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INVARIANT SUBMANIFOLDS OF LP-SASAKIAN MANIFOLDS

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ABSTRACT. The object of the present paper is to study some geometric conditions for an invariant submanifold of an LP-Sasakian manifold to be totally geodesic. Further we consider concircular curvature tensor satisfying some geometric conditions of an invariant submanifold of an LP-Sasakian manifold to be totally geodesic. In extension, we build an example of LP-Sasakian manifold to verify our main result totally geodesic.

1. INTRODUCTION

In 1989, Matsumoto [15] defined the notion of Lorentzian para-Sasakian manifold. The same notion was independently defined by Mihai and Rosca [18] and obtained several results. In the modern analysis, the geometry of submanifold has turned into a subject of growing interest for its significant applications in applied mathematics and theoretical physics. For instance, the notion of invariant submanifold is used to study the properties of non-linear autonomous system; see [12]. Also the notion of geodesics plays an important role in the theory of relativity; see [17]. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Therefore totally geodesic submanifolds are also very much important in physical sciences. The study of the geometry of invariant submanifolds was introduced by Bejancu and Papaghuic [4] and again Papaghuic has worked on semi-invariant submanifold of LP-Sasakian manifold. In general, the geometry of an invariant submanifold inherits almost all properties of the ambient manifold.

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The generalization of LP-Sasakian manifold is (LCS)-manifold and it was investigated by Shaikh in 2003 [22]. A $(2n+1)$-dimensional Lorentzian manifold M is smooth connected para contact Hausdorff manifold with Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : TpM \times TpM \longrightarrow R$ is a nondegenerate inner product of signature $(-,+,\dots,-,+)$, where TpM denotes the tangent space of M at p and R is the real number space.

Lorentzian α-Sasakian manifold is the extended version of α-Sasakian manifold with Lorentzian metric, and it was first studied by the Yildiz [28] in 2005. In this article, we study the conformally flat, quasi conformally flat, and Wely semisymmetric and it is shown that they are locally isometric to a sphere. Further many geometers studied the Lorentzian α-Sasakian manifolds with different curvature tensors and different connections.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M is said to be a Lorentzian para-Sasakian manifold, if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η, and a Lorentzian metric g that satisfies [15, 16];

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi) = 0, \quad g(X,\xi) = \eta(X), \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X, Y, Z on M.

Also in an LP-Sasakian manifold, the following relations hold [15, 16]:

$$\begin{aligned} g(R(X,Y)Z,\xi) &= \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), & (2.3)\\ R(\xi,X)Y &= g(X,Y)\xi - \eta(Y)X, & (2.4)\\ R(X,Y)\xi &= \eta(X)Y - \eta(Y)X, & (2.5)\\ R(\xi,X)\xi &= X + \eta(X)\xi, & (2.6)\\ S(X,\xi) &= 2n\eta(X), & (2.7)\\ S(\phi X,\phi Y) &= S(X,Y) + 2n\eta(X)\eta(Y), & (2.8) \end{aligned}$$

for all vector fields X, Y, Z and where S is the Ricci tensor and R is the Riemannian curvature tensor.

Let $\tilde{M}$ be a submanifold of a $(2n+1)$-dimensional LP-Sasakian manifold M. We denote ∇ and $\tilde{\nabla}$ as the Levi–Civita connections of M and $\tilde{M}$ in that order. Then for each vector fields X and Y on M, the second fundamental form α is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X,Y). \tag{2.9}$$

Furthermore, for any section N of the normal bundle $T^{\perp}M$, we have

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.10}$$

where $\nabla^{\perp}$ denotes the normal bundle connection on M. The second fundamental form α and shape operator A_N are related by

$$g(A_N X, Y) = g(\alpha(X,Y), N). \tag{2.11}$$

A submanifold $\tilde{M}$ is said to be totally geodesic if $\alpha(X,Y) = 0$ for any vector fields X and Y on M, which means that the geodesics in M are also geodesics in $\tilde{M}$. A submanifold $\tilde{M}$ is said to be semiparallel [10] (resp. 2-semi-parallel, see [2]) if

$$\tilde{R}(X,Y)\cdot\alpha = 0, \quad (resp. \quad \tilde{R}(X,Y)\cdot\tilde{\nabla}\alpha = 0) \;\text{ for all }\; X,Y \in M. \tag{2.12}$$

