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SOME PROPERTIES OF PRIME AND Z-SEMI-IDEALS IN POSETS

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Abstract. We define the notion of z-semi-ideals in a poset P and we show that if a z-semi-ideal J satisfies $(*)$-property, then every minimal prime semi-ideal containing J is also a z-semi-ideal of P. We also show that every prime semi-ideal is a z-semi-ideal or the maximal z-semi-ideals contained in it are prime z-semi-ideals. Further, we characterize some properties of union of prime semi-ideals of P provided the prime semi-ideals are contained in the unique maximal semi-ideal of P.

1. Preliminaries

Throughout this paper, $(P, \leq)$ denotes a poset with smallest element 0. For the basic terminology and notation for posets, we refer the reader to [8, 12]. For $M \subseteq P$, let $L(M) := \{x \in P : x \leq m \text{ for all } m \in M\}$ denote the lower cone of M in P, and dually let $U(M) := \{x \in P : m \leq x \text{ for all } m \in M\}$ be the upper cone of M in P. Let $A, B \subseteq P$; then we write $L(A, B)$ instead of $L(A \cup B)$ and dually for the upper cones. If $M = \{x_1, \ldots, x_n\}$ is finite, then we use the notation $L(x_1, \ldots, x_n)$ instead of $L(\{x_1, \ldots, x_n\})$ (and dually). It is clear that for any subset A of P, we have $A \subseteq L(U(A))$ and $A \subseteq U(L(A))$. If $A \subseteq B$, then $L(B) \subseteq L(A)$ and $U(B) \subseteq U(A)$. Moreover, $LUL(A) = L(A)$ and $ULU(A) = U(A)$.

Following [6], a nonempty subset I of P is called a semi-ideal if $b \in I$ and $a \leq b$ imply that $a \in I$. A non-empty subset I of P is said to be an ideal if $LU(a, b) \subseteq I$ for all $a, b \in I$. A proper semi-ideal (ideal) I of P is called a prime

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semi-ideal (prime ideal) if for any $a, b \in P$, $L(a, b) \subseteq I$ implies $a \in I$ or $b \in I$; see [8]. An ideal I of a poset P is called semiprime if $L(a, b) \subseteq I$ and $L(a, c) \subseteq I$ together imply $L(a, U(b, c)) \subseteq I$. Let I be a semi-ideal of P and let $J \subseteq P$. Then the extension of I by $A \in P$ is meant the set $\langle A, I\rangle = \{x \in P \ : \ L(a, x) \subseteq I \ for \ all \ a \in A\}$. Clearly $\langle A, I\rangle$ is a semi-ideal of P but not an ideal of P, in general. If $A = \{x\}$, then $\langle x, I\rangle = \{a \in P \ : \ L(a, x) \subseteq I\}$; see [4]. For $a \in P$, the subset $L(a) = \{x \in P : x \leq a\}$ is the ideal generated by a. For any subset A of P, we denote $A^* = A\backslash\{0\}$.

In 1973, Mason [9] defined the notion of z-ideal for an arbitrary commutative ring R as follows: An ideal I of R is called a z-ideal if $M_a = M_b$ and $b \in I$ imply $a \in I$, where M_a is the intersection of all maximal ideals of R containing a. Equivalently, since $M_b \subseteq M_a$ if and only if $M_a = M_{ab}$, I is a z-ideal if and only if $M_b \subseteq M_a$ and $b \in I$ which imply $a \in I$. Later in 2013, Aliabad, Azarpanah, and Taherifar [1] have shown that for any ideal I of R, I is a z-ideal if and only if $M_a \subseteq I$ for any $a \in I$.

Following [1], a semi-ideal I of P is called a z-semi-ideal if $M_a \subseteq I$ for any $a \in I$, where M_a is the intersection of all maximal semi-ideals of P containing a. It is easy to see that I is a z-semi-ideal if and only if whenever $b \in P$, $a \in I$, and $M_b \subseteq M_a$, then $b \in I$. A z-semi-ideal that is prime is called prime z-semi-ideal. It is clear that an arbitrary union of z-semi-ideals and an arbitrary intersection of z-semi-ideals of P are z-semi-ideals of P.

The following example shows that prime semi-ideals and z-semi-ideals are independent concepts.

Example 1.1. Consider the set $P = \{0, a, b, c, d, e, f\}$ and define a relation $\leq$ on P as follows:

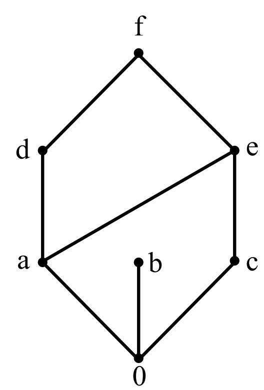

Figure 1

Then $(P, \leq)$ is a poset. Here $R = \{0, a, b, c, d, e\}$ and $S = \{0, a, c, d, e, f\}$ are the maximal semi-ideals of P. The semi-ideal $I = \{0, a, c, d, e\}$ is a z-semi-ideal of P but not a prime semi-ideal as $L(b, f) \subseteq I$ with $b, f \notin I$. Also, $J = \{0, b, c\}$ is a prime semi-ideal of P but not a z-semi-ideal as $M_b \nsubseteq I$.

