^{'})\{-Y\\ &\quad +\eta(Y)\xi\}-(3f^2+3f^{'})\eta(Y)\phi^2(\nabla_X\xi).\end{aligned}$$

Thus if Y is orthogonal to ξ, then above equation implies that

$$\phi^2((\nabla_X Q)(Y)) = \frac{1}{2}dr(X)-(2f(Xf))+(Xf^{'})Y. \tag{3.8}$$

In particular, if f is constant, then from (3.8), we have

$$\phi^2\left((\nabla_X Q)(Y)\right) = \frac{1}{2}dr(X).$$

Thus we are in a condition to state the following.

Theorem 3.2. *A three-dimensional noncosymplectic f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$ with f being constant, is locally ϕ-Ricci symmetric if and only if the scalar curvature is constant.*

Also in [35], it is proved that if in a three-dimensional noncosymplectic f-Kenmotsu manifold, the Ricci tensor S is η-parallel, then the scalar curvature is constant, provided f is constant. Thus in view of (2.5), a three-dimensional noncosymplectic f-Kenmotsu manifold bearing η-parallel Ricci tensor is an η-Einstein manifold. Conversely, if the f-Kenmotsu manifold is η-Einstein, then $(\nabla_X S)(\phi Y,\phi Z)=0$. Thus we can state the following.

Theorem 3.3. *In a three-dimensional noncosymplectic f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$ with f being constant, the Ricci tensor is η-parallel if and only if the manifold is η-Einstein.*

Again from the equation (2.7), we have

$$\begin{aligned}(\nabla_W S)(\phi X,\phi Y) &= \left(\frac{dr(W)}{2}+2f(Wf)+(Wf^{'})\right)\{g(X,Y)\\ &\quad -\eta(X)\eta(Y)\}+\left(\frac{r}{2}+f^2+f^{'}\right)\{-fg(W.X)\eta(Y)\\ &\quad +fg(W,Y)\eta(X)\}.\end{aligned} \tag{3.9}$$

As a consequence of (3.9), we are in a condition to state the following.

Theorem 3.4. *In a three-dimensional noncosymplectic f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$ with f being constant, the Ricci tensor is η-parallel if and only if the scalar curvature is $r=-2f^2$ being constant.*

From Theorems 3.2 and 3.4, we can state the following.

Theorem 3.5. *In a three-dimensional noncosymplectic f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$ with f being constant, the Ricci tensor is η-parallel if and only if the manifold is locally ϕ-Ricci symmetric.*

4. Cyclic Ricci Parallel three-dimensional f-Kenmotsu manifold

Gray [10] introduced two classes of Riemannian manifold that determined by covariant derivative of Ricci tensor. The class A consisting of all Riemannian manifolds whose Ricci tensor S is of Codazzi type, that is,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z),$$

whereas class B consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, that is,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y) = 0. \tag{4.1}$$

A Riemannian manifold is called cyclic Ricci parallel if the Ricci tensor is nonzero and bears the condition (4.1). There are certain hypersurfaces bearing the condition (4.1) [19]. It is clear from (4.1) that the scalar curvature is constant. Taking covariant derivative of (2.5) along Z and using (2.13), we get

$$\begin{aligned}(\nabla_Z S)(X,Y) &= \left\{\frac{dr(Z)}{2} + 3f(Zf) + (Zf^{'})\right\} g(X,Y) - \{\frac{dr(Z)}{2} \\ &\quad +6f(Zf) + 3(Zf^{'})\}\eta(X)\eta(Y) - f\big(\frac{r}{2} + 3f^2 \\ &\quad +3f^{'}\big)\{g(\phi Z,\phi X)\eta(Y) + g(\phi Z,\phi Y)\eta(X)\}.\end{aligned} \tag{4.2}$$

From (4.2), it is obvious that, for $r = -6f^2$ being constant, the condition (4.1) is satisfied. Thus we are in a position to state the following.