On a Riemannian manifold M, for a $(0,k)$-type tensor field T $(k > 1)$ and a $(0,2)$-type tensor field E, we denote the $Q(E,T)$ as a $(0,k+2)$-type tensor field [25], defined as follows:

$$\begin{aligned} Q(E,T)(X_1,X_2,\ldots,X_k;X,Y) \;=\; & -T((X\wedge_E Y)X_1,X_2,\ldots,X_k) \\ & -T(X_1,(X\wedge_E Y)X_2,\ldots,X_k) \\ & -\cdots - T(X_1,X_2,\ldots,X_{k-1},(X\wedge_E Y)X_k), \end{aligned} \tag{2.13}$$

where $(X\wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y$. Moreover, a submanifold $\tilde{M}$ is said to be pseudo-parallel [3] if

$$\tilde{R}(X,Y)\cdot\alpha = f_Q(g,\alpha) \tag{2.14}$$

holds for any vector fields X and Y tangent to M and a smooth function f. Similarly, a submanifold $\tilde{M}$ is said to be 2-pseudo-parallel if $R(X,Y)\cdot\tilde{\nabla}\alpha = f_Q(S,\tilde{\nabla}\alpha)$, and it is Ricci generalized pseudo-parallel [19] if $\tilde{R}(X,Y)\cdot\alpha = f_Q(S,\alpha)$ for any $X,Y \in M$.

3. Invariant submanifold of LP-Sasakian manifold

Let $f : \tilde{M} \to M$ be an immersion. If $\tilde{M}$ is an immersed manifold of M and $\phi(f(T_p\tilde{M})) \subset T_p\tilde{M}$, for any $p \in \tilde{M}$, then $\tilde{M}$ is called an invariant submanifold of M.

Let $\tilde{M}$ be a submanifold of an LP-Sasakian manifold M. The submanifold $\tilde{M}$ of M is said to be invariant if the structure vector field ξ is tangent to $\tilde{M}$, at every point of $\tilde{M}$, and ϕX is tangent to $\tilde{M}$ for any vector field X tangent to $\tilde{M}$, at every point on $\tilde{M}$, that is, $\phi(T_p\tilde{M}) \subset T_p\tilde{M}$ at every point on $\tilde{M}$; see [21]. Every invariant submanifold of an LP-Sasakian manifold satisfies $\alpha(X,\xi) = 0$ for any vector field X on M.
In [20] Özgür and Murathan proved the following lemma.

Lemma 3.1. *Let $\tilde{M}$ be an invariant submanifold of an LP-Sasakian manifold M. Then the following relations hold on $\tilde{M}$:*

$$\nabla_X\xi = \phi X, \tag{3.1}$$

$$\alpha(X,\xi) = 0, \tag{3.2}$$

$$(\nabla_X\phi)Y = g(X,Y)\xi + \eta(X)Y + 2\eta(X)\eta(Y)\xi, \tag{3.3}$$

$$\alpha(X,\phi Y) = \phi\alpha(X,Y). \tag{3.4}$$

Now we can state the following statement.

Theorem 3.2. *An invariant submanifold of an LP-Sasakian manifold is an LP-Sasakian manifold.*