A semi-ideal J of P is said to have $(*)$-property if for any $a, b \in P\backslash J$, we have either $a = b$ or $L(a, b) = \{0\}$. Following [2], a non-empty subset M of P is called an m-system if for any $x_1, x_2 \in M$ there exists $t \in L(x_, x_2)$ such that $t \in M$. It is trivial that for any $x \in P$, $U(x)$ is an m-system of P. Also for any semi-ideal I of P, we have I is a prime semi-ideal of P if and only if $P \backslash I$ is an m-system of P.

2. Main results

In this section, we study some important properties of z-semi-ideals and prime semi-ideals of P. Some elementwise characterizations of smallest z-semi-ideals and largest z-semi-ideals of P are given. Further, we discuss some important properties of union of z-semi-ideals of P.

The following theorems and lemma are very useful to prove our main results.

Theorem 2.1. ([2, Theorem 2.6]). *Let M be a non-void m-system in P and let J be a semi-ideal of P with $J \cap M = \phi$. Then J is contained in a prime semi-ideal I of P with $I \cap M = \phi$.*

Theorem 2.2. ([2, Theorem 2.7]). *Let I and J be semi-ideals of P, and let I be prime with $J \subseteq I$. If J has $(*)$-property, then the following conditions are equivalent:*

(a) *I is a minimal prime semi-ideal of J.*
(b) *For each $x \in I$, there exist $y \in P \backslash I$ and $t \in U(x)$ such that $L(t, y) \subseteq J$.*

Lemma 2.3. ([2, Lemma 2.9]). *For any semi-ideal I of P, we have $P(I) = I$.*

Theorem 2.4. *Let P be a poset. Then we have the followings:*

(a) *For any $a, b \in P$, we have $M_t = M_a \cap M_b$ for any $t \in L(a, b)$.*
(b) *For any $a, b, c \in P$, we have $b \in M_a$ if and only if $M_b \subseteq M_a$ if and only if $M_{t_1} \subseteq M_{t_2}$ for any $t_1 \in L(b, c)$ and $t_2 \in L(a, c)$.*

Proof. (a) Let $a, b \in P$. Consider $X = \{M \in Max(P) : L(a, b) \subseteq M\}$, $Y = \{M \in Max(P) : a \in M\}$, and $Z = \{M \in Max(P) : b \in M\}$. Then $Y \subseteq X$ and $Z \subseteq X$, which imply $M_t \subseteq M_a$ and $M_t \subseteq M_b$, so $M_t \subseteq M_a \cap M_b$ for any $t \in L(a, b)$. It is trivial that $M_a \cap M_b \subseteq M_t$. So $M_t = M_a \cap M_b$ for any $t \in L(a, b)$.

(b) Let $a, b, c \in P$. Consider $X = \{M \in Max(P) : a \in M\}$, $X_1 = \{M \in Max(P) : L(a, c) \subseteq M\}$, $Y = \{M \in Max(P) : b \in M\}$, and $Y_1 = \{M \in Max(P) : L(b, c) \subseteq M\}$. If $b \in M_a$, then $X \subseteq Y$, which implies $M_b \subseteq M_a$.

If $M_b \subseteq M_a$, then $b \in M_a$ as $b \in M_b$.

If $M_b \subseteq M_a$, then $M_b \cap M_c \subseteq M_a \cap M_c$. By part (a), we have $M_{t_1} \subseteq M_{t_2}$ for any $t_1 \in L(b, c)$ and $t_2 \in L(a, c)$. □

Theorem 2.5. *Let P be a poset and J be a semi-ideal of P. If J is a z-semi-ideal of P and has $(*)$-property, then every minimal prime semi-ideal containing J is also a z-semi-ideal of P.*

Proof. Let J be a z-semi-ideal of P and has $(*)$-property. Let $I \in Min(J)$. If I is not a z-semi-ideal of P, then there exist $b \notin I$ and $a \in I$ such that $M_b \subseteq M_a$.

Let $S = (P \backslash I) \cup \{a_{ij} : a_{ij} \in L(y_i, t_j) \backslash J$ for $y_i \in P \backslash J$ and let $t_j \in U(a)\}$. Then by Theorem 2.2, S is an m-system with $S \cap J = \phi$. By Theorem 2.1, there exists $I' \in Spec(P)$ such that $J \subseteq I'$ and $I' \cap S = \phi$. Let $x \in I'$. If $x \notin I$, then $x \in S$ and so $s \in I' \cap S$, a contradiction. So $I' \subseteq I$. If $a \in J$, then $b \in J$, a contradiction. So $a \notin J$. Since $y_i = a$, we have $a \in S$, which implies $a \notin I'$ and so $I' \subset I$, a contradiction to the minimality of I. Thus I is a z-semi-ideal of P. □

Let I be a semi-ideal of P. The set of all zero-divisors with respect to I, denoted by $Z_I(P)$, is defined as $Z_I(P) = \{x \in P \mid L(x, y) \subseteq I \text{ for some } y \notin I\}$.