Theorem 4.1. *A three-dimensional noncosymplectic f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$ with f being constant, satisfies cyclic parallel Ricci tensor if and only if the scalar curvature is constant.*

Keeping in mind of Theorems 3.2 and 4.1, we can state the following results.

Theorem 4.2. *A three-dimensional noncosymplectic f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$ with f being constant, satisfies cyclic parallel Ricci tensor if and only if it is locally ϕ-Ricci symmetric.*

Again from Theorems 3.5 and 4.2, we have the following results.

Theorem 4.3. *A three-dimensional noncosymplectic f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$ with f being constant, satisfies cyclic parallel Ricci tensor if and only if it bears η-parallel Ricci tensor.*

5. Conformal Ricci soliton on f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$

Let (g,ξ,λ) be a conformal Ricci soliton on f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$. Making use of (2.4), we write $\pounds_\xi g$ in term of the Levi–Civita connection ∇, that is,

$$(\pounds_\xi g)(X,Y) = g(\nabla_X\xi,Y) + g(X,\nabla_Y\xi) = 2f\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{5.1}$$

From (1.3) and (5.1), we get

$$S(X,Y) = \beta_1 g(X,Y) - fg(X,Y) + f\eta(X)\eta(Y), \tag{5.2}$$

where $\beta_1 = \frac{1}{2}\left[2\lambda + \left(p + \frac{2}{3}\right)\right]$.
On the other hand, from (5.2), we have

$$QX = \beta_1 X - fX + f\eta(X)\xi. \tag{5.3}$$

Thus from (1.3), we obtain

$$S(X,Y) = [\beta_2 + \lambda - f]\, g(X,Y) + f\eta(X)\eta(Y), \tag{5.4}$$

where $\beta_2 = -\frac{1}{2}\left(p + \frac{2}{3}\right)$. Hence, we can state the following.

Theorem 5.1. *If a three-dimensional f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$ admits the conformal Ricci soliton (g,ξ,λ) or almost conformal Ricci soliton (g,ξ,λ), then the manifold is η-Einstein.*

Corollary 5.2. *If a three-dimensional cosymplectic manifold $M^3(\phi,\xi,\eta,g)$ admits the conformal Ricci soliton (g,ξ,λ) or almost conformal Ricci soliton (g,ξ,λ), then the manifold is Einstein.*

It is well known that $\nabla g = 0$ and since $2\lambda - \left(p + \frac{2}{3}\right)$, we get $\nabla(2\lambda - \left(p + \frac{2}{3}\right))g = 0$. Thus from (1.3), it follows that $\pounds_V g + 2S$ is parallel.

In [5] Calin and Crasmareanu proved that a symmetric parallel second order tensor in an f-Kenmotsu manifold is a constant multiple of the metric tensor on the set of which f is nonzero. Therefore we conclude that $\pounds_V g + 2S = \Phi(\text{say})$ is a constant multiple of the metric tensor g, that is,

$$(\pounds_V g + 2S)(X,Y) = \Phi(X,Y) = \Phi(\xi,\xi)g(X,Y),$$

where $\Phi(\xi,\xi)$ is given by

$$\Phi(\xi,\xi) = (\pounds_V g + 2S)(\xi,\xi) = -4(f^2 + f^{'}).$$

Then (1.3) reduces to

$$\pounds_V g + 2S + \left[2\lambda - (p + \frac{2}{n})\right] g = \left\{-4(f^2 + f^{'}) + 2\lambda - \left(p + \frac{2}{3}\right)\right\} g.$$

Again in view of (1.3), it gives $\lambda = \left[2(f^2 + f^{'}) + \frac{1}{2}\left(p + \frac{2}{3}\right)\right]$. Thus we are in a position to state the following.