Kowalczyk [14] studied the semi-Riemannian manifold satisfying $Q(S, R) = 0$ and $Q(S, g) = 0$, where S and R are the Ricci tensor and curvature tensor in that order. Also De and Majhi [8] investigated the invariant submanifolds of Kenmotsu manifolds and showed that geometric conditions of invariant submanifolds of Kenmotsu manifolds are totally geodesic. Özgür and Murathan [20] developed the invariant submanifolds of Lorentzian para-Sasakian manifolds of semiparallel and 2-semiparallel conditions. De and Samui [9] studied the recurrent, bi-recurrent, pseudo-parallel, generalized Ricci pseudo-parallel, $C(X,Y)\cdot\alpha = f_Q(g,\alpha)$, and $C(X,Y)\cdot\alpha = f_Q(S,\alpha)$ on invariant submanifolds of LP-Sasakian manifolds. Recently, Hu and Wang [13] obtained the geometric conditions of invariant submanifolds of a trans-Sasakian manifolds to be totally geodesic. Invariant submanifolds of different manifolds were studied by many geometers such as [1, 4–9, 11, 13, 20, 21, 23, 24]. Motivated by the above studies, we make an attempt to study the invariant submanifolds of LP-Sasakian manifolds satisfying some geometric conditions such has $Q(\alpha, R) = 0$, $Q(S,\alpha) = 0$, $Q(S, \tilde{\nabla}\alpha) = 0$, $Q(S, \tilde{R}\cdot\alpha) = 0$, $Q(g, C\cdot\alpha) = 0$, and $Q(S, C\cdot\alpha) = 0$, and we show that they are all totally geodesic.

First we have prove the invariant submanifold of LP-Sasakian manifold satisfying $Q(\alpha, R) = 0$. Consider an invariant submanifold of LP-Sasakian manifold satisfies $Q(\alpha, R) = 0$.

Then

$$\begin{aligned} Q(\alpha\cdot R)(X,Y,Z;U,V) &= 0 = ((U\wedge_\alpha V)\cdot R)(X,Y)Z \\ &= -R((U\wedge_\alpha V)X,Y)Z - R(X,(U\wedge_\alpha V)Y)Z \\ &\quad -R(X,Y)(U\wedge_\alpha V)Z. \end{aligned} \tag{3.5}$$

Here $(U\wedge_\alpha V)$ is the endomorphism and it is denoted by

$$(U\wedge_\alpha V)W = \alpha(V,W)U - \alpha(U,W)V. \tag{3.6}$$

Taking (3.6) in (3.5), we get

$$\begin{aligned} &-\ \alpha(V,X)R(U,Y)Z + \alpha(U,X)R(V,Y)Z \\ &-\ \alpha(V,Y)R(X,Y)Z + \alpha(U,Y)R(X,V)Z \\ &-\ \alpha(V,Z)R(X,Y)Z + \alpha(U,Z)R(X,Y)V = 0. \end{aligned} \tag{3.7}$$

Setting $V = Z = \xi$ in (3.7) and with the help of (3.2), one can get

$$\alpha(U,X)R(\xi,Y)\xi + \alpha(U,Y)R(X,\xi)\xi = 0. \tag{3.8}$$

Using (2.4) and then contracting over Y and W, (3.8) yields

$$\alpha(U,X) = 0.$$

Thus the manifold is totally geodesic. Conversely, if $\alpha(X,Y) = 0$, for any vector fields X and Y on M, then it follows from (3.7) that $Q(\alpha, R) = 0$.

Therefore, we can state the following.

Theorem 3.3. *An invariant submanifold of LP-Sasakian manifold satisfies $Q(\alpha, R) = 0$ if and only if it is totally geodesic.*

By using the above result, we can state the following corollaries.

Corollary 3.4. *An invariant submanifold of an $(LCS)_{(2n+1)}$ manifold satisfies $Q(\alpha, R) = 0$ if and only if it is totally geodesic, provided $\alpha^2 \neq \rho$.*

Corollary 3.5. *An invariant submanifold of a Lorentzian α-Sasakian manifold satisfies $Q(\alpha, R) = 0$ if and only if it is totally geodesic.*

Next we prove the invariant submanifold of LP-Sasakian manifold satisfying $Q(S, \alpha) = 0$. Consider the invariant submanifold $\tilde{M}$ of LP-Sasakian manifold M satisfies $Q(S, \alpha) = 0$. Then

$$\begin{aligned} 0 &= Q(S,\alpha)(X,Y;U,V) \\ &= -\alpha((U \wedge_S V)X, Y) - \alpha(X, (U \wedge_S V)Y), \end{aligned} \tag{3.9}$$

where $(U \wedge_S V)W$ is defined as

$$(U \wedge_S V)W = S(V,W)U - S(U,W)V. \tag{3.10}$$

In view of (3.10), (3.9) gives

$$\begin{aligned} &- \quad S(V,X)\alpha(U,Y) + S(U,X)\alpha(V,Y) \\ &- \quad S(V,Y)\alpha(X,U) + S(U,Y)\alpha(X,V) = 0. \end{aligned} \tag{3.11}$$