Proposition 2.6. *Let P be a poset and let I be a semi-ideal of P. If I is a z-semi-ideal of P, then $I \subseteq Z_I(P)$.*

Proof. Let I be a z-semi-ideal of P. Assume that $I \nsubseteq Z_I(P)$. Then there exists $a \in I \backslash Z_I(P)$ such that $M_a \subseteq I$ and $L(a, x) \nsubseteq I$ for all $x \in P \backslash I$. Then for $t \in L(a, x) \backslash I$, by Theorem 2.4, we have $M_t = M_a \cap M_x \subseteq M_a \subseteq I$, a contradiction. □

Every semi-ideal I of P is contained in at least a z-semi-ideal of P, namely I_z, which is defined as $I_z = \cap\{I \subseteq J \mid J \text{ is z-semi-ideal}\}$. It is clear that $I \subseteq I_z$. For semi-ideals I and J, if $I \subseteq J$, then $I_z \subseteq J_z$. If I is not a z-semi-ideal and contains a prime semi-ideal, then from Theorem 2.10, we have I_z is a prime z-semi-ideal of P.

Now we give an elementwise characterization of I_z.

Proposition 2.7. *For a semi-ideal I of P, we have*

$$I_z = \{a \in P \mid \text{there exists } b \in I \text{ with } M_a \subseteq M_b\}.$$

Proof. Let $a, b \in P$ with $a \leq b$ and $b \in I_z$. Then there exists $y \in I$ such that $M_b \subseteq M_y$. Since $M_a \subseteq M_b$, we have $M_a \subseteq M_y$ and so $a \in I_z$, and hence I_z is a semi-ideal of P. Also, $I \subseteq I_z$.

We now claim that I_z is the smallest z-semi-ideal containing I. Let $a \in I_z$ with $M_b \subseteq M_a$. Then there exists $y \in I$ such that $M_a \subseteq M_y$, which implies $M_b \subseteq M_y$ and so $b \in I_z$. Hence I_z is a z-semi-ideal of P.

Let J be a z-semi-ideal of P with $I \subset J \subset I_z$. Let $b \in I_z$ and $b \notin J$. Then there exists $y \in I$ such that $M_b \subseteq M_y$, which implies $b \in J$, a contradiction. So I_z is the smallest z-semi-ideal containing I. □

Lemma 2.8. *Let P be a poset. Then*

(a) *if $I = L(a)$ for $a \in P$, then $(I)_z = M_a$;*

(b) *if I and J are two semi-ideals of P, then $(\bigcup_{i \in I, j \in J} L(i, j))_z = (I \cap J)_z = I_z \cap J_z$.*

Proof. (a) It is trivial that M_a is a z-semi-ideal for any $a \in P$.

(b) Let I and J be two semi-ideals of P. Then $(\bigcup_{i \in I, j \in J} L(i, j))_z \subseteq (I \cap J)_z \subseteq I_z \cap J_z$. We now prove that $I_z \cap J_z \subseteq (\bigcup_{i \in I, j \in J} L(i, j))_z$. Let K be a z-semi-ideal of P with $\bigcup_{i \in I, j \in J} L(i, j) \subseteq K$. For each $P_1 \in Min(K)$, by Theorem 2.5, we have P_1 is a z-semi-ideal of P. Also by Lemma 2.3, we have $P_1 = \bigcap_{P' \in Min(K)} P'$. Since

$\bigcup_{i\in I,j\in J} L(i,j) \subseteq K$, we have $\bigcup_{i\in I,j\in J} L(i,j) \subseteq P'$ for every $P' \in Min(K)$, which implies either $I \subseteq P'$ or $J \subseteq P'$, so $I_z \subseteq P'$ or $J_z \subseteq P'$. Thus $I_z \cap J_z \subseteq P'$ and hence $I_z \cap J_z \subseteq K$. So $I_z \cap J_z$ is the smallest z-semi-ideal containing $\bigcup_{i\in I,j\in J} L(i,j)$.