Theorem 5.3. *If a three-dimensional regular f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$ with the symmetric $(0,2)$ tensor field $\pounds_V g + 2S = \Phi$ is parallel, then the conformal Ricci soliton (g,ξ,λ) is shrinking, steady, and expanding according as $2(f^2 + f^{'}) + \frac{1}{2}\left(p + \frac{2}{3}\right)$ is nonpositive , zero, and positive, respectively.*

In particular, we suppose that V is a pointwise collinear with ξ, that is, $V = \rho\xi$, where ρ is a function on $M^3(\phi,\xi,\eta,g)$. Then applying the property of Lie derivative and Levi–Civita connection the conformal Ricci soliton equation (1.3) reduces to

$$\begin{aligned} &\rho g(\nabla_X \xi, Y) + (X\rho)\eta(Y) + \rho g(\nabla_Y \xi, X) \\ &\quad +(Y\rho)\eta(X) + 2S(X,Y) - \left[2\lambda - (p + \frac{2}{3})\right] g(X,Y) = 0. \end{aligned} \tag{5.5}$$

Making use of (2.4), equation (5.5) yields

$$\begin{aligned}2\rho f g(X,Y) - 2f\rho\eta(X)\eta(Y) + (X\rho)\eta(Y) + (Y\rho)\eta(X) \\ +2S(X,Y) - \left[2\lambda - (p+\frac{2}{3})\right] g(X,Y) = 0.\end{aligned} \tag{5.6}$$

Replacing $Y = \xi$ in (5.6) and using (2.10), we have

$$(X\rho) + (\xi\rho)\eta(X) - 4(f^2 + f^{'})\eta(X) - \left[2\lambda - (p+\frac{2}{3})\right]\eta(X) = 0. \tag{5.7}$$

Again putting $X = \xi$ in (5.7), we get

$$(\xi\rho) = \frac{1}{2}\left[2\lambda - \left(p+\frac{2}{3}\right)\right] + 2\left(f^2 + f^{'}\right). \tag{5.8}$$

From (5.7) and (5.8), we obtain

$$(X\rho) = \left[-\frac{1}{2}\left\{2\lambda - \left(p+\frac{2}{3}\right)\right\} + 2(f^2+f^{'})\right]\eta(X) + \left[2\lambda - (p+\frac{2}{3})\right]\eta(X). \tag{5.9}$$

After straightforward calculation, this implies that, for $d\eta \neq 0$,
$-\frac{1}{2}\left[2\lambda - \left(p+\frac{2}{3}\right)\right] + 2\left(f^2+f^{'}\right) + \left[2\lambda - (p+\frac{2}{3})\right] = 0$
or $\lambda = \frac{1}{2}\left[-4(f^2+f^{'}) + \left(p+\frac{2}{3}\right)\right]$.
In view of the above equation and from (5.9), we get $X\rho = 0$ which means that ρ is constant. Thus from (5.6), we have

$$S(X,Y) = \frac{1}{2}\left[2\lambda - (p+\frac{2}{3})\right] g(X,Y) - \rho f g(X,Y) + \rho f\eta(X)\eta(Y). \tag{5.10}$$

As the above consequence, we are going to state the result in the following way.

Theorem 5.4. *If a three-dimensional regular f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$ admits the conformal Ricci soliton (g,V,λ) such that V is pointwise collinear with ξ, then V is a constant multiple of ξ and the underlying manifold is an η-Einstein manifold of the type* (5.10). *Also the conformal Ricci soliton (g,ξ,λ) is shrinking, steady, and expanding according as the value $(p+\frac{2}{3}) - 4(f^2+f^{'})$ is negative, zero, and positive, respectively.*

Corollary 5.5. *If a three-dimensional cosymplectic manifold $M^3(\phi,\xi,\eta,g)$ admits the conformal Ricci soliton (g,V,λ) such that V is pointwise collinear with ξ, then V is a constant multiple of ξ and the underlying manifold is Einstein. Also the conformal Ricci soliton (g,ξ,λ) becomes expanding or steady according as the conformal pressure $p \geq 0$.*

We suppose that λ is a smooth function for almost conformal Ricci soliton. Then applying exterior derivative in (5.9), we obtain

$$4\left(f^2+f^{'}\right) + \left[2\lambda - (p+\frac{2}{3})\right] = 0 \tag{5.11}$$

and $\lambda = 0$, if f is constant. This indicate that λ is a constant function. Thus from (5.9) and (5.11), we get ρ is constant. Therefore we can state the following.