Inserting $U = Y = \xi$ in (3.11) and by virtue of (3.2), we get

$$-2n\alpha(X,V) = 0. \tag{3.12}$$

It follows that $\alpha(X, V) = 0$, for any vector fields X and V on M. Thus $\tilde{M}$ is totally geodesic.
Conversely, if $\tilde{M}$ is totally geodesic $\alpha(X, V) = 0$ for any vector fields X and V on M, then

$$Q(S,\alpha)(X,Y;U,V) = -\alpha((U \wedge_S V)X, Y) - \alpha(X, (U \wedge_S V)Y). \tag{3.13}$$

With the help of (3.10) and (3.12), we have

$$\begin{aligned} Q(S,\alpha)(X,Y;U,V) &= -S(V,X)\alpha(U,Y) + S(U,X)\alpha(V,Y) \\ &\quad -S(V,Y)\alpha(X,U) + S(U,Y)\alpha(X,V) = 0. \end{aligned}$$

Thus we can state the following.

Theorem 3.6. *An invariant submanifold of an LP-Sasakian manifold satisfies $Q(S, \alpha) = 0$ if and only if it is totally geodesic.*

In view of the above theorem, we can state the following corollaries.

Corollary 3.7. *An invariant submanifold of an $(LCS)_{2n+1}$ manifold satisfies $Q(S, \alpha) = 0$ if and only if it is totally geodesic, provided $\alpha^2 \neq \rho$.*

Corollary 3.8. *An invariant submanifold of a Lorentzian α-Sasakian manifold satisfies $Q(S, \alpha) = 0$ if and only if it is totally geodesic.*

Further, we prove that all invariant submanifolds of LP-Sasakian manifolds satisfy $Q(S, \tilde{\nabla}\alpha) = 0$ and $Q(S, \tilde{R} \cdot \alpha) = 0$.

Theorem 3.9. *An invariant submanifold of LP-Sasakian manifold is totally geodesic if and only if $Q(S, \tilde{\nabla}\alpha) = 0$.*

Proof. We assume that $Q(S, \tilde{\nabla}\alpha) = 0$, which implies

$$0 = Q(S, \tilde{R}(X, Y) \cdot \alpha)(W, K, U, V)$$

for any vector fields X, W, K, U on M.

In view of (2.13), the above relation can be written as

$$\begin{aligned} 0 \;\; = \;\; & -(\tilde{\nabla}_X\alpha)(S(V, W)U, K) + (\tilde{\nabla}_X\alpha)(S(U, W)V, K) \\ & -(\tilde{\nabla}_X\alpha)(W, S(V, K)U) + (\tilde{\nabla}_X\alpha)(W, S(U, K)V). \end{aligned} \tag{3.14}$$

Expanding the above term and inserting $V = K = W = \xi$ in (3.14) and then using (3.1), we get

$$S(\xi, \xi)\alpha(U, \nabla_X\xi) = 0, \quad \Rightarrow \quad \alpha(U, \phi X) = 0. \tag{3.15}$$

□

Using the above proof, we state the following corollaries.

Corollary 3.10. *An invariant submanifold of $(LCS)_{2n+1}$manifold is totally geodesic if and only if $(S, \tilde{\nabla}\alpha) = 0$, provided $\alpha^2 \neq \rho$.*

Corollary 3.11. *An invariant submanifold of Lorentzian α-Sasakian manifold satisfies $(S, \tilde{\nabla}\alpha) = 0$ if and only if it is totally geodesic.*

Theorem 3.12. *An invariant submanifold of LP-Sasakian manifold is totally geodesic if and only if $Q(S, \tilde{R} \cdot \alpha) = 0$.*