Therefore $(\bigcup_{i\in I,j\in J} L(i,j))_z = (I \cap J)_z = I_z \cap J_z$. □

Theorem 2.9. *Let J be a semi-ideal of P. If J is prime, then either J is a z-semi-ideal or the maximal z-semi-ideals contained in J are prime z-semi-ideals.*

Proof. Let $S = \{K \mid K$ be a z-semi-ideal, let $K \subseteq J$ and let $K \cap (P \backslash J) = \{\phi\}\}$. Clearly $S \neq \{\phi\}$, since $0 \in S$. By Zorn's lemma, S has maximal elements. Let I be the maximal element of S. By Theorem 2.1, there exists a prime semi-ideal P_1 of P such that $I \subseteq P_1$ and $P_1 \cap (P \backslash J) = \{\phi\}$. So $I \subseteq P_1 \subseteq J$. By Theorem 2.5, P_1 is a z-semi-ideal and so $P_1 \neq J$. Thus either $I = P_1$ or $I \subset P_1$. Here $I \subset P_1$ gives a contradiction to maximality of I. So $I = P_1$ is prime. □

Following [5, 6], a semi-ideal I of P is said to be strongly irreducible if for semi-ideals A and B, $A \cap B \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$. It is easy to observe that every prime semi-ideal is strongly irreducible.

Theorem 2.10. *Let I be a z-semi-ideal of P. Then I is prime if and only if I is strongly irreducible and contains a prime semi-ideal of P.*

Proof. Assume that I is strongly irreducible and contains a prime semi-ideal P_1 of P. Suppose that $L(a,b) \subseteq I$ for $a, b \in P$. If $L(a,b) \subseteq P_1$, then either $a \in P_1$ or $b \in P_1$, which implies $a \in I$ or $b \in I$. If $L(a,b) \nsubseteq P_1$, then there exists $t \in L(a,b)$ such that $t \in I \backslash P_1$. By Theorem 2.4, $M_t = M_a \cap M_b$. Since I is a z-semi-ideal, we have $M_a \cap M_b = M_t \subseteq I$, which implies $a \in M_a \subseteq I$ or $b \in M_b \subseteq I$. □

The following example shows that we cannot drop the condition I is strongly irreducible in Theorem 2.10.

Example 2.11. Consider $P = \{1, 2, 3, 4, 5, 6\}$. Then P is a poset under the relation division and $I = \{1, 2, 3, 5\}$, $A = \{1, 2, 3, 4, 5\}$, and $B = \{1, 2, 3, 5, 6\}$ are semi-ideals of P. Clearly $A \cap B \subseteq I$, but $A \nsubseteq I$ and $B \nsubseteq I$. Also $J = \{1, 3, 5\}$ is a prime semi-ideal of P with $J \subseteq I$ and I is a z-semi-ideal of P, but I is not a prime semi-ideal of P.

Theorem 2.12. *Let I and J be two semi-ideals of P. Then we have the followings:*

(a) *If J is a strongly irreducible semi-ideal and $I \cap J$ is a z-semi-ideal of P, then either I is a z-semi-ideal or J is a z-semi-ideal of P.*
(b) *If I and J are strongly irreducible semi-ideals of P and $I \cap J$ is a z-semi-ideal of P, then either I is a z-semi-ideal or J is a z-semi-ideal of P.*
(c) *If I and J are strongly irreducible semi-ideals of P which are not in a chain and $I \cap J$ is a z-semi-ideal of P, then both I and J are z-semi-ideals of P.*

Proof. (a) Suppose that $I \cap J$ is a z-semi-ideal of P, where I is a semi-ideal and J is a strongly irreducible semi-ideal of P. If $I \subseteq J$ or $J \subseteq I$, then either I or J is a z-semi-ideal of P. Suppose that $I \nsubseteq J$ and $J \nsubseteq I$. Then there exist $a \in I \backslash J$ and $b \in J \backslash I$. Clearly $L(a,b) \subseteq I \cap J$. If $t \in L(a,b)$, then by Theorem 2.4, $M_t = M_a \cap M_b$. Since $I \cap J$ is a z-semi-ideal, we have $M_a \cap M_b \subseteq I \cap J$, which implies $M_a \cap M_b \subseteq J$. Since J is a strongly irreducible semi-ideal and $a \notin J$, we have $M_b \subseteq J$ and so J is a z-semi-ideal of P.

(b) It follows from (a).

(c) Let I and J be strongly irreducible semi-ideals of P with $I \nsubseteq J$ and $J \nsubseteq I$, and let $I \cap J$ is a z-semi-ideal of P. If $I \nsubseteq J$, then by part (a), J is a z-semi-ideal of P. Also, if $J \nsubseteq I$, then by part (a), I is a z-semi-ideal of P. □

Corollary 2.13. *Let I and J be semi-ideals of P. Then we have the followings:*

(a) *If J is a prime semi-ideal and $I \cap J$ is a z-semi-ideal of P, then either I is a z-semi-ideal or J is a z-semi-ideal of P.*
(b) *If I and J are prime semi-ideals of P and $I \cap J$ is a z-semi-ideal of P, then either I is a z-semi-ideal or J is a z-semi-ideal of P.*
(c) *If I and J are prime semi-ideals of P that are not in a chain and $I \cap J$ is a z-semi-ideal of P, then both I and J are z-semi-ideals of P.*

Following [3], for each subset A of P, we define $V(A) = \{P' \in Spec(P) : A \subseteq P'\}$ to be the closed subset of P. If $A = \{a\}$, then we define $V(a) = \{P' \in Spec(P) : a \in P'\}$ and $D(a) = Spec(P) \backslash V(a)$. It is clear that $V(A) = \bigcap_{a \in A} V(a)$.