Theorem 5.6. *If a three-dimensional regular f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$ admits the almost conformal Ricci soliton (g,V,λ) such that V is a pointwise collinear with ξ, then V is a constant multiple of ξ as well as λ becomes a constant function, that is, an almost conformal Ricci soliton becomes a conformal Ricci soliton provided f is constant.*

6. Application of torqued vector field to conformal Ricci soliton

In this section, we are going to discuss about the application of torqued vector field, that is, the potential field ξ is a torqued vector field τ for the conformal Ricci soliton. We suppose that the conformal Ricci soliton (g,ξ,λ), where the potential field ξ is a torqued vector field τ. From the definition of Lie-derivative, we have

$$\begin{aligned}(\pounds_\tau g)(X,Y) &= g(\nabla_X\tau,Y)+g(X,\nabla_Y\tau) \\ &= \alpha(X)g(\tau,Y)+\alpha(Y)g(\tau,X)\end{aligned} \tag{6.1}$$

for vector fields X and Y being tangent to M.
In view of (1.3) and (6.1), we get

$$S(X,Y)=\frac{1}{2}\left[2\lambda-(p+\frac{2}{3})\right]g(X,Y)-\frac{1}{2}\left[\alpha\otimes\gamma+\gamma\otimes\alpha\right]. \tag{6.2}$$

Here we denotes that the dual 1-form of τ by γ. Hence the manifold is an almost quasi-Einstein manifold.
Putting $X=Y=e_i$ in (6.2), where $\{e_i\}$ is the orthonormal basis of the tangent space TM, and summing over i, we get

$$r=\frac{3}{2}\left[2\lambda-(p+\frac{2}{3})\right]-1.$$

It is known that, for the conformal Ricci soliton $r=-1$, thus we get $\lambda=\frac{1}{2}[p+\frac{2}{3}]$. So we assert the following.

Theorem 6.1. *Let (g,ξ,λ) be a conformal Ricci soliton on a three-dimensional f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$. If the potential field ξ is a torqued vector field, then $M^3(\phi,\xi,\eta,g)$ is an almost quasi-Einstein manifold.*

Corollary 6.2. *Let (g,ξ,λ) be a conformal Ricci soliton on a three-dimensional f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$. If the potential field ξ is a torqued vector field, then such soliton is expanding if and only if the conformal pressure p vanishes.*

Also from (1.3), (2.7) and (6.1), we get

$$\begin{aligned}&(\pounds_\tau g+2S+[2\lambda-(p+\frac{2}{n})]g)(X,Y) \\ &\quad=[2(\frac{r}{2}+f^2+f')+(2\lambda-(p+\frac{2}{3}))]g(X,Y) \\ &\qquad-2(\frac{r}{2}+3f^2+3f'))\eta(X)\eta(Y)+\alpha(X)g(\tau,Y)+\alpha(Y)g(\tau,X).\end{aligned} \tag{6.3}$$

In particular, if $\lambda = 2f^2 + 2f' + \frac{1}{2}\left(p + \frac{2}{3}\right)$, equation (6.3) implies that

$$(\pounds_\tau g + 2S + [2\lambda - (p + \frac{2}{n})]g)(X,Y) \tag{6.4}$$
$$= \{r + 6(f^2 + f')\}\{g(X,Y) - \eta(X)\eta(Y)\} + \alpha(X)g(\tau,Y) + \alpha(Y)g(\tau,X).$$

It is clear from (6.4) that $M^3(\phi,\xi,\eta,g)$ bears the conformal Ricci soliton of the type $\left(g,\xi,\ 2f^2 + 2f' + \frac{1}{2}\left(p + \frac{2}{3}\right)\right)$ if and only if the torqued potential field τ is a concircular vector field provided $r = -6(f^2+f')$. Thus, we can state the following.