Proof. Let us assume $Q(S, \tilde{R} \cdot \alpha) = 0$; then it follows that

$$Q(S, \tilde{R}(X, Y) \cdot \alpha)(W, K, U, V) = 0, \tag{3.16}$$

for any vector fields X, Y, W, K, U, V on M. We can obtain directly from the above equation and (2.13) that

$$\begin{aligned} - \;\; & (\tilde{R}(X, Y) \cdot \alpha)(S(V, W)U, K) \\ + \;\; & (\tilde{R}(X, Y) \cdot \alpha)(S(U, W)V, K) \\ - \;\; & (\tilde{R}(X, Y) \cdot \alpha)(W, S(V, K)U) \\ + \;\; & (\tilde{R}(X, Y) \cdot \alpha)(W, S(U, K)V) = 0. \end{aligned} \tag{3.17}$$

Then by the definition of $\tilde{R} \cdot \alpha$ and putting $Y = W = K = V = \xi$ in (3.17), we get

$$S(\xi, \xi)\alpha(R(X, \xi)\xi, U) = 0. \tag{3.18}$$

By using the relation (2.5) in (3.18), we get

$$S(\xi, \xi)\{-\alpha(X, U) - \eta(X)\alpha(\xi, U)\} = 0. \tag{3.19}$$

In view of (3.2), (2.7), and (2.1), (3.19) yields

$$2n\alpha(X, U) = 0 \Rightarrow \alpha(X, U) = 0.$$

This completes the proof. □

Corollary 3.13. *An invariant submanifold of $(LCS)_{2n+1}$ manifold is totally geodesic if and only if $Q(S, \tilde{R} \cdot \alpha) = 0$, provided $\alpha^2 \neq \rho$.*

Corollary 3.14. *An invariant submanifold of a Lorentzian α-Sasakian manifold is totally geodesic if and only if $Q(S, \tilde{R} \cdot \alpha) = 0$.*

In 1940, Yano introduced the concircular curvature tensor. A $(2n+1)$-dimensional concircular curvature tensor C was given by [26, 27]

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\},$$

where R and r are the Riemannian curvature tensor and scalar curvature tensor, respectively.

Theorem 3.15. *An invariant submanifold of LP-Sasakian manifold is totally geodesic if and only if $Q(g, C \cdot \alpha) = 0$, provided $r \neq 2n(2n+1)$.*

Proof. Assuming that $Q(g, C \cdot \alpha) = 0$, we get

$$Q(g, C(X,Y) \cdot \alpha)(W,K,U,V) = 0$$

for any vector fields $X, Y, W, K, U, V \in M$.
From (2.13), the above equation reduces to

$$\begin{aligned} &- \ g(V,W)(C(X,Y) \cdot \alpha)(U,K) \\ &+ \ g(U,K)(C(X,Y \cdot \alpha)(V,K) \\ &- \ g(V,K)(C(X,Y) \cdot \alpha)(W,U) \\ &+ \ g(U,K)(C(X,Y) \cdot \alpha)(W,V) = 0. \end{aligned} \tag{3.20}$$

Putting $Y = K = W = U = \xi$ in (3.20), we have

$$S(\xi,\xi)\alpha(C(X,\xi)\xi, V) = 0, \quad \Rightarrow \quad 2n\left\{1 - \frac{r}{2n(2n+1)}\right\}\alpha(X,V) = 0.$$

□

Using the above proof, we can state the following corollaries.

Corollary 3.16. *An invariant submanifold of $(LCS)_{2n+1}$ manifold is totally geodesic if and only if $Q(g, C \cdot \alpha) = 0$, provided $r \neq (\alpha^2 - \rho)2n(2n+1)$.*

Corollary 3.17. *An invariant submanifold of a Lorentzian α-Sasakian manifold is totally geodesic if and only if $Q(g, C \cdot \alpha) = 0$, provided $r \neq \alpha^2 2n(2n+1)$.*

Now we prove our next theorem.