In general, the set of all prime semi-ideals of P does not necessarily form a chain. Indeed, consider the prime semi-ideals $P_1 = \{1,2\}$ and $P_2 = \{1,3\}$ of P as shown in Figure 2. Here both P_1 and P_2 does not form a chain. But the following theorem shows that prime semi-ideals of poset containing a given prime semi-ideal forms a chain.

Theorem 2.14. *Let I be a semi-ideal of P. If the intersection of two prime semi-ideals of P is again a prime semi-ideal of P, then all the elements of $V(I)$ form a chain.*

Proof. Let $V(I) = \{P_i \in Spec(P) : I \subseteq P_i\}$. Suppose that $P_i \nsubseteq P_j$ and $P_j \nsubseteq P_i$ for some $i \neq j$. Then there exist $a \in P_i \backslash P_j$ and $b \in P_j \backslash P_i$. Clearly $L(a,b) \subseteq P_i \cap P_j = P_k$ for some $P_k \in V(I)$, which implies $a \in P_k$ or $b \in P_k$, which implies $a \in P_j$ or $b \in P_i$, a contradiction. Thus either $P_i \subseteq P_j$ or $P_j \subseteq P_i$ for all $i \neq j$. □

The following example shows that we cannot drop the condition intersection of two prime semi-ideals is a prime semi-ideal of P in Theorem 2.14.

Consider the set $P = \{1,2,3,4,6,18\}$. Then P is a poset under the relation division. Here $I = \{1,3\}$, $P_1 = \{1,2,3,4\}$, $P_2 = \{1,2,3,4,6\}$, and $P_3 = \{1,2,3,6,18\}$ are prime semi-ideals containing I. But P_1, P_2 and P_3 does not form a chain and $P_2 \cap P_3 = \{1,2,3,6\}$ is not a prime semi-ideal of P.

Every z-semi-ideal I of P contains a greatest z-semi-ideal of P, namely I^z, which is defined as the union of all z-semi-ideals of P contained in I. For semi-ideals I and J, if $I \subseteq J$, then $I^z \subseteq J^z$.

Lemma 2.15. *For semi-ideals I_i of P, we have* $\bigcap_i I_i^z = (\bigcap_i I_i)^z$.

Proof. Let I be a semi-ideal of P. Since I^z is a z-semi-ideal contained in I, $\bigcap_i I_i^z$ is a z-semi-ideal contained in $\bigcap_i I_i$. If J is a z-semi-ideal of P contained in $\bigcap_i I_i$, then $J \subset I_i$ for all i. So $J \subset I_i^z$ for all i which implies $J \subset \bigcap_i I_i^z$. Thus $\bigcap_i I_i^z$ is the greatest z-semi-ideal contained in $\bigcap_i I_i$ and hence $\bigcap_i I_i^z = (\bigcap_i I_i)^z$. □

Now we can give an elementwise characterization of I^z corresponding to that for I_z.

Proposition 2.16. *For any semi-ideal I of P, we have*

$$I^z = \{a \in I_z \mid M_y \subseteq M_a \text{ implies } y \in I\}.$$

Proof. Let $S = \{a \in I_z \mid M_y \subseteq M_a \text{ implies } y \in I\}$. Clearly $S \neq \{\phi\}$. Let $a,\ b \in P$ with $a \leq b$ and $b \in S$. Then $b \in I_z$. Thus there exists $y \in I$ such that $M_b \subseteq M_y$. Since $M_a \subseteq M_b$, we have $a \in I_z$. Also, if $M_y \subseteq M_a$, then $M_y \subseteq M_b$ and so $y \in I$ and thus $a \in S$. Also, $S \subseteq I$.

Now we claim that S is a maximal z-semi-ideal contained in I.

If $a \in S$ and $M_b \subseteq M_a$, then $a \in I_z$. So there exists $y \in I$ such that $M_a \subseteq M_y$. Since $M_b \subseteq M_y$, we have $b \in I_z$. Also, if $M_x \subseteq M_b$, then $M_x \subseteq M_a$ and so $x \in I$ and hence $b \in S$.