Theorem 6.3. *Let (g,τ,λ) be a conformal Ricci soliton on a three-dimensional f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$, where τ is a torqued vector field. Then $\left(g,\tau,2f^2 + 2f' + \frac{1}{2}\left(p + \frac{2}{3}\right)\right)$ is a conformal Ricci soliton on M if and only if the torqued potential field τ is a concircular vector field provided $r = -6(f^2 + f')$.*

Corollary 6.4. *If the potential field ξ of a conformal Ricci soliton (g,ξ,λ) is a torqued vector field τ in a three-dimensional cosymplectic manifold $M^3(\phi,\xi,\eta,g)$, then $\lambda = \frac{1}{2}\left(p + \frac{2}{3}\right)$ and such soliton is always shrinking if and only if te torqued potential field τ is a concircular vector field provided the scalar curvature r vanishes.*

Corollary 6.5. *If a three-dimensional regular f-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$ admits a conformal gradient shrinking Ricci soliton, then the manifold is Einstein provided f is constant.*

7. Example

7.1. **When f is a constant function.** Let $M^3 = \{(u,v,w) \in \mathbb{R}^3 : u, v, z(\neq 0) \in \mathbb{R}\}$ be a Riemannian manifold, where (u,v,w) denotes the standard coordinates of a point in $\mathbb{R}^3$. Let us suppose that

$$e_1 = w\frac{\partial}{\partial u}, \quad e_2 = w\frac{\partial}{\partial v}, \quad e_3 = -w\frac{\partial}{\partial w}$$

are linearly independent vector fields at each point of M^3 and therefore it forms a basis for the tangent space $T(M^3)$. We also define the Riemannian metric g of the manifold M^3 as $g(e_i,e_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta and $i,j = 1,2,3$, and is given by

$$g = \frac{1}{w^2}\left[du \otimes du + dv \otimes dv + dw \otimes dw\right].$$

Let η be the 1-form having the significance

$$\eta(U) = g(U,e_3)$$

for any $U \in \Gamma(TM)$ and let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0.$$

By the linearity properties of ϕ and g, we can easily verify the following relations

$$\eta(e_3) = 1, \qquad \phi^2(U) = -U + \eta(U)e_3,$$
$$g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V)$$

for arbitrary vector fields $U, W \in T(M^3)$. This shows that $\xi = e_3$ and that the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M^3. If ∇ is the Levi–Civita connection with respect to the Riemannian metric g, then with the help of above, we can easily calculate that

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

Now we recall Koszul's formula as

$$\begin{aligned} 2g(\nabla_U V, W) &= U(g(V, W)) + V(g(W, X)) - W(g(U, V)) \\ &\quad -g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]) \end{aligned}$$

for arbitrary vector fields $U, V, W \in T(M^3)$. Making use of Koszul's formula, we get the following:

$$\begin{aligned} \nabla_{e_2} e_3 = e_2, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0, \\ \nabla_{e_1} e_3 = e_1, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = -e_3. \end{aligned}$$

From the above calculation, it is clear that M^3 satisfies the condition $\nabla_U \xi = f\{U - \eta(U)\xi\}$ for $e_3 = \xi$, where $f = 1 = \alpha$ is nonzero constant. Thus we conclude that M^3 leads to an f-Kenmotsu (Kenmotsu) manifold. Also $f^2 + f'$ is nonzero. That implies that M^3 is a three-dimensional regular f-Kenmotsu manifold.

It is know that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Therefore, we find the component of curvature tensor and Ricci tensor as follows:

$$\begin{aligned} R(e_2, e_3)e_3 &= -e_2, & R(e_3, e_2)e_2 &= -e_3, \\ R(e_1, e_3)e_3 &= -e_1, & R(e_3, e_1)e_1 &= -e_3, \\ R(e_1, e_2)e_2 &= -e_1, & R(e_1, e_2)e_3 &= 0, \\ R(e_2, e_1)e_1 &= -e_2, & R(e_3, e_1)e_2 &= 0, \\ S(e_1, e_1) &= -2, & S(e_2, e_2) &= -2, \\ S(e_3, e_3) &= -2, & S(\phi e_1, \phi e_1) &= -2, \\ S(\phi e_2, \phi e_2) &= -2, & S(\phi e_3, \phi e_3) &= 0, \\ S(\phi e_i, \phi e_j) &= 0 \end{aligned}$$