Theorem 3.18. *An invariant submanifold of LP-Sasakian manifold is totally geodesic if and only if $Q(S, C \cdot \alpha) = 0$, provided $r \neq 2n(2n+1)$.*

Proof. Assuming that $Q(S, C \cdot \alpha) = 0$, we get

$$Q(S, C(X,Y) \cdot \alpha)(W,K,U,V) = 0$$

for any vector fields $X, Y, W, K, U, V \in M$. By using (2.13), the above equation reduces to

$$\begin{aligned} &- \ S(V,W)(C(X,Y) \cdot \alpha)(U,K) \\ &+ \ S(U,K)(C(X,Y \cdot \alpha)(V,K) \\ &- \ S(V,K)(C(X,Y) \cdot \alpha)(W,U) \\ &+ \ S(U,K)(C(X,Y) \cdot \alpha)(W,V) = 0. \end{aligned} \tag{3.21}$$

Putting $Y = K = W = U = \xi$ in (3.21) and using (3.2), we have

$$S(\xi,\xi)\alpha(C(X,\xi)\xi,V) = 0 \Rightarrow 2n\left\{1 - \frac{r}{2n(2n+1)}\right\}\alpha(X,V) = 0.$$

This completes the proof. □

Corollary 3.19. *An invariant submanifold of $(LCS)_{2n+1}$ manifold is totally geodesic if and only if $Q(S, C\cdot\alpha) = 0$, provided $r \neq (\alpha^2 - \rho)2n(2n+1)$.*

Corollary 3.20. *An invariant submanifold of Lorentzian α-Sasakian manifold is totally geodesic if and only if $Q(S, C\cdot\alpha) = 0$, provided $r \neq \alpha^2 2n(2n+1)$.*

4. Example

We consider the 5-dimensional manifold $M = \{x, y, z, u, t \in R^5\}$, where x, y, z, u, t are the standard coordinates in R^5. We choose linearly independent global frame fields $\{e_1, e_2, e_3, e_4, e_5\}$ on M as

$$e_1 = e^t\frac{\partial}{\partial x}, \quad e_2 = e^t\frac{\partial}{\partial y}, \quad e_3 = e^t\frac{\partial}{\partial z}, \quad e_4 = e^t\frac{\partial}{\partial u}, \quad e_5 = \frac{\partial}{\partial t}.$$

Let g be the Lorentzian metric defined by

$$\begin{aligned} &g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = g(e_4,e_4) = 1, \quad g(e_5,e_5) = -1,\\ &g(e_i,e_j) = 0 \quad for \quad 1 \leq i,j \leq 5. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_5)$, for any $Z \in M$. We define a $(1,1)$-tensor field ϕ as

$$\begin{aligned} \phi(e_1) = -\,e_3,\ \ \phi(e_2) = -e_4,\ \ \phi(e_3) = -e_1,\\ \phi(e_4) = -e_2,\ \ \phi(e_5) = 0. \end{aligned} \tag{4.1}$$

The linearity of ϕ and g yields that

$$\begin{gathered} \eta(e_5) = -1,\\ \phi^2(Z) = Z + \eta(Z)\xi,\\ g(\phi U, \phi Z) = g(U,Z) + \eta(U)\eta(Z). \end{gathered}$$

For any $U, Z \in M$, let ∇ be the Levi–Civita connection with respect to the Lorentzian metric g and let R be the curvature tensor. Then we have

$$\begin{aligned} [e_1,e_2] = [e_1,e_3] = [e_1,e_4] = 0, &\quad [e_1,e_5] = -e_1,\\ [e_2,e_3] = [e_2,e_4] = [e_3,e_4] = 0, &\quad [e_2,e_5] = -e_2,\\ [e_3,e_5] = -e_3, &\quad [e_4,e_5] = -e_4. \end{aligned} \tag{4.2}$$

The Koszul formula is defined by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)\\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

By using the Koszul formula, we can get the followings:

$$\begin{array}{lllll} \nabla_{e_1}e_1 = e_5, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = 0, & \nabla_{e_1}e_5 = -e_1, \\ \nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = e_5, & \nabla_{e_2}e_3 = 0, & \nabla_{e_2}e_4 = 0, & \nabla_{e_2}e_5 = -e_2, \\ \nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = e_5, & \nabla_{e_3}e_4 = 0, & \nabla_{e_3}e_5 = -e_3, \\ \nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = 0, & \nabla_{e_4}e_4 = e_5, & \nabla_{e_4}e_5 = -e_4, \\ \nabla_{e_5}e_1 = 0, & \nabla_{e_5}e_2 = 0, & \nabla_{e_5}e_3 = 0, & \nabla_{e_5}e_4 = 0, & \nabla_{e_5}e_5 = 0. \end{array} \tag{4.3}$$

From the above calculation, it can be easily seen that $M^5(\phi, \xi, \eta, g)$ satisfies $\eta(\xi) = -1$ and $\nabla_X \xi = \phi X$. Hence the manifold is an LP-Sasakian manifold.