Suppose that there exists a z-semi-ideal J such that $S \subset J \subset I$. If $x \in J$ and $x \notin S$, then $x \in I_z$. If $M_y \subseteq M_x$, then $y \in J$, which implies $y \in I$ and so $x \in S$, a contradiction. Hence S is the greatest z-semi-ideal contained in I. □

Proposition 2.17. *Let P be a poset and let I and J be semi-ideals of P. Then the following properties hold:*

(a) *If I is a z-semi-ideal of P, so is $\langle J, I\rangle$ for any J.*
(b) *If I is a z-semi-ideal of P, so is $\langle L(a), I\rangle$ for any $a \in P$.*

Proof. (a) Let I be a z-semi-ideal of P and let $b \in \langle J, I\rangle$. Then $L(b, j) \subseteq I$ for all $j \in J$. Suppose that $M_b \nsubseteq \langle J, I\rangle$ for some $b \in \langle J, I\rangle$. Then there exists $a \in M_b$ such that $a \notin \langle J, I\rangle$, which implies $M_a \subseteq M_b$ and $L(a, j) \nsubseteq I$ for some $j \in J$. Therefore there exists $t \in L(a, j)$ such that $t \notin I$. By Theorem 2.4, $M_t \subseteq M_s$ for $s \in L(b, j)$, which implies $M_t \subseteq I$, a contradiction. So $\langle J, I\rangle$ is a z-semi-ideal of P.

(b) It follows from (a) that $\langle a, I\rangle = \langle L(a), I\rangle$ for all $a \in P$. □

Following [7], a semi-ideal I is said to be a u-semi-ideal if, for $x, y \in I$, $U(x, y) \cap I \neq \{\phi\}$. It is clear that an arbitrary union of u-semi-ideals and an arbitrary intersection of u-semi-ideals need not to be an u-semi-ideal of P. Indeed, consider the poset depicted in Figure 2. Here $I = \{1, 2, 3, 12\}$ and $J = \{1, 2, 3, 18\}$ are u-semi-ideals of P, but $I \cup J = \{1, 2, 3, 12, 18\}$ and $I \cap J = \{1, 2, 3\}$ are not u-semi-ideals of P.

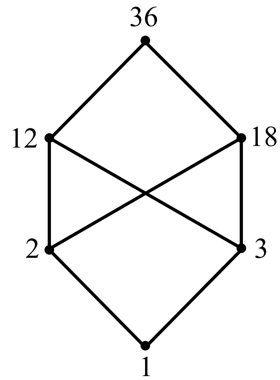

Figure 2

In general, the union of two prime semi-ideals need not be a prime semi-ideal of P. Indeed, consider the prime semi-ideals $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$ of P as shown in Figure 2. Here $P_1 \cup P_2 = \{1, 2, 3\}$ is not prime as $L(12, 18) \subseteq P_1 \cup P_2$, but $12, 18 \notin P_1 \cup P_2$.

In [11], Rudd showed that the sum of two prime ideals in $C(X)$ is prime. Now to prove union of prime u-semi-ideals is a prime semi-ideal of P, we assume that union of two u-semi-ideals is an u-semi-ideal and, distinct prime semi-ideals P_1 and P_2 are contained in the same maximal semi-ideal of P.

The following two theorems are useful to prove Theorems 2.20 and 2.22.

Theorem 2.18. *Let P_1 and P_2 be two u-semi-ideals of P. If P_1 and P_2 are prime and $P_1 \cup P_2$ is a u-semi-ideal of P, then $P_1 \cup P_2$ is a prime u-semi-ideal of P.*

Proof. Let P_1 and P_2 be two prime u-semi-ideals of P, and $P_1 \cup P_2$ be a u-semi-ideal of P. Let $L(a, b) \subseteq P_1 \cup P_2$ for any $a, b \in P$. If $L(a, b) \subseteq P_1$ or $L(a, b) \subseteq P_2$, then $a \in P_1 \cup P_2$ or $b \in P_1 \cup P_2$. If $L(a, b) \nsubseteq P_1$ and $L(a, b) \nsubseteq P_2$, then there exist $s, t \in L(a, b)$ such that $s \notin P_1$ and $t \notin P_2$, which imply $s \in P_2$ and $t \in P_1$. Since $P_1 \cup P_2$ is a u-semi-ideal, we have $U(s, t) \cap (P_1 \cup P_2) \neq \{\phi\}$. Let $r \in U(s, t) \cap (P_1 \cup P_2)$. If $r \in P_1$, then $s \in P_1$, a contradiction. If $r \in P_2$, then $t \in P_2$, a contradiction. So $P_1 \cup P_2$ is a prime u-semi-ideal of P. □

Note that the condition $P_1 \cup P_2$ is a u-semi-ideal in Theorem 2.18 is not superficial. Indeed, consider the poset depicted in Figure 2. Here $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$ are prime u-semi-ideals of P. But $P_1 \cup P_2 = \{1, 2, 3\}$ is not a u-semi-ideal as $U(2, 3) \cap (P_1 \cup P_2) = \{\phi\}$, and also $P_1 \cup P_2$ is not prime as $L(12, 18) \subseteq P_1 \cup P_2$ with $12, 18 \notin P_1 \cup P_2$.

Now we show that union of semiprime u-semi-ideal and prime u-semi-ideal is semiprime u-semi-ideal of poset.