for all $i, j = 1, 2, 3$ $(i \neq j)$. From the above consequence, it is clear that $\phi^2((\nabla_U Q)(V)) = 0$ for all vector fields $U, V \in T(M)$. Hence M is locally ϕ-Ricci symmetric. Also $r = -6$. Therefore the scalar curvature is constant, that is, Theorem 3.2 is verified. Moreover, $(\nabla_X S)(\phi e_i, \phi e_j) = 0$ for $X \in T(M), i, j = 1, 2, 3$. So M^3 is η-parallel, cyclic parallel, and η-Einstein.

7.2. **When f is a smooth function.** We consider the three-dimensional manifold $M = \{(u, v, w) \in \mathbb{R}^3, w \neq 0\}$, where (u, v, w) is the standard coordinate in

$\mathbb{R}^3$. Let e_1, e_3, e_3 be linearly independent vector fields at each point of M, given by

$$e_1 = \frac{1}{w}\frac{\partial}{\partial u}, \quad e_2 = \frac{1}{w}\frac{\partial}{\partial v}, \quad e_3 = -\frac{\partial}{\partial w}.$$

Let g be the Riemannian metric such that

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

and given by

$$g = w^2 \left[du \otimes du + dv \otimes dv + \frac{1}{w^2} dw \otimes dw \right].$$

Let η be the 1-form having the significance

$$\eta(U) = g(U, e_3)$$

for any $U \in \Gamma(TM)$ and let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

Making use of the linearity of ϕ and g, we have

$$\eta(e_3) = 1, \quad \phi^2(U) = -U + \eta(U)e_3$$
$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V)$$

for any $U, W \in \Gamma(TM)$. Now we can easily calculate

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -\frac{1}{w}e_2, \quad [e_2, e_3] = -\frac{1}{w}e_1.$$

Making use of Koszul's formula, we get the following:

$$\begin{aligned}
\nabla_{e_2} e_3 &= -\frac{1}{w}e_2, & \nabla_{e_2} e_2 &= \tfrac{1}{w}e_3, & \nabla_{e_2} e_1 &= 0,\\
\nabla_{e_3} e_3 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_1 &= 0,\\
\nabla_{e_1} e_3 &= -\frac{1}{w}e_1, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_1 &= \frac{1}{w}e_3.
\end{aligned}$$

Consequently it is clear that M satisfies the condition $\nabla_U \xi = f\{U - \eta(U)\xi\}$ for $e_3 = \xi$, where $f = -\frac{1}{w}$. Thus we conclude that M leads to an f-Kenmotsu manifold. Also $f^2 + f' = \frac{2}{w^2}$ is nonzero. That implies M is a three-dimensional regular f-Kenmotsu manifold.

7.3. **Example.** We consider the three-dimensional manifold $M = \{(u, v, w) \in \mathbb{R}^3, w \neq 0\}$, where (u, v, w) are the standard coordinates in $\mathbb{R}^3$. Let (e_1, e_3, e_3) be linearly independent vector fields at each point of M, given by

$$e_1 = \sin^2 w\frac{\partial}{\partial u}, \quad e_2 = \sin^2 w\frac{\partial}{\partial v}, \quad e_3 = \sin w\frac{\partial}{\partial w}$$

being linearly independent at each point of M. Let g be the Riemannian metric defined

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$$

and given by

$$g = \sin^4 w \left[du \otimes du + dv \otimes dv + \frac{1}{\sin^2 w} dw \otimes dw \right].$$

Let η be the 1-form having the significance

$$\eta(U) = g(U, e_3)$$

for any $U \in \Gamma(TM)$ and let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