Let $\tilde{M}$ be a submanifold of M and consider the isometric immersion $f : \tilde{M} \to M$ defined by $f(x_1, y_1, z) = (x_1, 0, y_1, 0, z)$. It can be easily proved that $\tilde{M} = \{(x_1, y_1, z) \in R^3 : (x_1, y_1, z) \neq 0\}$, where (x_1, y_1, z) is standard coordinates in R^3, is a 3-dimensional submanifold of the 5-dimensional LP-Sasakian manifold. We consider the vector fields

$$e_1 = e^t \frac{\partial}{\partial x}, \quad e_3 = e^t \frac{\partial}{\partial y}, \quad e_5 = \frac{\partial}{\partial z},$$

Thus, we have

$$\phi(e_1) = -e_3, \quad \phi(e_3) = -e_1, \quad \phi(e_5) = 0.$$

The linearity of ϕ and g yields that

$$\begin{gathered} \eta(e_5) = -1, \\ \phi^2(Z) = Z + \eta(Z)\xi, \\ g(\phi U, \phi Z) = g(U, Z) + \eta(U)\eta(Z) \end{gathered}$$

for any vector fields X, Y on $\tilde{M}$. Thus, for $e_5 = \xi$, $\tilde{M}(\phi, \xi, \eta, g)$ defines an almost contact metric manifold. The 1-forms η is closed. Similar to the above, it can be proved that $M(\phi, \xi, \phi, g)$ is a 3-dimensional LP-Sasakian manifold. Taking $e_5 = \xi$ and using Koszul's formulas for the metric g, it can be easily calculated that

$$\begin{array}{lll} \tilde{\nabla}_{e_1}e_1 = -e_5, & \tilde{\nabla}_{e_1}e_3 = 0, & \tilde{\nabla}_{e_1}e_5 = 0, \\ \tilde{\nabla}_{e_3}e_1 = 0, & \tilde{\nabla}_{e_3}e_3 = -e_5, & \tilde{\nabla}_{e_3}e_5 = 0, \\ \tilde{\nabla}_{e_5}e_1 = 0, & \tilde{\nabla}_{e_5}e_3 = 0, & \tilde{\nabla}_{e_5}e_5 = 0, \end{array} \tag{4.4}$$

$\tilde{\nabla}_{e_1}e_1 = \tilde{\nabla}_{e_3}e_3 = -e_5$, $\tilde{\nabla}_{e_5}e_5 = 0$, and the remaining term are zero.

Let us assume

$$TM = D \oplus D^{\perp} \oplus <\xi>,$$

where $D = <e_1>$ and $D^{\perp} = <e_3>$. Then we see that $\phi(e_1) = -e_3 \in D^{\perp}$ for $e_1 \in D$ and $\phi(e_3) = -e_1 \in D$, for $e_3 \in D^{\perp}$. Hence the submanifold is invariant. Now from the values of $\nabla_{e_i}e_j$ and $\tilde{\nabla}_{e_i}e_j$, we see that $\alpha(e_i, e_j) = 0$ for all $i, j = 1, 2, 3$. We see that the submanifold is totally geodesic. Thus, Theorems 3.3, 3.6, 3.9, 3.12, 3.15, and 3.18 are verified.

5. Conclusion

In this paper, we studied the geometric property of invariant submanifold of LP-Sasakian manifolds. We proved some geometric condition for an invariant submanifold of LP-Sasakian manifold to be totally geodesic. Further, we verified the same conditions to $(LCS)_{2n+1}$ and Lorentzian α-Sasakian manifold. Finally, we prepared an example of the invariant submanifold of LP-Sasakian, which is totally geodesic.

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