Theorem 2.19. *Let P_1 and P_2 be two semi-ideals and $P_1 \cup P_2$ be a u-semi-ideal of P. If P_1 is a semiprime u-semi-ideal and P_2 is a prime u-semi-ideal, then $P_1 \cup P_2$ is a semiprime u-semi-ideal of P.*

Proof. Let $I = P_1 \cup P_2$ where P_1 is a semiprime u-semi-ideal and P_2 is prime u-semi-ideal of P. By the assumption, I is a u-semi-ideal of P. Suppose that $L(a, b) \subseteq I$ and $L(a, c) \subseteq I$. Now we claim that $L(a, U(b, c)) \subseteq I$. Suppose

$L(a,b) \nsubseteq P_1$ or $L(a,c) \nsubseteq P_2$. Then there exist $s \in L(a,b)$ and $t \in L(a,c)$ such that $s \notin P_1$ and $t \notin P_2$, which imply $s \in P_2$ and $t \in P_1$. Since $P_1 \cup P_2$ is a u-semi-ideal, we have $U(s,t) \cap (P_1 \cup P_2) \neq \{\phi\}$. Let $r \in U(s,t) \cap (P_1 \cup P_2)$. If $r \in P_1$, then $s \in P_1$, a contradiction. If $r \in P_2$, then $t \in P_2$, a contradiction. Thus $L(a,b) \subseteq P_1$ and $L(a,c) \subseteq P_2$.

Since P_2 is prime, either $a \in P_2$ or $c \in P_2$. If $a \in P_2$ or $a \in P_1$, then $L(a,U(b,c)) \subseteq I$. Suppose $a \notin P_2$ and $c \in P_2$. Let $r \in L(a,b)$. Since $P_1 \cup P_2$ is a u-semi-ideal, $U(r,c) \cap (P_1 \cup P_2) \neq \{\phi\}$. Let $q \in U(r,c) \cap (P_1 \cup P_2)$. If $q \in P_1$, then $c \in P_1$. So $L(a,c) \subseteq P_1$. Since P_1 is semiprime, we have $L(a,U(b,c)) \subseteq P_1$, which implies $L(a,U(b,c)) \subseteq I$. If $q \in P_2$, then $r \in P_2$. So $L(a,b) \subseteq P_2$. Since P_2 is prime, we have $b \in P_2$. So $L(a,U(b,c)) \subseteq P_2$, which implies $L(a,U(b,c)) \subseteq I$. □

Although the proof of the following two theorems and proposition are just similar to that of Theorem 3.2, Proposition 3.6, and Theorem 3.7 in [10] for a commutative ring R, for the sake of completeness, we present the proof of the same for poset P.

Theorem 2.20. *If the union of minimal prime u-semi-ideals of P is a z-semi-ideal and the intersection of two prime semi-ideals of P is again a prime semi-ideal of P, then the union of every two prime u-semi-ideals that are not in a chain is a prime z-semi-ideal of P. In fact, if $\{P_i\}_{i\in A}$ is a family of prime u-semi-ideals not all in a chain, then $\bigcup_{i\in A} P_i$ is a prime z-semi-ideal of P.*

Proof. Let P_1 and P_2 be prime u-semi-ideals of P. Let I and J be two distinct minimal prime u-semi-ideals contained in P_1 and P_2, respectively. By the hypothesis, $I \cup J$ is a z-semi-ideal and is prime by Theorem 2.18. Also $I \cup J \subseteq P_1 \cup P_2$. On the other hand, $I \cup J$ is a prime ideal containing I and J. By Theorem 2.14, $P_1 \subseteq I \cup J$ or $I \cup J \subseteq P_1$ and, $P_2 \subseteq I \cup J$ or $I \cup J \subseteq P_2$.

If $P_1 \subseteq I \cup J$ and $I \cup J \subseteq P_2$, then $P_1 \subseteq I \cup J \subseteq P_2$, a contradiction. If $I \cup J \subseteq P_1$ and $P_2 \subseteq I \cup J$, then $P_2 \subseteq I \cup J \subseteq P_1$, a contradiction. If $P_1 \subseteq I \cup J$ and $P_2 \subseteq I \cup J$, then $P_1 \cup P_2 \subseteq I \cup J$ and so $P_1 \cup P_2 = I \cup J$ is a prime z-semi-ideal of P.

If I_i is a minimal prime u-semi-ideal contained in P_i, then $\cup I_i \subseteq \cup P_i$. Conversely, if $x \in \cup P_i$, then there is a finite subfamily $P_1, P_2, P_3, \ldots, P_n$ of $\{P_i\}$ such that $x \in \bigcup_{i=1}^{n} P_i$. Without loss of generality, assume that no two $P_i's$ are in a chain and $n > 1$. Then I_1 is a minimal prime u-semi-ideal contained in the prime u-semi-ideal $\bigcup_{i=1}^{n-1} P_i$. So $\bigcup_{i=1}^{n} P_i = I_1 \cup I_n$. Thus $x \in I_1 \cup I_n \subset \bigcup I_i$ and so $\bigcup_{i=1}^{n} P_i = \bigcup_{i=1}^{n} I_i$, a prime z-semi-ideal of P. □