Making use of the linearity of ϕ and g, we have

$$\eta(e_3) = 1, \quad \phi^2(U) = -U + \eta(U)e_3$$
$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V)$$

for any $U, W \in \Gamma(TM)$. Now we can easily calculate

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -2\cos w\, e_2, \quad [e_2, e_3] = -2\cos w\, e_1.$$

Making use of Koszul's formula, we get the following:

$$\begin{aligned} \nabla_{e_2} e_3 &= -2\cos w e_2, & \nabla_{e_2} e_2 &= 2\cos w e_3, & \nabla_{e_2} e_1 &= 0, \\ \nabla_{e_3} e_3 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_1 &= 0, \\ \nabla_{e_1} e_3 &= -2\cos w e_1, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_1 &= 2\cos w e_2. \end{aligned}$$

Consequently it is clear that M satisfies the condition $\nabla_U \xi = f\{U - \eta(U)\xi\}$ for $e_3 = \xi$, where$f = -2\cos w$. Thus we conclude that M leads to an f-Kenmotsu manifold. Also $f^2 + f^{'} = 2(2\cos^2 w + \sin w)$ is nonzero, which implies M is a three-dimensional regular f-Kenmotsu manifold.
It is know that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Therefore, we find the component of curvature tensor as follows:

$$\begin{aligned} R(e_2, e_3)e_3 &= -2(\sin w + 2\cos^2 w)e_2, & R(e_2, e_3)e_1 &= 0, \\ R(e_3, e_2)e_2 &= -2(\sin w + 2\cos^2 w)e_3, & R(e_1, e_2)e_3 &= 0, \\ R(e_3, e_1)e_1 &= -2(\sin w + 2\cos^2 w)e_2, & R(e_3, e_1)e_2 &= 0, \\ R(e_1, e_3)e_3 &= -2(\sin w + 2\cos^2 w)e_1, & & \\ R(e_1, e_2)e_2 &= -4\cos^2 w e_1, & R(e_2, e_1)e_1 &= 4\cos^2 w e_3. \end{aligned}$$

From the above expressions of the curvature tensor, we evaluate the value of the Ricci tensor as follows:

$$\begin{aligned} S(e_1, e_1) &= -2(\sin w + 4\cos^2 w), \\ S(e_2, e_2) &= -2(\sin w + 2\cos^2 w), \\ S(e_3, e_3) &= -2(\sin w + 2\cos^2 w). \end{aligned} \tag{7.1}$$

Also the value of $r = -(6\sin w + 16\cos^2 w)$. We suppose that the conformal Ricci soliton (g, ξ, λ), where the potential field ξ is a torqued vector field τ. Then in view of (2.16) and (6.1), we get from (1.3) that

$$S(e_1, e_1) = -\tfrac{1}{2}\left\{2\lambda - \left(p + \tfrac{2}{3}\right)\right\}, \qquad S(e_2, e_2) = -\frac{1}{2}\left\{2\lambda - \left(p + \frac{2}{3}\right)\right\}, \tag{7.2}$$
$$S(e_3, e_3) = -\tfrac{1}{2}\left\{2\lambda - \left(p + \tfrac{2}{3}\right)\right\}.$$

In view of (7.1) and (7.2), we can easily calculate

$$\lambda = \frac{1}{3}\left(p + \frac{2}{3}\right) + 2\sin w + \frac{32}{6}\cos^2 w. \tag{7.3}$$

It is clear from (7.3) that if the conformal pressure p is zero at an equilibrium point, then g always expands the Ricci soliton. Hence Corollary 6.2 is verified.

8. Conclusion

As a generalization of Ricci soliton, Basu and Bhattacharyya [2] introduced the notion of conformal Ricci soliton. Also almost Ricci soliton is another generalization introduced by Pigola et al. [23]. The theory of above solitons plays an important role in mathematical physics. In this paper, we have studied conformal Ricci soliton and almost conformal Ricci soliton on f-Kenmotsu manifolds satisfying various conditions. The applications of torque vector field to such solitons are also considered. Finally, some examples have been constructed to illustrate the obtained results.

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