Proposition 2.21. *Let Q and R be semi-ideals of P. If $Q \subset R$ are prime semi-ideals which are not z-semi-ideals and Q_z, R_z are strongly irreducible, then we have the followings:*

(a) *Either* (i) $Q \subset Q_z \subset R^z \subset R$ *or* (ii) $R^z \subset Q \subset R \subset Q_z$.
(b) *In case* (i), $Q^z = R^z = I$ *if and only if* I *is the unique z-semi-ideal between* Q *and* R.
(c) *In case* (ii), *if* J *is any prime semi-ideal with* $R^z \subset J \subset Q_z$, *then* $R^z = J^z = Q^z$ *and* $R_z = J_z = Q_z$.

Proof. (a) Let Q and R be prime semi-ideals that are not z-semi-ideals and $Q \subset R$. Then R_z is a prime z-semi-ideal. Similarly, R^z is a prime z-semi-ideal by Theorem 2.10, since $R^z \supset Q^z$ and Q^z is prime. Since R^z, Q and Q_z are primes containing Q^z, they are in a chain and the possible cases are as follows:

Case (i): $Q^z \subset Q \subset Q_z \subseteq R^z \subset R$ (or) $Q^z \subset Q \subset R^z \subseteq Q_z$; but this violates the minimality of Q_z unless $Q_z = R^z$.

Case (ii): $Q^z \subseteq R^z \subset Q \subset Q_z$, which contradicts the maximality of Q^z unless $Q^z = R^z$. Also, R and Q_z are primes containing Q and so must be in a chain. If $Q_z \subset R$, then it contradicts maximality of R^z. So $R \subset Q_z$.Thus $Q \subset Q_z \subset R^z \subset R$ or $R^z \subset Q \subset R \subset Q_z$.

(b) If I is any z-semi-ideal between Q and R, then I lies between Q^z and R_z. The result follows.

(c) This follows from the maximality of R^z and the minimality of Q_z. □

Let P be a poset. Then P is said to satisfy the descending chain condition if every non-empty subset of P has a minimal element.

Theorem 2.22. *Let* I *and* J *be u-semi-ideals of* P *satisfying descending chain condition and the intersection of two prime semi-ideals of* P *is again a prime semi-ideal of* P. *If* I *and* J *are not in a chain and if the union of two minimal prime u-semi-ideals of* P *is a z-semi-ideal of* P, *then we have the followings:*

(a) *If* I *is a z-semi-ideal and* J *is prime, then* $I \cup J$ *is a prime z-semi-ideal of* P.
(b) *If* I *is semiprime and* J *is prime, then* $I \cup J = I \cup (I \cup J)^z$. *Moreover, if* $I \cup J$ *is not a z-semi-ideal of* P, *then* $I \cup J$ *is a minimal prime semi-ideal of* P *containing* I *and* I *does not contain any prime semi-ideal of* P.

Proof. (a) If I is prime, then by Theorem 2.20, $I \cup J$ is a prime z-semi-ideal. If I is not prime, then $I \subset (I \cup J)^z$, since I is a z-semi-ideal. Clearly $J \cup (I \cup J)^z \subseteq I \cup J$. So $I \cup J = J \cup (I \cup J)^z$ is a prime z-semi-ideal of P.

(b) Let I be a semiprime u-semi-ideal of P. Then $I = \bigcap_i P_i$, where P_i are minimal prime u-semi-ideals containing I. Since $J \subseteq I \cup J$, we have either $J \subseteq (I \cup J)^z \subset I \cup J$ or $(I \cup J)^z \subseteq J \subset I \cup J$, by Theorem 2.14

Case (i): If $J \subseteq (I \cup J)^z \subset I \cup J$, then $I \cup J = I \cup (I \cup J)^z$.

Case (ii): Suppose that I and $(I \cup J)^z$ are in a chain. If $I \subset (I \cup J)^z$, then $I \subset J$, which is a contradiction. If $(I \cup J)^z \subset I = \bigcap_i P_i$, then each P_i contain the prime semi-ideal $(I \cup J)^z$ and so form a chain, a contradiction.

If I and $(I \cup J)^z$ are not in a chain, then $I \cup (I \cup J)^z$ is a prime semi-ideal containing $(I \cup J)^z$ and so is in a chain with J. If $J \subset I \cup (I \cup J)^z \subset I \cup J$, then

$I \cup J = I \cup (I \cup J)^z$. If $I \cup (I \cup J)^z \subset J$, then $I \subset J$, a contradiction. If I contains a prime semi-ideal, then it is prime and we are done by Theorem 2.20.

Finally, suppose that $I \cup J$ is not a z-semi-ideal and there exists a prime semi-ideal K with $I \subset K \subset I \cup J$. Then $I \cup J = K \cup J$ is a z-semi-ideal, a contradiction. So $I \cup J$ is a minimal prime semi-ideal containing I. □

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